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Numerical Methods for Solving Nonlinear Fisher Equation Using Backward Differentiation Formula

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Abstract

This paper examines the numerical solution of the nonlinear Fisher equation that is used to find the growth of tumour cells in the brain. By employing new methods that transform nonlinear partial differential equations (PDE) into nonlinear ordinary differential equations (ODE) through spatial discretization. The stability of the resulting nonlinear system is evaluated using Lyapunov's criteria. Implicit stiff solvers, including various orders of backward differentiation formulas, are used to address the ODE system. The efficiency of these numerical methods is demonstrated through two examples, and a comparison with existing methods from the literature is conducted. Compared to traditional methods, the proposed numerical techniques are distinguished by their simplicity, precision, and remarkable efficiency.

Keywords: Method of lines; Fisher equation; Stability analysis; Backward differentiation formula

MSC 2010 No.: 65M20, 92D25, 65L06, 65L07

1. Introduction

The Fisher equation, formulated in Fisher (1937), is discussed in Section 2. Equation (1) has been explained in more detail in the paper by Kolmogorov et al. (1937). This equation is also known as the Fisher-Kolmogorov-Petrovsky-Piscunov (Fisher-KPP) equation. The exact solution of the Fisher equation was initially discovered by Ablowitz and Zeppetella (1979). Fisher's equation (1) has broad applications, including nuclear chemistry, chemical kinetics (Malfliet (1992)), and a multitude of scientific and engineering fields. In this chapter, a wide range of topics has been investigated, including population growth models, flame propagation, neurophysiology, autocatalytic reactions, and branching Brownian motion, as cited in the references Frank-Kamenetskii (1969), Ammerman and Cavalli-Sforza (1971), and Bramson (1978); Canosa (1969). These various applications emphasize its role in understanding physical, chemical, biological, and medical phenomena (Vimal et al. (2024a); Wang (1988); Polyanin and Zaitsev (2003); Sherratt (1998)).

Many researchers have explored the theory behind Fisher's equation, and computational methods have been widely used to study its traveling wave solutions. This has resulted in extensive discussions and the publication of valuable research. Significantly, Tyson and Brazhnik (2000), Kawahara and Tanaka (1983), Larson (1978), and Vimal et al. (2024b) provide in-depth overviews of Fisher's equation. Ablowitz and Zeppetella (1979) offers explicit solutions for specific wave speeds, and Chandraker et al. (2015) introduces a semi-implicit difference scheme. Hussan and Mebrate (2022) presents semi-implicit time discretization methods, such as Crank-Nicolson schemes.

Mickens (1994) proposes an optimized finite-difference scheme tailored for Fisher's equation. Moreover, modified versions of Fisher's reaction-diffusion equation have been addressed using radial basis functions in conjunction with the differential quadrature method (Hanaç et al. (2022)). Fisher's equation's applicability extends to bounded domains through Faedo-Galerkin's method (Hamrouni et al. (2021)), and it has been solved using an extended homogeneous balance method for various nonlinear equations, including Fisher's (Fares et al. (2021)). Additionally, symmetries of the generalized Fisher equation have been explored (Rosa et al. (2020)), and Fisher's equation has been extended to the fuzzy fractional Fisher equation in a Caputo sense (Ahmad et al. (2021)). In the numerical domain, Qiu and Sloan (1998) have utilized a moving mesh method to provide solutions for Fisher's equation. These thorough investigations and applications highlight the broad importance and flexibility of Fisher's equation in scientific research and problem-solving across multiple disciplines.

This research paper presents a systematic and innovative approach by directly applying the method of lines (MOL) to tackle nonlinear time-dependent Fisher's equations. In Section 2, the mathematical formulation of the Fisher equation is described. In Section 3, the derivatives along the X-direction are approximated by finite differences in the method of lines. The suggested approach, the method of lines (MOL), involves converting the nonlinear PDEs into a system of nonlinear ODEs. In Section 4, the stability of this nonlinear system is evaluated through Lyapunov's indirect method. In Section 5, the nonlinear system is linearized using Taylor series expansion and subsequently solved with three different schemes of backward differentiation formulas. In Section

6, comparison with other existing numerical methods, two test problems have been successfully solved and their numerical solutions have been thoroughly discussed. The results are presented through an error table, as well as two- and three-dimensional figures. In Section 7, the conclusion is given.

2. Mathematical Formulation

The Fisher equation is expressed as:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \alpha u(1-u), \quad 0 \le x \le 1, \quad t > 0, \tag{1}$$

with initial condition

$$u(x,0) = u_0(x), \quad 0 \le x \le 1.$$

Additionally, consider the boundary conditions,

$$u(0,t) = f_1(t), \quad 0 \le t \le T,$$
$$u(1,t) = f_2(t), \quad 0 \le t \le T.$$

In this framework, the symbol α represents a factor affecting reactions, while T denotes the final time in our mathematical analysis. Additionally, the functions $u_0(x)$, f_1, and f_2 are used to define initial and boundary conditions for the model.

3. Proposed Scheme

In this paper, we present an effective solution approach for time-dependent nonlinear Fisher equations through creative schemes we have developed. Our approach begins with the discretization of the X-direction using the method of lines technique. To achieve this goal, we can partition the spatial dimension into $K+1$ evenly spaced points using a uniform interval of $h = \frac{1}{K}$. As clearly illustrated, partial derivatives are estimated using the central difference method, as shown below:

$$\frac{\partial u}{\partial x}(x_j,t) = \frac{u_{j+1}(t) - u_{j-1}(t)}{2h}, \quad j = 1,2,\ldots,K-1,$$
$$\frac{\partial^2 u}{\partial t^2}(x_j,t) = \frac{u_{j-1}(t) - 2u_j(t) + u_{j+1}(t)}{(h)^2}, \quad j = 1,2,\ldots,K-1.$$

Upon substituting the boundary conditions $u_0(t) = 0$ and $u_K(t) = 0$ into Fisher's equation (1), we effectively convert it into a system of nonlinear ordinary differential equations (ODEs). These ODEs are defined by their initial conditions, shaping the system's behavior and evolution over time:

$$\frac{du_j(t)}{dt} = \frac{u_{j-1}(t) - 2u_j(t) + u_{j+1}(t)}{(h)^2} + \alpha u_j(1-u_j),$$
$$u_j(0) = u_0(x_j), \quad j = 1,2,\ldots,K-1.$$

Here, with $u_j(t) = u(x_j, t)$ representing the system's variables, this set of $(K-1) \times (K-1)$ nonlinear differential equations can be efficiently expressed in matrix form as follows:

$$\frac{dU}{dt} = F(U,t), \tag{2}$$

$$U(0) = U_0.$$

In our formulation, we define the state vector as $U(t) = [u_1(t), u_2(t), ..., u_{K-1}(t)]^T$, and the nonlinear function F consists of elements f_j given by the equation:

$$f_j(u_1, u_2, ..., u_{N-1}, t) = \lambda u_{j-1}(t) + u_j(\alpha - \alpha u_j - 2\lambda) + \lambda u_{j+1}(t). \tag{3}$$

Here, λ represents a constant calculated as $\frac{1}{(\Delta x)^2}$, and the index j ranges from 1 to $K-1$. As part of our research, we will thoroughly examine this nonlinear system of ordinary differential equations (referred to as Equation (3)) with specific emphasis on its stability, uniqueness, and existence characteristics. This analysis will yield valuable insights into the behavior and attributes of the system under investigation.

Theorem 3.1.

Investigating the initial value problem (IVP) involves treating $\mathbf{F(U,t)}$ as a continuous function and exploring it in the following manner:

$$\frac{dU}{dt} = F(U,t), \quad U(t_0) = a,$$

allows for the existence of a solution denoted as $U = f(t)$ within the interval $|(t-t_0)| \leq \delta$, where $\delta > 0$.

Proof:

Consider the functions defined as follows:

$$f_j(u_1, u_2, ..., u_{K-1,t}) = \lambda u_{j-1}(t) + u_j(\alpha - \alpha u_j - 2\lambda) + \lambda u_{j+1}(t).$$

Given that j ranges from 1 to $K-1$, and recognizing that the functions are clearly continuous, we can confidently assert the existence of a solution for this initial value problem (IVP). ∎

Theorem 3.2.

Let $\mathbf{C^1}$ denote the set of functions that are differentiable and have continuous first derivatives.

If $\mathbf{F(U,t)} \in \mathbf{C^1}$, then a unique solution exists for the initial value problem (IVP).

Proof:

The partial derivatives of the functions described in Equation (4) can be expressed as follows:

$$\frac{\partial f_j}{\partial u_j} = \alpha - 2\alpha u_j - 2\lambda, \quad j = 1, 2, ..., K-1, \tag{4}$$

$$\frac{\partial f_j}{\partial u_{j+1}} = \lambda, \quad j = 1, 2, ..., K-1, \tag{5}$$

$$\frac{\partial f_j}{\partial u_{j-1}} = \lambda, \quad j = 1, 2, ..., K-1, \tag{6}$$

$$\frac{\partial f_j}{\partial u_i} = 0, \quad j = 1, 2, ..., K-1, \quad i \neq j-1, j, j+1. \tag{7}$$

Every partial derivative of the function is in existence and remains continuous across the entire domain, thereby confirming that $\mathbf{F}(\mathbf{U}, \mathbf{t}) \in \mathbf{C^1}$. Consequently, the IVP possesses a distinct solution. The forthcoming section will explore an examination of the stability of the nonlinear system. ■

4. Stability Analysis

In the context of nonlinear stability analysis, Lyapunov's stability theory stands out as a fundamental mathematical instrument. To assess stability, a significant approach involves determining the eigenvalues of the Jacobian matrix at the equilibrium point of a nonlinear autonomous system.

When we look at the nonlinear system described in Equation (3), because it operates on its own without external influences, we observe the following:

$$\frac{dU}{dt} = F(U),$$
$$U(0) = U_0.$$

Where F represents a nonlinear function of U, the elements f_j can be expressed as follows:

$$f_j(u_1, u_2, ..., u_{K-1,t}) = \lambda u_{j-1}(t) + u_j(\alpha - \alpha u_j - 2\lambda) + \lambda u_{j+1}(t).$$

For $j = 1, 2, \dots, K-1$, we can expand F as a Taylor series centered around the equilibrium point $U^* = 0$,

$$F(U) \approx F(U^*) + F(U^*)(U - U^*)$$
$$\approx f'(U^*)U.$$

We will investigate the system's stability as described in Equation (5) by employing Lyapunov's Indirect Method.

4.1. Lyapunov's Indirect Method

Consider the equilibrium point at $x = 0$ for the equation $\dot{x} = f(x)$. Here, $f : D \longrightarrow \mathbb{R}^k$ is a continuously differentiable function, and D represents a neighborhood around the origin. Let

$$A = \frac{\partial f}{\partial x}|_{x=0}.$$

Then,
1. The origin exhibits asymptotic stability when the real part of each eigenvalue λ_i of matrix A satisfies $Re(\lambda_i) \leq 0$,
2. The origin is deemed unstable if there exists at least one eigenvalue A_i of matrix A such that $Re(\lambda_i) > 0$.

For the nonlinear system described in Equation (5), we can provide the Jacobian matrix as follows:

$$F(U^{(k+1)}) = F(U^{(k)}) + J_F^{(k)}(U^{(k+1)} - U^k) + O(\Delta t^2), \tag{8}$$

where

$$J_F^{(k)} = \begin{pmatrix} (\frac{\partial f_1}{\partial u_1})^{(k)} & (\frac{\partial f_1}{\partial u_2})^{(k)} & \cdots (\frac{\partial f_1}{\partial u_{K-1}})^{(k)} \\ \vdots & & \\ (\frac{\partial f_{K-1}}{\partial u_1})^{(k)} & (\frac{\partial f_{K-1}}{\partial u_2})^{(k)} & \cdots (\frac{\partial f_{K-1}}{\partial u_{K-1}})^{(k)} \end{pmatrix}.$$

The Jacobian matrix, denoted as matrix 'A' and evaluated at the equilibrium point, can be represented as a tridiagonal matrix, and its specific form is

$$A = \begin{pmatrix} R_1 & \lambda & & & \\ \lambda & R_2 & \lambda & & \\ & \ddots & \ddots & \ddots & \\ & & \lambda & R_{K-2} & \lambda \\ & & & \lambda & R_{K-1} \end{pmatrix},$$

given that $R_j = \alpha - 2\alpha u_j - 2\lambda$, where $j = 1, 2, ..., K - 1$.

The eigenvalues of the matrix are expressed as:

$$\mu_s = -2\lambda + 2\lambda \cos\left(\frac{s\pi}{n+1}\right), \quad \text{for } s = 1, 2, \ldots, n.$$

Given that

$$-1 < \cos\left(\frac{s\pi}{n+1}\right) < 1,$$

it follows that

$$2\lambda \cos\left(\frac{s\pi}{n+1}\right) < 2\lambda.$$

Thus, we have:

$$-2\lambda + 2\lambda \cos\left(\frac{s\pi}{n+1}\right) < 0.$$

This implies that all eigenvalues of the Jacobian matrix lie in the open left half of the complex plane. Consequently, the origin is asymptotically stable for the nonlinear system.

5. Numerical Integration

Partition the time span $[0, T]$ into $M + 1$ equidistant intervals, each with a time step of $\Delta t = T/M$. Then, employ backward differentiation formulas (BDFs) for implicit time integration, such as BDF1 (backward Euler), BDF2, or BDF3, to approximate the solution iteratively at each time step.

5.1. Backward Differentiation Formula of order one (BDF1)

$$U^{k+1} = U^k + (\Delta t)F(U^{k+1}, t^{k+1}), \quad k = 0, 1, ..., M-1, \tag{9}$$

we denote the initial condition as U^0, and for the purpose of iterative solution, the nonlinear system (7) can be linearized by employing a Taylor series expansion centered at the vector $U^{(k)}$.

$$F(U^{k+1}) = F(U^{(k)}) + J_F^{(k)}(U^{(k+1)} - U^{(k)}) + O(\Delta t)^2, \tag{10}$$

$J_F^{(k)}$ denotes the Jacobian matrix corresponding to the system at the k^{th} time step.

By substituting Equation (10) into Equation (9), we obtain:

$$\begin{aligned} U^{k+1} &= U^k + (\Delta t)[F(U^{(k)}) + J_F^{(k)}(U^{(k+1)} - U^{(k)})], \\ U^{k+1} &= U^k + (1 - \Delta t J_F^{(k)})^{-1}\Delta t[F(U^{(k)})]. \end{aligned} \tag{11}$$

5.2. Backward Differentiation Formula of order two (BDF2)

$$U^{k+1} = \frac{4}{3}U^k - \frac{1}{3}U^{k-1} + \frac{2}{3}(\Delta t)F(U^{k+1}, t^{k+1}), \quad k = 0, 1, ..., M-1, \tag{12}$$

U^1 is obtained using BDF1, and for linearization, we employ a Taylor series expansion

$$F(U^{k+1}) = F(U^{(k)}) + J_F^{(k)}(U^{(k+1)} - U^{(k)}) + O(\Delta t)^2, \tag{13}$$

given that $J_F^{(k)}$ represents the Jacobian matrix at the k^{th} time level. When we insert Equation (13) into Equation (12), we get

$$U^{(k+1)} = \frac{4}{3}U^{(k)} - \frac{1}{3}U^{k-1} + \frac{2\Delta t[F(U^{(k)}) + J_F^{(k)}(U^{(k+1)}) - U^{(k)}]}{3}, \quad k = 2, \ldots, M \tag{14}$$

$$\left(I - \frac{2\Delta t}{3}J_F^{(k)}\right)U^{(k+1)} = \left(\frac{4}{3}I - \frac{2\Delta t}{3}J_F^{(k)}\right)U^{(k)} + \frac{2\Delta t}{3}F(U^{(k)}) - \frac{1}{3}(U^{(k-1)}),$$

$$\begin{aligned} U^{(k+1)} &= (I - \frac{2\Delta t}{3}J_F^{(k)})^{-1}\left(\frac{4}{3}I - \frac{2\Delta t}{3}J_F^{(k)}\right)U^{(k)} \\ &+ \left(I - \frac{2\Delta t}{3}J_F^{(k)}\right)^{-1}\frac{2\Delta t}{3}F(U^{(k)}) - \left(I - \frac{2\Delta t}{3}J_F^{(k)}\right)^{-1}\frac{1}{3}(U^{(k-1)}). \end{aligned} \tag{15}$$

Therefore, the presented scheme has been linearized, simplifying the task to solving the linear algebraic equations described in Equation (15), which results in reduced computational time.

5.3. Backward Differentiation Formula of order three (BDF3)

$$U^{k+1} = \frac{18}{11}U^k - \frac{9}{11}U^{k-1} + \frac{2}{11}U^{k-2} + \frac{6}{11}(\Delta t)F(U^{k+1}, t^{k+1}), \quad k = 0, 1, ..., M-1, \tag{16}$$

U^1 and U^2 are obtained through BDF1, and for the purpose of linearization, we employ a Taylor series expansion

$$F(U^{k+1}) = F(U^{(k)}) + J_F^{(k)}(U^{(k+1)} - U^{(k)}) + O(\Delta t)^2, \tag{17}$$

with $J_F^{(k)}$ symbolizing the Jacobian matrix at the k^{th} temporal stage.

Replacing Equation (17) into Equation (16) yields:

$$\begin{aligned} U^{k+1} &= \frac{18}{11}U^k - \frac{9}{11}U^{k-1} + \frac{2}{11}U^{k-2} \\ &+ \frac{6\Delta t[F(U^{(k)}) + J_F^{(k)}(U^{(k+1)}) - U^{(k)}]}{11}, \quad k = 0, 1, \ldots, M-1 \end{aligned} \tag{18}$$

$$\left(I - \frac{6\Delta t}{11}J_F^{(k)}\right)U^{(k+1)} = \left(\frac{18}{11}I - \frac{6\Delta t}{11}J_F^{(k)}\right)U^{(k)} + \frac{6\Delta t F(U^{(k)})}{11} - \frac{9}{11}U^{(k-1)} + \frac{2}{11}U^{(k-2)},$$

$$\begin{aligned} U^{(k+1)} &= \left(I - \frac{6\Delta t}{11}J_F^{(k)}\right)^{-1}\left(\frac{18}{11}I - \frac{6\Delta t}{11}J_F^{(k)}\right)U^{(k)} + \left(I - \frac{6\Delta t}{11}J_F^{(k)}\right)^{-1}\frac{6\Delta t F(U^{(k)})}{11} \\ &- \left(I - \frac{6\Delta t}{11}J_F^{(k)}\right)^{-1}\frac{9}{11}U^{(k-1)} + \left(I - \frac{6\Delta t}{11}J_F^{(k)}\right)^{-1}\frac{2}{11}U^{(k-2)}. \end{aligned} \tag{19}$$

By representing $J_F^{(k)}$ as the Jacobian matrix at the k^{th} time step, this approach becomes linearized, simplifying the problem to solving linear algebraic equations and thereby reducing computational time significantly.

6. Numerical Findings and Discussion

The effectiveness and versatility of the proposed numerical method have been thoroughly tested through multiple trials. Various combinations of α values and different final time settings were used to compute solutions, which were then systematically compared with the exact solutions from illustrative examples. Furthermore, a comparative analysis was conducted to compare the numerical results with those available in existing literature. All computational tasks were executed using MATLAB codes developed specifically for this purpose, ensuring accuracy and reliability.

Example 6.1.

Consider the Fisher equation

$$u_t = u_{xx} + \alpha u(1-u), \tag{20}$$

subject to the given initial condition,

$$u(x,0) = \frac{1}{(1+e^{\sqrt{\frac{\alpha}{6}}x})^2}. \tag{21}$$

Additionally, considering the prescribed boundary conditions,

$$u(0,t) = \frac{1}{(1+e^{-5t})^2}, \qquad 0 \le t \le T,$$
$$u(1,t) = \frac{1}{(1+e^{1-5t})^2}, \qquad 0 \le t \le T.$$

The exact solution, as provided in Wazwaz and Gorguis (2004), is expressed as:

$$u(x,t) = \frac{1}{(1+e^{\sqrt{\frac{\alpha}{6}}x-\frac{5}{6}\alpha t})^2}. \tag{22}$$

In our approach utilizing the method of lines semi-discretization, we have directly solved the Fisher equation. We used first, second, and third-order implicit solvers of backward differentiation formulas (BDFs) and compared the numerical solutions to the exact solutions, at time intervals of $\Delta t = 0.000005$, for two different values of α: specifically, $\alpha = 6$ and $\alpha = 1$. Figures 1, 2 and 3 show the comparison between numerical and exact solutions for $\Delta t = 0.000005$ and $\alpha = 6$, while Figures 4, 5 and 6 display the same for $\alpha = 1$. Furthermore, Table 3 and Table 4 provide the error analysis for both $\alpha = 6$ and $\alpha = 1$ at $\Delta t = 0.000005$. For a visual representation of the absolute error, Figures 7 and 8 illustrate graphs for various Δt values, for $\alpha = 6$ and $\alpha = 1$.

To evaluate the performance of our proposed numerical schemes, Tables 5, 6, 7, 10, and 11 compare our results with previously published schemes for Example 1, at $\alpha = 6$ for different values of time steps. Similarly, Tables 8 and 9 conduct the same comparison for Example 2 at $\alpha = 1$.

Remarkably, the numerical solutions achieved through the implementation of BDF2 and BDF3 display superior accuracy when compared to the outcomes produced by BDF1. The proposed numerical schemes include implicit solvers, specifically, BDF1, BDF2, and BDF3, designed to enhance accuracy in addressing the Fisher equation. Compared to the methods described by Bastani and Salkuyeh (2012), Mittal and Jiwari (2009), amd Hussan and Mebrate (2022), our proposed numerical schemes exhibit a remarkable level of agreement with exact solutions.

Example 6.2.

Let's consider Fisher's equation within the domain [0, 1]:

$$u_t = u_{xx} + u(1 - u^\alpha), \tag{23}$$

subject to the given initial condition,

$$u(x,0) = \left\{ \frac{1}{2} \tanh\left(-\frac{\alpha}{2\sqrt{2\alpha+4}} x \right) + \frac{1}{2} \right\}^{\frac{2}{\alpha}}. \tag{24}$$

The precise solution is elaborated as follows (Wang (1988); Bastani and Salkuyeh (2012)):

$$u(x,t) = \left\{ \frac{1}{2} \tanh\left(-\frac{\alpha}{2\sqrt{2\alpha+4}} \left(x - \frac{\alpha+4}{\sqrt{2\alpha+4}} t \right) \right) + \frac{1}{2} \right\}^{\frac{2}{\alpha}}. \tag{25}$$

7. Conclusion

This paper examines the use of semidiscretization techniques in combination with backward differentiation formulas to solve Fisher's equation. This equation is found in various scientific and engineering fields, with its main application in the biomedical sciences. It is used to determine the size of brain tumours. Two examples are compared between numerical results and exact solutions. BDF2 and BDF3 are found to be more accurate than BDF1. The results obtained with this approach are more precise and closer to the exact solution compared to those described in the comparison table. The numerical methods proposed in this paper demonstrate their effectiveness and reliability in solving Fisher's equation.

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Appendix

Table 1. We compared numerical (BDF1, BDF2, BDF3) and exact results for Example 1 at various spatial points, with a time step of $\Delta t = 0.000005$, a total time of $T = 0.1$, and $\alpha = 6$

	Calculated Solution			**Exact solution**
x	BDF1	BDF2	BDF3	
0	0.387455619	0.387455619	0.387455619	0.387455619
0.1	0.358418057	0.358420710	0.358420160	0.358426914
0.2	0.329970861	0.329972600	0.329974683	0.329984205
0.3	0.302300596	0.302301568	0.302305675	0.302317425
0.4	0.275584019	0.275584418	0.275589827	0.275603147
0.5	0.249979873	0.249979929	0.249985848	0.250000000
0.6	0.225625036	0.225625003	0.225630619	0.225644772
0.7	0.202631559	0.202631703	0.202636242	0.202649430
0.8	0.181084751	0.181085339	0.181088115	0.181099172
0.9	0.161042341	0.161043628	0.161044094	0.161051594
1	0.142536957	0.142536957	0.142536957	0.142536957

Table 2. We compared numerical (BDF1, BDF2, BDF3) and exact results for Example 2 at various spatial points, with a time step of $\Delta t = 0.000005$, a total time of $T = 0.1$, and $\alpha = 1$

	Calculated Solution			**Exact solution**
x	BDF1	BDF2	BDF3	
0	0.271254811	0.271254811	0.271254811	0.271254811
0.1	0.260737784	0.260738230	0.260738056	0.260738428
0.2	0.250420391	0.250420750	0.250420905	0.250421096
0.3	0.240310950	0.240311240	0.240311656	0.240311688
0.4	0.230417634	0.230417876	0.230418460	0.230418385
0.5	0.220747899	0.220748116	0.220748764	0.220748648
0.6	0.211308466	0.211308683	0.211309286	0.211309201
0.7	0.202105305	0.202105547	0.202106001	0.202106010
0.8	0.193143625	0.193143913	0.193144130	0.193144276
0.9	0.184427868	0.184428219	0.184428133	0.184428430
1	0.175962132	0.175962132	0.175962132	0.175962132

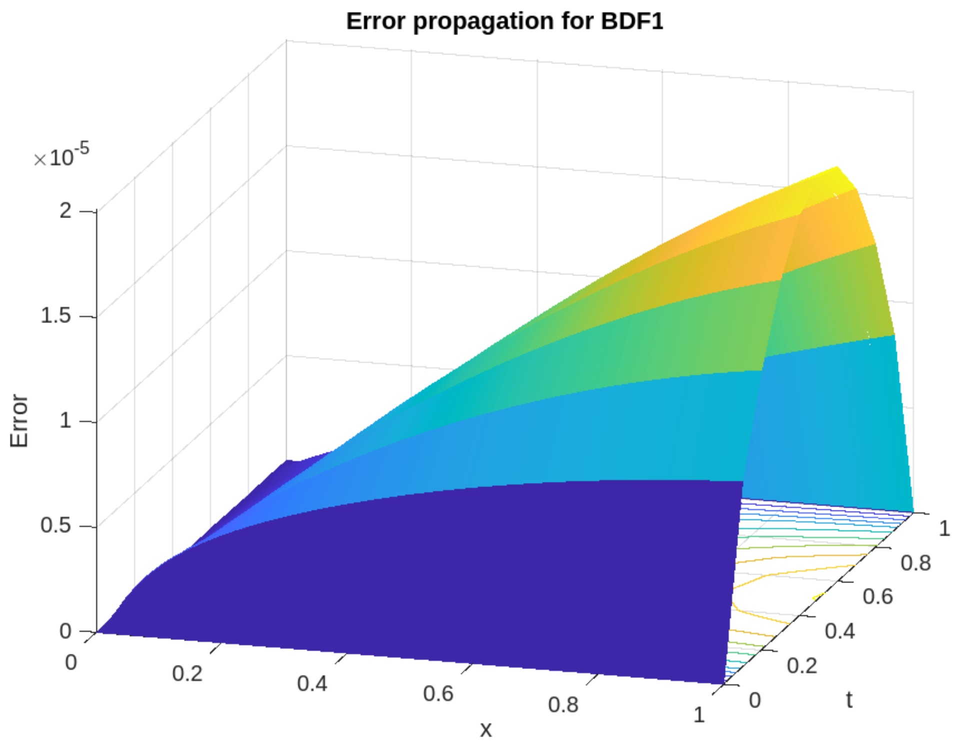

Figure 1. Solution at $\Delta t = 0.000005$, $\alpha = 6$, and $T = 0.1$ for Example 1

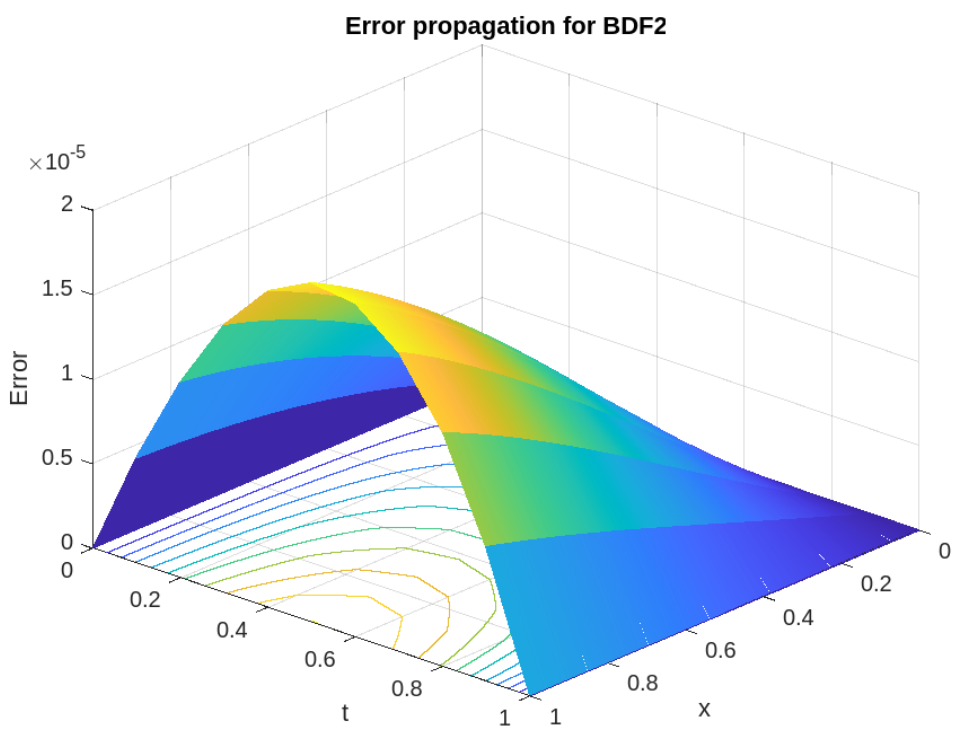

Figure 2. Solution at $\Delta t = 0.000005$, $\alpha = 6$, and $T = 0.1$ for Example 1

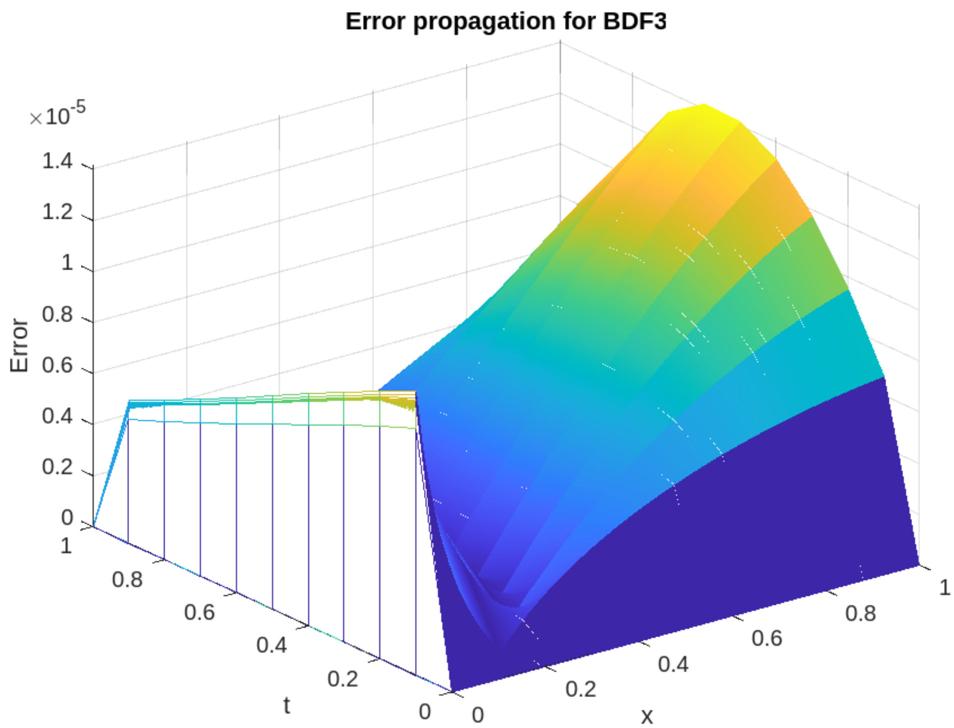

Figure 3. Solution at $\Delta t = 0.000005$, $\alpha = 6$, and $T = 0.1$ for Example 1

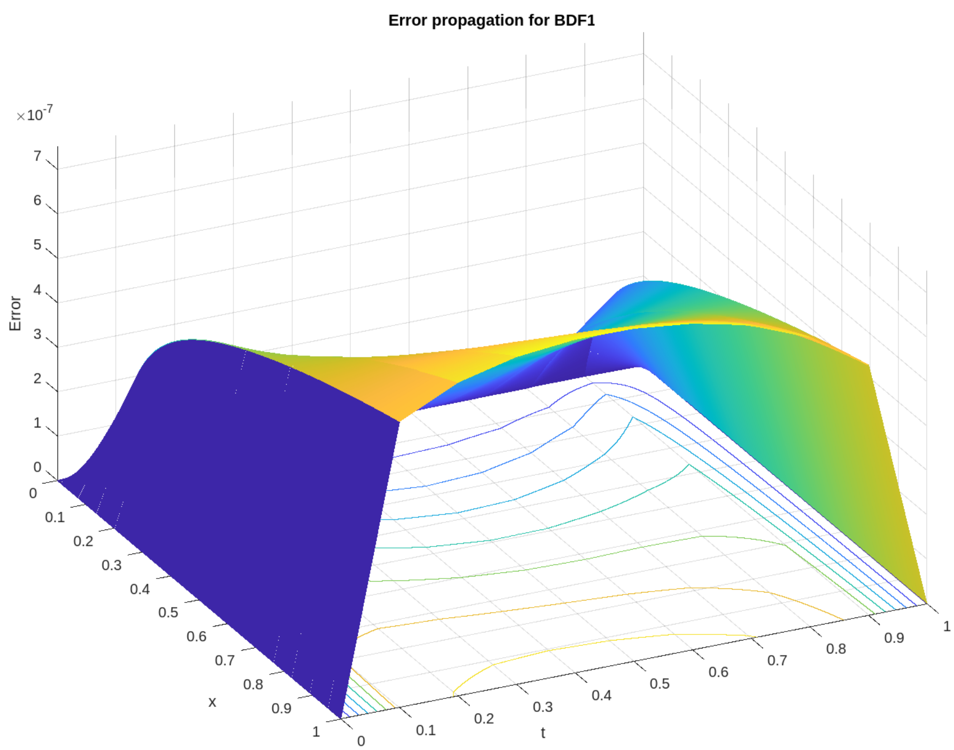

Figure 4. Solution at $\Delta t = 0.000005$, $\alpha = 1$, and $T = 0.1$ for Example 2

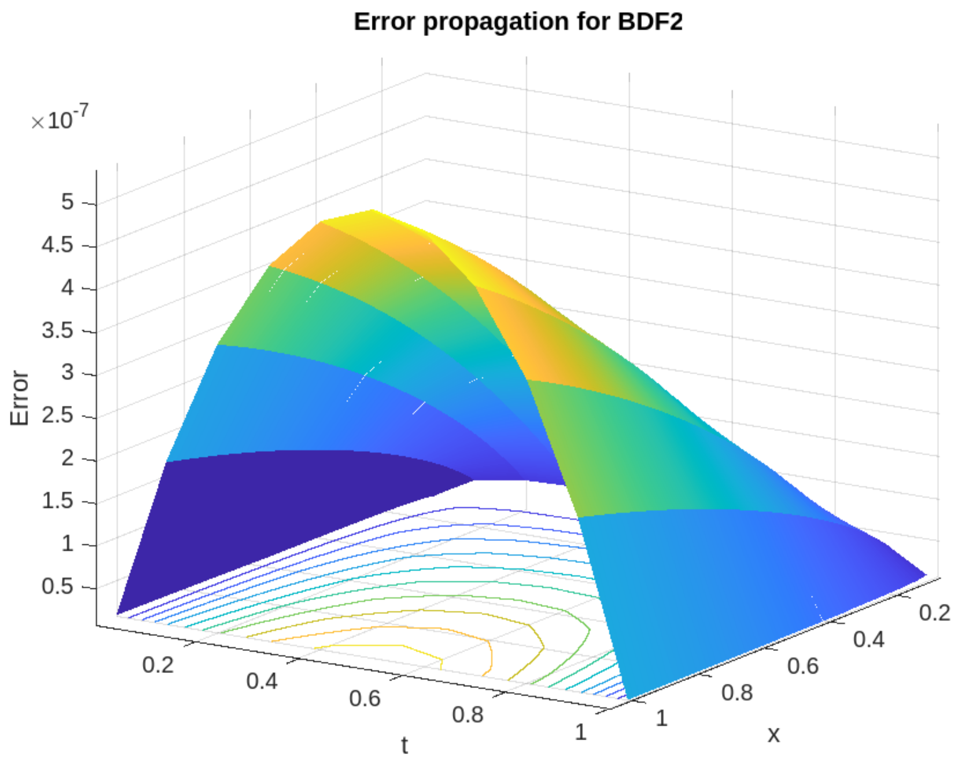

Figure 5. Solution at $\Delta t = 0.000005$, $\alpha = 1$, and $T = 0.1$ for Example 2

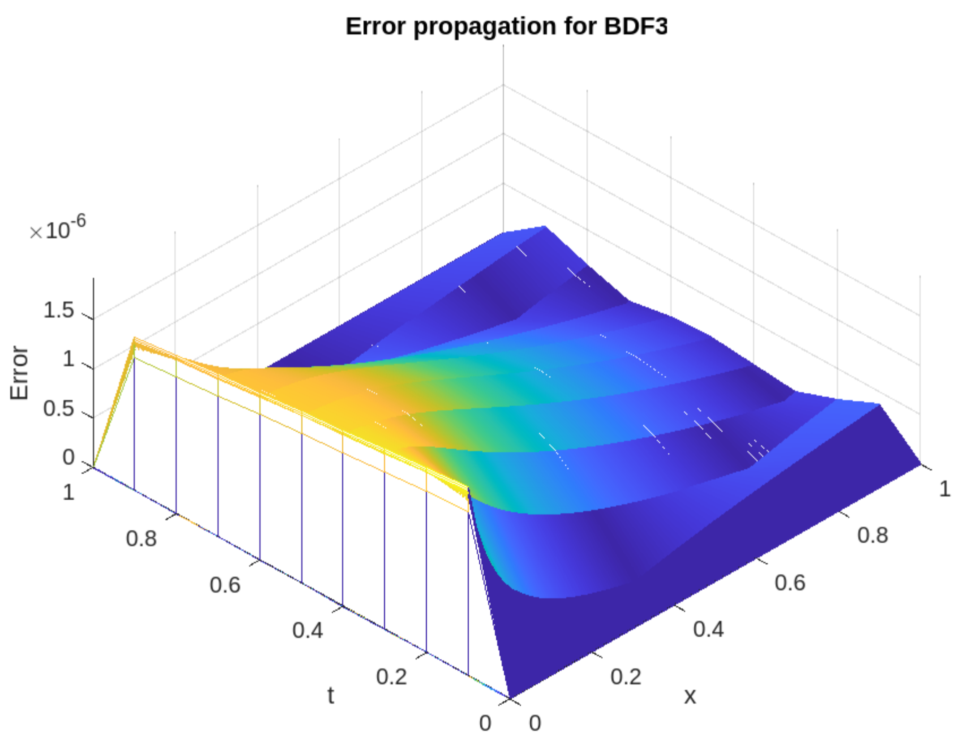

Figure 6. Solution at $\Delta t = 0.000005$, $\alpha = 1$, and $T = 0.1$ for Example 2

Table 3. We analyzed the errors (BDF1, BDF2, BDF3) at various spatial points in Example 1, comparing them to the exact solution. This evaluation was carried out with $\Delta t = 0.000005$, $T = 0.1$, and $\alpha = 6$

	Absolute error		
x	BDF1	BDF2	BDF3
0.1	8.85790E-06	6.20440E-06	6.75430E-06
0.2	1.33443E-05	1.16049E-05	9.52180E-06
0.3	1.68285E-05	1.58563E-05	1.17495E-05
0.4	1.91282E-05	1.87290E-05	1.33201E-05
0.5	2.01267E-05	2.00705E-05	1.41522E-05
0.6	1.97368E-05	1.97697E-05	1.41538E-05
0.7	1.78707E-05	1.77267E-05	1.31883E-05
0.8	1.44206E-05	1.38330E-05	1.10570E-05
0.9	9.25270E-06	7.96580E-06	7.50020E-06

Table 4. We analyzed the errors (BDF1, BDF2, BDF3) at various spatial points in Example 2, comparing them to the exact solution. This evaluation was carried out with $\Delta t = 0.000005$, $T = 0.1$, and $\alpha = 1$

	Absolute error		
x	BDF1	BDF2	BDF3
0.1	6.437E-07	1.983E-07	3.716E-07
0.2	7.053E-07	3.458E-07	1.905E-07
0.3	7.378E-07	4.480E-07	3.260E-08
0.4	7.505E-07	5.092E-07	7.500E-08
0.5	7.491E-07	5.322E-07	1.157E-07
0.6	7.350E-07	5.174E-07	8.510E-08
0.7	7.048E-07	4.625E-07	8.700E-09
0.8	6.511E-07	3.628E-07	1.461E-07
0.9	5.626E-07	2.113E-07	2.972E-07

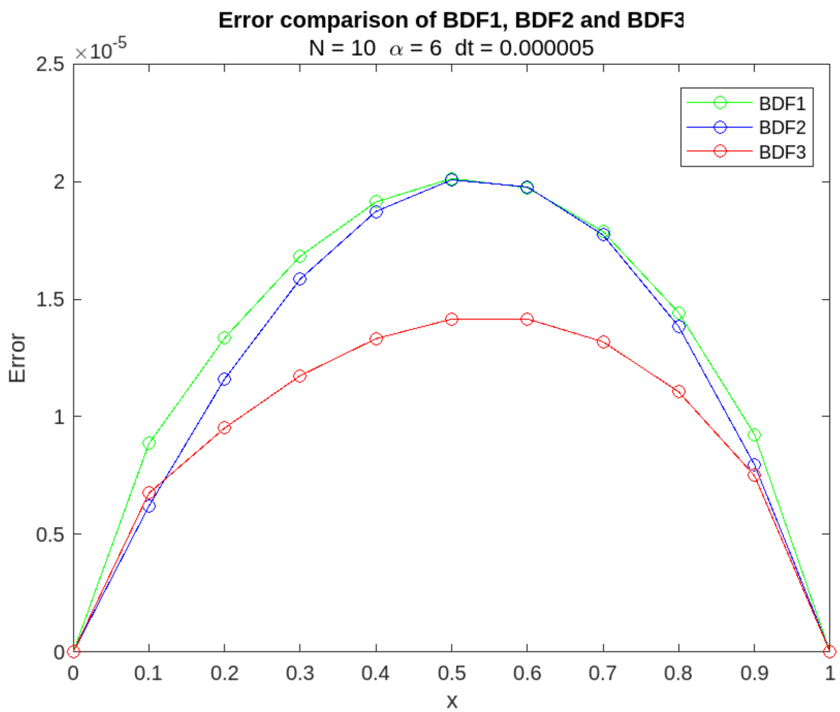

Figure 7. Absolute error at $\Delta t = 0.000005$, $T = 0.1$, and $\alpha = 6$ for Example 1

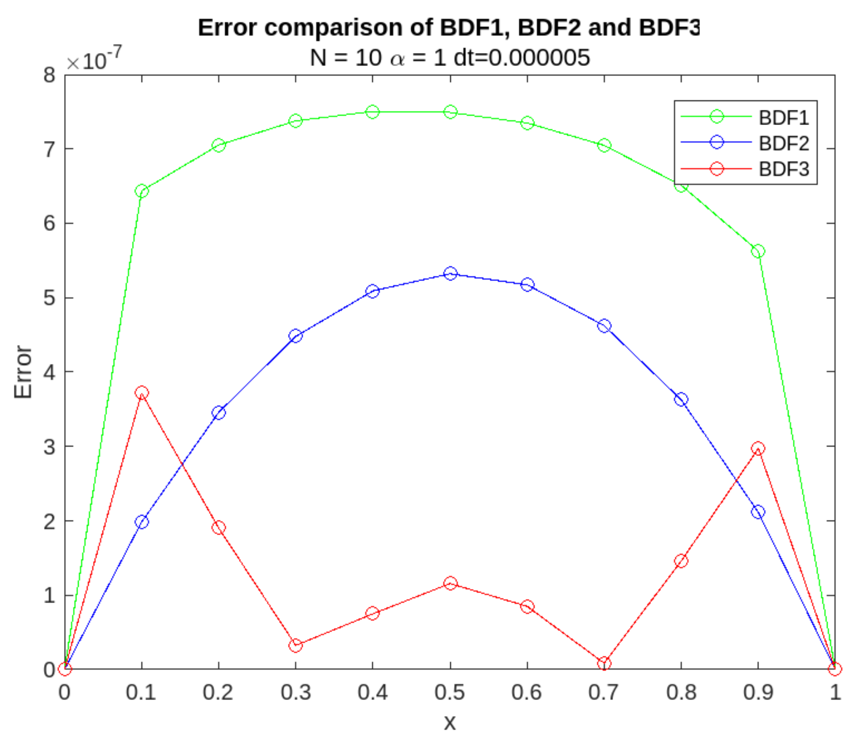

Figure 8. Absolute error at $\Delta t = 0.000005$, $T = 0.1$, and $\alpha = 1$ for Example 2

Table 5. We conducted a spatial comparison between numerical results and exact solutions for Example 1 at $\Delta t = 0.00005$, $\alpha = 6$, and a spatial resolution of N = 20

T	x	**DQM(Bastani and Salkuyeh (2012))**	**BDF1**	**BDF2**	**BDF3**	**Exact Solution**
0.5	0.25	0.81847	0.818375	0.818403	0.818413	0.818393
	0.75	0.72592	0.725799	0.725835	0.725840	0.725824
1.0	0.25	0.98293	0.982916	0.982922	0.982922	0.982919
	0.75	0.97208	0.972067	0.972074	0.972074	0.972071

Table 6. Numerical and exact results for Example 1 were compared at various spatial locations with a time step of $\Delta t = 0.0001$ and $\alpha = 6$

x	T	**Mittal and Jiwari (2009)**	**BDF1**	**BDF2**	**BDF3**	**Exact Solution**
0.25	0.5	0.81847	0.818399	0.818455	0.818475	0.818393
	1.0	0.98293	0.982909	0.982917	0.982921	0.982919
	2.0	0.99988	0.999883	0.999883	0.999883	0.999883
	5.0	1.00000	1.000000	1.000000	1.000000	1.000000
0.5	0.5	0.77590	0.775828	0.775896	0.775932	0.775828
	1.0	0.97816	0.978134	0.978145	0.978151	0.978147
	2.0	0.99985	0.978134	0.999850	0.999850	0.978147
	5.0	1.00000	1.000000	1.000000	1.000000	1.000000
0.75	0.5	0.72594	0.725830	0.725902	0.725913	0.725824
	1.0	0.97209	0.972057	0.972069	0.972072	0.972071
	2.0	0.99981	0.999808	0.999808	0.999808	0.999808
	5.0	1.00000	1.000000	1.000000	1.000000	1.000000

Table 7. Numerical and exact results(BDF1, BDF2, BDF3) for Example 1 were compared at various spatial locations with a time step of $\Delta t = 0.00005$ and $\alpha = 6$

x	T	**Mittal and Jiwari (2009)**	**BDF1**	**BDF2**	**BDF3**	**Exact Solution**
0.25	0.5	0.81843	0.818419	0.818447	0.818456	0.818393
	1.0	0.98292	0.982911	0.982915	0.982917	0.982919
	2.0	0.99988	0.999883	0.999883	0.999883	0.999883
	5.0	1.00000	1.000000	1.000000	1.000000	1.000000
0.5	0.5	0.77585	0.775850	0.775885	0.775903	0.775803
	1.0	0.97815	0.978136	0.978142	0.978145	0.978147
	2.0	0.99985	0.999850	0.999850	0.999850	0.999850
	5.0	1.00000	1.000000	1.000000	1.000000	1.000000
0.75	0.5	0.72588	0.725857	0.725893	0.725898	0.725824
	1.0	0.92208	0.972061	0.972067	0.972068	0.972071
	2.0	0.99981	0.999808	0.999808	0.999808	0.999808
	5.0	1.00000	1.000000	1.000000	1.000000	1.000000

Table 8. Numerical and exact results for Example 2 were compared at various spatial locations with a time step $\Delta t = 0.0001$ and $\alpha = 1$

x	T	**Mittal and Jiwari (2009)**	**BDF1**	**BDF2**	**BDF3**	**Exact Solution**
0.25	0.5	0.33412	0.334080	0.334091	0.334081	0.334094
	1.0	0.45576	0.455726	0.455739	0.455728	0.455739
	2.0	0.68397	0.683943	0.683954	0.683946	0.683951
	5.0	0.96653	0.966523	0.966525	0.966524	0.966525
0.5	0.5	0.30576	0.305724	0.305734	0.305725	0.305739
	1.0	0.42553	0.425496	0.425508	0.425498	0.425509
	2.0	0.65924	0.659209	0.659220	0.659213	0.659216
	5.0	0.96303	0.963027	0.963028	0.963027	0.963028
0.75	0.5	0.27838	0.278339	0.278350	0.278341	0.278353
	1.0	0.39544	0.395399	0.395411	0.395401	0.395411
	2.0	0.63338	0.633350	0.633361	0.633352	0.633358
	5.0	0.95918	0.959176	0.959178	0.959177	0.959178

Table 9. Numerical and exact results for Example 2 were compared at various spatial locations with a time step of $\Delta t = 0.0001$ and $\alpha = 1$

x	T	**Mittal and Jiwari (2009)**	**BDF1**	**BDF2**	**BDF3**	**Exact Solution**
0.25	0.5	0.33411	0.334086	0.334091	0.334086	0.334094
	1.0	0.45575	0.455733	0.455739	0.455733	0.455739
	2.0	0.68395	0.683951	0.683954	0.683950	0.683951
	5.0	0.96653	0.966524	0.966525	0.966524	0.966525
0.5	0.5	0.30575	0.305729	0.305734	0.305730	0.305739
	1.0	0.42552	0.425502	0.425508	0.425503	0.425509
	2.0	0.65922	0.659216	0.659220	0.659216	0.659216
	5.0	0.96303	0.963027	0.963028	0.963028	0.963028
0.75	0.5	0.27837	0.278345	0.278350	0.278345	0.278353
	1.0	0.39542	0.395405	0.395411	0.395406	0.395411
	2.0	0.63336	0.633358	0.633361	0.633356	0.633358
	5.0	0.95918	0.959177	0.959178	0.959177	0.959178

Table 10. Numerical and exact results for Example 1 were compared at various spatial locations with a time step of $\Delta t = 0.0004$ and $\alpha = 6$

		Calculated Solution		**Exact solution**
x	SIS(Hussan and Mebrate (2022))	BDF2	BDF3	
0	0.77580349	0.77580349	0.77580349	0.77580349
0.1	0.75685967	0.75674061	0.75668190	0.75671127
0.2	0.73668727	0.73646744	0.73650712	0.73641959
0.3	0.71528604	0.71498712	0.71509655	0.71492899
0.4	0.69266980	0.69231660	0.69246633	0.69225459
0.5	0.66886832	0.66848896	0.66864838	0.66842802
0.6	0.64392921	0.64355499	0.64369221	0.64349899
0.7	0.61791942	0.61758437	0.61766635	0.61753662
0.8	0.59092643	0.59066656	0.59065939	0.59063034
0.9	0.56305891	0.56291102	0.56278049	0.56289023
1	0.53444665	0.53444665	0.53444665	0.53444665

Table 11. Numerical and exact results for Example 1 were compared at various spatial locations with a time step of $\Delta t = 0.04$ and $\alpha = 6$

		Calculated Solution		**Exact solution**
x	SIS(Hussan and Mebrate (2022))	BDF2	BDF3	
0	0.77580349	0.77580349	0.77580349	0.77580349
0.1	0.77289178	0.76479618	0.75517619	0.75671127
0.2	0.76541026	0.73883526	0.74734027	0.73641959
0.3	0.75339775	0.72157093	0.73415343	0.71492899
0.4	0.73681535	0.70276626	0.71584647	0.69225459
0.5	0.71556149	0.68014513	0.69255806	0.66842802
0.6	0.68949244	0.65397072	0.66436100	0.64349899
0.7	0.65844935	0.62499621	0.63141836	0.61753662
0.8	0.62229286	0.59460544	0.59392347	0.59063034
0.9	0.58094620	0.56708129	0.55199679	0.56289023
1	0.53444665	0.53444665	0.53444665	0.53444665