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Optimal Range for Value of Two-person Zero-sum Game Models with Uncertain Payoffs

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Abstract

Game theory deals with the decision-making of individuals in conflicting situations with known payoffs. However, these payoffs are imprecisely known, which means they have uncertainty due to vagueness in the data set of most real-world problems. Therefore, we consider a two-person zero-sum game model on a larger scale where the payoffs are imprecise and lie within a closed interval. We define the pure and mixed strategy as well as value for the game models. The proposed method computes the optimal range for the value of the game model using interval analysis. To derive some important results, we establish some lemmas that relate the value of interval game models to their payoffs and then prove some important theorems. Furthermore, we establish the min-max theorem for the most and least mixed strategies of the game model. Then, we obtain the bounds of optimal mixed strategies as well as the approximate value of the game model. The developed theories are verified and demonstrated through realistic two-person zero-sum game models with interval payoffs.

Keywords: Non-cooperative game; Zero-sum game; Matrix norms; Partial order relation; Saddle point; Game value; Interval analysis; Interval inequalities

MSC 2010 No.: 91A05, 91A10, 91A27, 91A35, 91A80

1. Introduction

Decision-making often poses a challenge as individuals frequently grapple with choices that may not align with their best interests, often navigating uncertainty regarding the outcomes. The ability to accurately assess potential outcomes would streamline planning processes and enhance efficiency. Game theory, a discipline focusing on decision-making amidst conflicting scenarios where one player's choices influence others' outcomes, offers a framework for identifying optimal strategies to achieve favorable results. Its application extends across various domains including economics, where it sheds light on consumer behavior, market dynamics, and corporate strategies, as well as in sports, social sciences, and military strategies. Rational decision-making, increasingly pivotal in these arenas, underscores the significance of leveraging game theory for informed and advantageous decision-making (see Lou et al. (2004); Rass et al. (2017); Petrosian et al. (2018); Sohrabi and Azgomi (2020); Afreen et al. (2023)). Given the interconnected nature of decisions in such scenarios, game theory serves as a clarifying force in decision-making, mitigating the complexities individuals encounter in their decision-making processes.

Von Neumann and Morgenstern (1920) pioneered the development of game theory and formulated a highly impactful theory on two-person zero-sum games in their seminal work, *"Theory of Games and Economic Behavior"* (Von Neumann and Morgenstern (2007)). John Nash introduced a cornerstone concept, *"Equilibrium points in N-person games"* which came to be known as Nash equilibrium (see (Nash et al. (1950); Nash (1951))). In real-world game scenarios, uncertainties surrounding payoffs often arise due to factors like human behavior or environmental conditions. In such cases, where exact payoff values are not ascertainable, the game is termed an approximation game or uncertain game model. Literature addresses this uncertainty through various approaches, including probabilistic and fuzzy set theory, to provide frameworks for analysis and decision-making (Dengfeng (1999); Collins and Hu (2008); Hu et al. (2008); Dutta and Gupta (2014); Rass et al. (2015); Jana and Roy (2019); Bhattacharya and De (2021); Wu and Lisser (2023); Dong (2024)).

In probabilistic and fuzzy set theories, decision-makers treat payoffs as either random numbers or fuzzy numbers, respectively. Through the use of appropriate probability distributions and membership functions, they aim to convert the problem into a deterministic form. Selecting suitable probability distributions and membership functions poses a challenge for decision-makers. An alternative approach to address uncertainty in game model payoffs is by treating them as closed intervals, leading to the concept of an interval game model. Interval analysis theory stands out as a superior approach compared to probabilistic and fuzzy set theories in addressing uncertainty in game model payoffs. While probabilistic and fuzzy set theories require decision-makers to select suitable probability distributions and membership functions, interval analysis theory simplifies the process by treating payoffs as closed intervals. This method provides a more straightforward and robust framework for managing uncertainty in game scenarios (Moore (1963); Moore (1979); Moore et al. (2009)). Interval analysis theory finds extensive application across various fields, including portfolio optimization, waste management, and interval optimization problems. Its versatility allows for effective problem-solving and decision-making in diverse domains, making it a valuable tool for addressing uncertainty and optimizing outcomes in practical scenarios (Alefeld and Mayer (2000); Bhurjee and Panda (2012); Zhu and Qiu (2018); Wang et al. (2019); Bhaumik and Roy (2021); Jangid et al. (2022); Temelcan et al. (2022)).

Over the past two decades, numerous studies have been conducted on interval game models, leading to the development of various methods aimed at determining interval game values and mixed strategies for players. Researchers have explored diverse avenues, advancing the understanding and application of interval analysis theory in game theory (Liu and Kao (2009); Levin (1999); Gok et al. (2011); Deng et al. (2016)). Nayak and Pal (2009) solved an interval matrix game through a pair of two-level deterministic linear programming problems. Li (2011) developed two deterministic linear programming problems to solve a two-person zero-sum game model with interval payoffs and established the lower and upper bounds of the game's value. Akyar et al. (2011) obtained a mixed strategy equilibrium for a two-person interval matrix game using Brown Robinsons approach. Li et al. (2012) considered an interval matrix game and obtained multiple strategies by addressing a bi-objective linear programming problem for different choices of satisfying factors. Roy and Mondal (2016) obtained the value of the fuzzy interval matrix game by solving two-level multi-objective programming problems. Bhurjee and Panda (2017) considered a normalized matrix game with interval payoffs and studied the optimal value as well as mixed strategies of the game for both players. Dey and Zaman (2020) proposed robust optimization methodologies to obtain the value of two-person zero-sum and nonzero-sum games with some or all payoffs as intervals. Recently, Bhurjee and Yadav (2021) proposed a method for determining the existence of the Nash equilibrium point in the generalized game model with interval payoffs.

The motivation for this paper stems from the prevailing trend in literature, where the predominant approach involves solving two-level linear programming problems to derive mixed strategies and the game value for two-person zero-sum games with interval payoffs. However, this process is often time-consuming, prompting the exploration of alternative methods to streamline the analysis and decision-making process in such games. However, this approach becomes increasingly cumbersome, particularly for larger zero-sum game models. In response, presented work introduces a practical method aimed at expediting the process of obtaining an approximate solution using matrix norms of the interval payoff matrix. Specifically, we utilize the I_1-norm and I_∞-norm of the interval payoff matrix due to their ease of evaluation. Through this novel approach, we successfully achieve an approximate solution for a two-person zero-sum game models without the need to solve any equations. We address a general two-person zero-sum finite game with interval payoffs and propose a methodology to determine the optimal range of value for the game through interval analysis. Our approach involves defining the saddle point, pure strategy, and game value using partial ordering applied to a set of closed intervals. We then establish lemmas and theorems that establish connections between the payoff matrix and the value of the interval matrix game, utilizing definitions of the I_1-norm and I_∞-norm for the interval matrix.

Further, we prove some important results for the set of mixed strategies of the game model and find the optimal bounds for the set of mixed strategies as $p_{\sup}$ and $p_{\inf}$ for the $m \times n$ interval matrix game using these results. Thereafter, we use the $\inf - \sup$ theorem to develop the correlation between $p_{\sup}$ and $p_{\inf}$ for the set of game strategies. Moreover, we highlight the significance of our results by backing all theoretical developments with a variety of numerical examples. The crucial advantage of our method offers a rapid solution process without requiring equation solving. Importantly, the interval matrix norm approach, to our knowledge, has not been utilized in interval matrix game theory.

The remaining paper is organized in the following way. Section 2 describes certain interval analysis prerequisites that will be used in the sub-sequence sections. The interval matrix game and its saddle point are defined in Section 3. A methodology is presented in Section 4 for obtaining the optimal range of game value and establish some important results are developed to connect the payoff matrix and game's value. We prove some results to obtain the supremum and infimum of the set of mixed strategies of the game in Section 5. Finally, Section 6 concludes the paper by summarizing the key findings and suggests potential directions for future research.

2. Preliminaries

Throughout the paper, $\mathbf{m} = [m^L, m^R]$ denotes a closed interval on real line $\mathbb{R}$, where m^L and m^R are the ends point of the interval, respectively. A closed interval $[m^L, m^R]$ is called as non-negative if $m^L \geq 0$; it is negative if $m^R < 0$, and if $m^L = m^R = m$ then $[m, m]$ is real. An interval matrix is represented by $\mathcal{M} = ([m_{ij}^L, m_{ij}^R])_{m\times n} = [\mathbf{m}_{ij}]_{m\times n}$. In 1963, Moore and Yang introduced a binary operation $* \in \{+, -, ., /\}$ on $\mathbb{R}$ (Moore and Yang (1959)).

2.1. I_1-norm and $I_\infty-$norm for an Interval Matrix

For $\mathcal{M} \in I(\mathbb{R})^{m\times n}$, the I_1-norm and $I_\infty-$norm of $\mathcal{M}$ are defined as follows:

$$\|\mathcal{M}\|_{I_1} = \max_j \sum_i |\mathbf{m}_{ij}| \quad \text{and} \quad \|\mathcal{M}\|_{I_\infty} = \max_i \sum_j |\mathbf{m}_{ij}|,$$

where $\|\mathcal{M}\|_{I_1}$ and $\|\mathcal{M}\|_{I_\infty}$ represent the maximum absolute column and row sums, respectively.

Example 2.1.

Let $\mathcal{M} \in I(\mathbb{R})^{3\times 4}$ be an interval payoff matrix.

$$\mathcal{M} = \begin{bmatrix} [0,1] & [4,5] & [-1,1] & [-3,2] \\ [3,5] & [2,4] & [2,6] & [2,6] \\ [-5,2] & [-1,0] & [-1,2] & [2,4] \end{bmatrix},$$

Since, $\|\mathcal{M}\|_{I_\infty} = \max\{u_1, u_2, u_3\}$ and $\|\mathcal{M}\|_{I_1} = \max\{t_1, t_2, t_3, t_4\}$, where u_i is the total of the absolute values of the constituents of i^{th} rows and t_j is the total of the absolute values of the constituents of j^{th} columns. Therefore,

$$\|\mathcal{M}\|_{I_\infty} = \max\{10, 21, 12\} = 21 \text{ and } \|\mathcal{M}\|_{I_1} = \max\{11, 10, 9, 13\} = 13.$$

Since the set $I(\mathbb{R})$ is not absolutely ordered, a partial ordering should be specified to compare to the intervals. There are a lot of partial orders defined in the literature. A partial ordering is considered in which we compare two closed intervals by comparing their bounds due to Ishibuchi and Tanaka (1990).

3. Interval Matrix Game

Let $\mathcal{M}$ be a normalized matrix of two-person zero-sum interval game model. It is assumed that the pure strategies for two players P_1 and P_2 are sets $s_1 = \{\alpha_1, \alpha_2, ..., \alpha_m\}$ and $s_2 = \{\beta_1, \beta_2, ..., \beta_n\}$, respectively. If player P_1 chooses $\alpha_1 \in s_1$ and P_2 chooses $\beta_1 \in s_2$, the payoff for player P_1 is $m(\alpha_1, \beta_1) = [m^L_{11}, m^R_{11}]$. This interval normalized matrix game is denoted by $(\mathcal{M}; s_1, s_2)$, where $\mathcal{M} = ([m^L_{ij}, m^R_{ij}])_{m\times n} = [\mathbf{m}_{ij}]_{m\times n}$ represents the payoff matrix for player P_1. The interval payoff matrices' lower and upper bounds matrix are $M^L = (m^L_{ij})_{m\times n}$ and $M^R = (m^R_{ij})_{m\times n}$, respectively (Liu and Kao (2009)).

3.1. Saddle Point

Let (α_1, β_1) and (α_2, β_2) be the saddle points with pure strategy for the lower and upper matrices M^L and M^R, respectively and the corresponding game values v^L and v^R. If $(\alpha_1, \beta_1) = (\alpha_2, \beta_2) = (\alpha, \beta)$ say, then (α, β) is known as a saddle point with pure strategy (α, β) for the game $(\mathcal{M}; s_1, s_2)$ and the interval $\boldsymbol{\nu} = \boldsymbol{\nu}$ referred to as value of the game (Bhurjee and Panda (2017)). If $(\alpha_1, \beta_1) \neq (\alpha_2, \beta_2)$, then the saddle point exists with mixed strategy for the game.

Example 3.1.

Consider an interval matrix game $(\mathcal{M}; s_1, s_2)$ whose payoff matrix is as follows:

$$\mathcal{M} = \begin{bmatrix} [1,2] & [2,4] & [3,7] \\ [2,4] & [4,5] & [6,8] \\ [3,7] & [6,8] & [8,9] \end{bmatrix},$$

The interval payoff matrices' lower and upper bounds matrix

$$M^L = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 8 \end{bmatrix} \text{ and } M^R = \begin{bmatrix} 2 & 4 & 7 \\ 4 & 5 & 8 \\ 7 & 8 & 9 \end{bmatrix}.$$

respectively. The saddle point for M^L is $(\alpha_1, \beta_1) = (3, 1)$ and its value is $v^L = 3$; the saddle point for M^R is $(\alpha_2, \beta_2) = (3, 1)$ and its value is $v^R = 7$. Since $(\alpha_1, \beta_1) = (\alpha_2, \beta_2) = (3, 1)$, then there exists a saddle point with pure strategy at the point $(3, 1)$ and the value is $\boldsymbol{\nu} = [3, 7]$ for the matrix game.

Example 3.2.

Consider the interval matrix game $(\mathcal{M}; s_1, s_2)$, where

$$\mathcal{M} = \begin{bmatrix} [-7,-4] & [-2,4] & [3,7] \\ [4,10] & [0,8] & [-8,9] \\ [7,8] & [6,7] & [7,9] \end{bmatrix},$$

The lower and upper payoff matrices of the game are

$$M^L = \begin{bmatrix} -7 & -2 & 3 \\ 4 & 0 & -8 \\ 7 & 6 & 7 \end{bmatrix} \text{ and } M^R = \begin{bmatrix} -4 & 4 & 7 \\ 10 & 8 & 9 \\ 8 & 7 & 9 \end{bmatrix}.$$

respectively. The saddle point with pure strategy for M^L is $(\alpha_1, \beta_1) = (3, 2)$ and its value is $v^L = 6$; and the saddle point with pure strategy for M^R is $(\alpha_2, \beta_2) = (2, 2)$ and its value is $v^R = 8$. Since $(\alpha_1, \beta_1) \neq (\alpha_2, \beta_2)$, then the saddle point with pure strategy does not exist for the game. Then, there exist mixed strategies for the game model.

3.2. Mixed Strategy

Let $p = (p_1, p_2, \ldots, p_m)^T$ and $q = (q_1, q_2, \ldots, q_n)^T$ be player's mixed strategies for players P_1 and P_2, respectively. Here, $p_i, i \in \Lambda_m$ and $q_j, j \in \Lambda_n$ are probabilities in which players P_1 and P_2 choose their pure strategies $\alpha_i \in s_1, i \in \Lambda_m$ and $\beta_j \in s_2, j \in \Lambda_n$, respectively. The sets

$$P = \left\{ p \in \mathbb{R}^m \mid \sum_i p_i = 1, p_i \geq 0, i \in \Lambda_m \right\} \text{ and } Q = \left\{ q \in \mathbb{R}^m \mid \sum_j q_j = 1, q_j \geq 0, j \in \Lambda_n \right\},$$

are denoted as the set of mixed strategies for players P_1 and P_2, respectively (Li (2011)). The interval payoff of player P_1 is defined as follows:

$$E(p,q) = p^T \mathcal{M} q = \left[\sum_i \sum_j m_{ij}^L p_i q_j, \sum_i \sum_j m_{ij}^R p_i q_j \right].$$

4. Optimal Range for Value of the Interval Matrix Game

In this section, we establish the relationship between the value of the interval game $(\mathcal{M}; P, Q)$ and its payoff matrix using interval analysis. Several lemmas and theorems are proved to discuss the range of values of the game in the present section.

Lemma 4.1.

Let $(\mathcal{M}; P, Q)$ be an interval matrix game. If $p = (p_1, p_2)$ and $q = (q_1, q_2)$ are the players mixed strategy sets, and $\boldsymbol{\nu}$ is the value of the game where

$$\mathcal{M} = \begin{bmatrix} \mathbf{m}_{11} & \mathbf{m}_{12} \\ \mathbf{m}_{21} & \mathbf{m}_{22} \end{bmatrix},$$

if $v^L > 0$, then $\frac{k}{\|\mathcal{M}\|_{I_\infty}} \preceq \boldsymbol{\nu} \preceq \|\mathcal{M}\|_{I_1}$,

if $v^R < 0$, then $-\|\mathcal{M}\|_{I_1} \preceq \boldsymbol{\nu} \preceq \frac{k}{\|\mathcal{M}\|_{I_\infty}}$,

where $k = \inf\{\boldsymbol{\nu}(|\mathbf{m}_{11}| + |\mathbf{m}_{12}|), \boldsymbol{\nu}(|\mathbf{m}_{21}| + |\mathbf{m}_{22}|)\}$.

Proof:

We have $\|\mathcal{M}\|_{I_\infty} = |\mathbf{m}_{11}| + |\mathbf{m}_{12}|$. This gives $|\mathbf{m}_{11}| + |\mathbf{m}_{12}| \geq |\mathbf{m}_{21}| + |\mathbf{m}_{22}|$. Then,

$$\frac{|\mathbf{m}_{21}| + |\mathbf{m}_{22}|}{\|\mathcal{M}\|_{I_\infty}} \leq 1, \tag{1}$$

Case 4.1.

if $v^L > 0$, then $\boldsymbol{\nu} \succ 0$. We have

$$\frac{\boldsymbol{\nu}(|\mathbf{m}_{21}| + |\mathbf{m}_{22}|)}{\|\mathcal{M}\|_{I_\infty}} \preceq \boldsymbol{\nu}.$$

Let $k = \boldsymbol{\nu}(|\mathbf{m}_{21}| + |\mathbf{m}_{22}|)$,

$$\frac{k}{\|\mathcal{M}\|_{I_\infty}} \preceq \boldsymbol{\nu}. \tag{2}$$

From the definition of value of the game

$$\boldsymbol{\nu} = p\,\mathbf{m}_{11} + (1-p)\,\mathbf{m}_{21} \preceq |\mathbf{m}_{11}| + |\mathbf{m}_{21}|, \qquad p \in [0,1].$$

Then, we get

$$\boldsymbol{\nu} \preceq \|\mathcal{M}\|_{I_1}, \tag{3}$$

where

$$\|\mathcal{M}\|_{I_1} = \sup\{|\mathbf{m}_{11}| + |\mathbf{m}_{21}|, |\mathbf{m}_{12}| + |\mathbf{m}_{22}|\}.$$

From (2) and (3),

$$\frac{k}{\|\mathcal{M}\|_{I_\infty}} \preceq \boldsymbol{\nu} \preceq \|\mathcal{M}\|_{I_1}.$$

Case 4.2.

if $v^R < 0$, then $\boldsymbol{\nu} \prec 0$,

(a.) While $\mathbf{m}_{11} \preceq 0$, $\mathbf{m}_{21} \succeq 0$,

$$0 \succeq \mathbf{m}_{11}p \succeq \mathbf{m}_{11}, \qquad p \in [0,1]. \tag{4}$$

Also, we get

$$\mathbf{m}_{21} \succeq \mathbf{m}_{21}(1-p) \succeq 0. \tag{5}$$

From (4) and (5),

$$\mathbf{m}_{21} \succeq \mathbf{m}_{11}p \oplus \mathbf{m}_{21}(1-p) \succeq \mathbf{m}_{11} \succeq \mathbf{m}_{11} \ominus \mathbf{m}_{21}.$$

$$\succeq -(|\mathbf{m}_{11}| + |\mathbf{m}_{21}|) = -\|\mathcal{M}\|_{I_1}.$$

Therefore,

$$\boldsymbol{\nu} \succeq -\|\mathcal{M}\|_{I_1}. \tag{6}$$

From (1), we have

$$\frac{k}{\|\mathcal{M}\|_{I_\infty}} \succeq \boldsymbol{\nu}. \tag{7}$$

From (6) and (7), we obtain

$$-\|\mathcal{M}\|_{I_1} \preceq \boldsymbol{\nu} \preceq \frac{k}{\|\mathcal{M}\|_{I_\infty}}.$$

(b.) While $\mathbf{m}_{11} \preceq 0$ and $\mathbf{m}_{21} \preceq 0$,

$$0 \succeq \mathbf{m}_{11}p \succeq \mathbf{m}_{11}, \qquad p \in [0,1]. \tag{8}$$

Also, we get

$$0 \succeq (1-p)\mathbf{m}_{21} \succeq \mathbf{m}_{21}, \text{ since } \mathbf{m}_{21} \preceq 0. \tag{9}$$

From (8) and (9), we have

$$0 \succeq \mathbf{m}_{11}p \oplus \mathbf{m}_{21}(1-p) \succeq \mathbf{m}_{11} \oplus \mathbf{m}_{21}$$

$$\succeq -\{\ominus\mathbf{m}_{11} \oplus (\ominus\mathbf{m}_{21})\} \succeq -(|\mathbf{m}_{11}| + |\mathbf{m}_{21}|) - \|\mathcal{M}\|_{I_1}.$$

Therefore,

$$\boldsymbol{\nu} \succeq - \|\mathcal{M}\|_{I_1}. \tag{10}$$

From (7) and (10), we have

$$-\|\mathcal{M}\|_{I_1} \preceq \boldsymbol{\nu} \preceq \frac{k}{\|\mathcal{M}\|_{I_\infty}}.$$

Similarly, we can prove the result for other cases **(c.)** For $\mathbf{m}_{11} \succeq 0$ and $\mathbf{m}_{21} \succeq 0$ and **(d.)** For $\mathbf{m}_{11} \succeq 0$ and $\mathbf{m}_{21} \preceq 0$. ∎

Lemma 4.2.

Let $(\mathcal{M}; P, Q)$ be an interval matrix game. If $p = (p_1, p_2)$ and $q = (q_1, q_2)$ are represents the mixed strategy for the players and $0 \in \boldsymbol{\nu}$. Then,

$$\frac{k_1}{\|\mathcal{M}\|_{I_\infty}} \ominus (\|\mathcal{M}\|_{I_1} \oplus [0, v^R]) \preceq \boldsymbol{\nu} \preceq \frac{k_2}{\|\mathcal{M}\|_{I_\infty}} \oplus \|\mathcal{M}\|_{I_1} \oplus [0, -v^L],$$

where

$$k_1 = \inf\{[0, v^R](|\mathbf{m}_{11}| + |\mathbf{m}_{12}|), [0, v^R](|[m_{21}^L, m_{11}^R]| + |\mathbf{m}_{22}|)\},$$

and

$$k_2 = \inf\{[v^L, 0](|\mathbf{m}_{11}| + |\mathbf{m}_{12}|), [v^L, 0](|\mathbf{m}_{21}| + |\mathbf{m}_{22}|)\}.$$

Proof:

Consider the following 2×2 interval payoff matrix

$$\mathcal{M} = \begin{bmatrix} \mathbf{m}_{11} \; \mathbf{m}_{12} \\ \mathbf{m}_{21} \; \mathbf{m}_{22} \end{bmatrix},$$

if $0 \in \boldsymbol{\nu}$, then $v^L \leq 0$ and $v^R \geq 0$.

Multiply by $[0, v^R]$ in (1), then

$$\frac{[0, v^R](|\mathbf{m}_{21}| + |\mathbf{m}_{22}|)}{\|\mathcal{M}\|_{I_\infty}} \preceq [0, v^R],$$

$$\frac{k_1}{\|\mathcal{M}\|_{I_\infty}} \preceq [0, v^R], \tag{11}$$

where $k_1 = [0, v^R](|\mathbf{m}_{21}| + |\mathbf{m}_{22}|)$,

We know that,

$$\boldsymbol{\nu} = p\, \mathbf{m}_{11} \oplus (1-p)\, \mathbf{m}_{21} \preceq \mathbf{m}_{11} \oplus \mathbf{m}_{21},$$

$$[0, v^R] \preceq \mathbf{m}_{11} \oplus \mathbf{m}_{21} \oplus [0, -v^L],$$

$$[0, v^R] \preceq \|\mathcal{M}\|_{I_1} \oplus [0, -v^L]. \tag{12}$$

From (11) and (12),

$$\frac{k_1}{\|\mathcal{M}\|_{I_\infty}} \preceq [0, v^R] \preceq \|\mathcal{M}\|_{I_1} \oplus [0, -v^L]. \tag{13}$$

Now multiply by $[v^L, 0]$ in (1), we have

$$\frac{[v^L, 0]\; (|\mathbf{m}_{21}| + |\mathbf{m}_{22}|)}{\|\mathcal{M}\|_{I_\infty}} \succeq [v^L, 0],$$

$$\frac{k_2}{\|\mathcal{M}\|_{I_\infty}} \succeq [v^L, 0], \tag{14}$$

where $k_2 = [v^L, 0]\,(|\mathbf{m}_{21}| + |\mathbf{m}_{22}|)$.

(a.)

While $\mathbf{m}_{11} \preceq 0$, $\mathbf{m}_{21} \succeq 0$. Then, we have

$$0 \succeq p\,\mathbf{m}_{11} \succeq \mathbf{m}_{11},\ \text{since } \mathbf{m}_{11} \preceq 0, \tag{15}$$

$$\mathbf{m}_{21} \succeq (1-p)\,\mathbf{m}_{21} \succeq 0,\ \text{since } \mathbf{m}_{21} \succeq 0. \tag{16}$$

From (15) and (16),

$$\mathbf{m}_{21} \succeq p\,\mathbf{m}_{11} \oplus (1-p)\,\mathbf{m}_{21} \succeq \mathbf{m}_{11},$$

$$\succeq \mathbf{m}_{11} \ominus \mathbf{m}_{21} \succeq -(|\mathbf{m}_{11}| + |\mathbf{m}_{21}|) = -\|\mathcal{M}\|_{I_1}.$$

Therefore,

$$\boldsymbol{\nu} \succeq -\|\mathcal{M}\|_{I_1},$$

$$[v^L, 0] \succeq \ominus(\|\mathcal{M}\|_{I_1} \oplus [0, v^R]). \tag{17}$$

From (14) and (17),

$$\ominus(\|\mathcal{M}\|_{I_1} \oplus [0, v^R]) \preceq [v^L, 0] \preceq \frac{k_2}{\|\mathcal{M}\|_{I_\infty}}. \tag{18}$$

Finally, from (13) and (18), we have

$$\frac{k_1}{\|\mathcal{M}\|_{I_\infty}} \ominus (\|\mathcal{M}\|_{I_1} \oplus [0, v^R]) \preceq \boldsymbol{\nu} \preceq \frac{k_2}{\|\mathcal{M}\|_{I_\infty}} \oplus \|\mathcal{M}\|_{I_1} \oplus [0, -v^L].$$

Similarly, we can prove the result for other cases **(b.)** While $\mathbf{m}_{11} \preceq 0$ and $\mathbf{m}_{21} \preceq 0$;
(c.) For $\mathbf{m}_{11} \succeq 0$ and $\mathbf{m}_{21} \succeq 0$;
(d.) For $\mathbf{m}_{11} \succeq 0$ and $\mathbf{m}_{21} \preceq 0$. ■

Example 4.1.

Consider an interval matrix game whose payoff matrix is given as follows.

$$\mathcal{M} = \begin{bmatrix} [2,6] & [-3,1] \\ [-4,7] & [-3,-2] \end{bmatrix},$$

and its value of the game is $\boldsymbol{\nu} = [-3, 1]$.

Since $\|\mathcal{M}\|_{I_\infty} = \max\{u_1, u_2\}$ and $\|\mathcal{M}\|_{I_1} = \max\{t_1, t_2\}$, where u_i is the total of the absolute values of the constituents of i^{th} rows and t_j is the total of the absolute values of the constituents of j^{th} columns. Here,

$$u_1 = 9,\ u_2 = 10,\ t_1 = 13,\ t_2 = 6.$$

Therefore, $\|\mathcal{M}\|_{I_\infty} = \max\{9, 10\} = 10$ and $\|\mathcal{M}\|_{I_1} = \max\{13, 6\} = 13$. We have the following by Lemma 4.2. Therefore,

$$\frac{[0,9]}{10} \ominus (13 \oplus [0,1]) \preceq [-3,1] \preceq \frac{[-27,0]}{10} \oplus 13 \oplus [0,3],$$

$$[-14, -12.1] \preceq [-3, 1] \preceq [10.3, 16].$$

Thus, we verify Lemma 4.2 for the interval game model.

Now, we generalize our results in Lemma 4.1 for payoff matrix $\mathcal{M} \in I(\mathbb{R})^{2\times 2}$ to $\mathcal{M} \in I(\mathbb{R})^{m\times n}$ and establish the similar results.

Lemma 4.3.

Let $\mathcal{M} \in I(\mathbb{R})^{m\times n}$ be a interval payoff matrix and $\boldsymbol{\nu}$ be the value of the interval matrix game $(\mathcal{M}; P, Q)$. Then,

if $v^L > 0$, $\frac{k}{\|\mathcal{M}\|_{I_\infty}} \preceq \boldsymbol{\nu} \preceq \|\mathcal{M}\|_{I_1}$,

if $v^R < 0$, $-\|\mathcal{M}\|_{I_1} \preceq \boldsymbol{\nu} \preceq \frac{k}{\|\mathcal{M}\|_{I_\infty}}$,

where $k = \sup_{1\leq i\leq m, i\neq p} \sum_{j=1}^{n} \boldsymbol{\nu}\, |\mathbf{m}_{ij}|$.

Proof:

Consider the interval payoff matrix of the game

$$\mathcal{M} = \begin{bmatrix} \mathbf{m}_{11} & \mathbf{m}_{12} & \cdots & \mathbf{m}_{1n} \\ \mathbf{m}_{21} & \mathbf{m}_{22} & \cdots & \mathbf{m}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{m}_{m1} & \mathbf{m}_{m2} & \cdots & \mathbf{m}_{mn} \end{bmatrix},$$

Case 4.3.

if $v^L > 0$, then $\boldsymbol{\nu} \succ 0$. Suppose $\|\mathcal{M}\|_{I_\infty} = \sum_{j=1}^{n} |\mathbf{m}_{pj}|$, for fixed p. From the definition of $\infty-$norm, we have

$$\|\mathcal{M}\|_{I_\infty} = \sum_{j=1}^{n} |\mathbf{m}_{ij}| \succeq \sup_{1\leq i\leq m, i\neq p} \sum_{j=1}^{n} |\mathbf{m}_{ij}| = r.$$

This gives,

$$\frac{r}{\|\mathcal{M}\|_{I_\infty}} \preceq 1. \qquad (19)$$

Multiply by $\boldsymbol{\nu}$ in (19), we get

$$\frac{k}{\|\mathcal{M}\|_{I_\infty}} \preceq \boldsymbol{\nu}, \tag{20}$$

where $k = r\boldsymbol{\nu}$. Then, we have

$$\boldsymbol{\nu} = \sum_{i=1}^{m} p_i \mathbf{m}_{ij}, \quad \textit{for any } j \textit{ and } p_i \in [0,1],$$

$$\preceq \sum_{j=1}^{m} |\mathbf{m}_{ij}| = \|\mathcal{M}\|_{I_1}.$$

Then,

$$\boldsymbol{\nu} \preceq \|\mathcal{M}\|_{I_1}. \tag{21}$$

From (20) and (21), we have

$$\frac{k}{\|\mathcal{M}\|_{I_\infty}} \preceq \boldsymbol{\nu} \preceq \|\mathcal{M}\|_{I_1}.$$

Case 4.4.

if $v^R < 0$, then $\boldsymbol{\nu} \prec 0$. From (19), we have

$$\frac{k}{\|\mathcal{M}\|_{I_\infty}} \succeq \boldsymbol{\nu}. \tag{22}$$

Also, the game value

$$\boldsymbol{\nu} = \sum_{i=1}^{m} p_i \, \mathbf{m}_{ij}, \qquad \textit{for any } j,$$

$$\boldsymbol{\nu} \succeq \sum_{i=1}^{m} -p_i(|\mathbf{m}_{ij}|) \succeq -\sum_{i=1}^{m} |\mathbf{m}_{ij}| \succeq -\|\mathcal{M}\|_{I_1}. \tag{23}$$

From (22) and (23),

$$-\|\mathcal{M}\|_{I_1} \preceq \boldsymbol{\nu} \preceq \frac{k}{\|\mathcal{M}\|_{I_\infty}}.$$

■

Lemma 4.4.

Let $\mathcal{M} \in I(\mathbb{R})^{m\times n}$ be a interval payoff matrix and $\boldsymbol{\nu}$ be the value of the interval matrix game $(\mathcal{M}; P, Q)$. If $0 \in \boldsymbol{\nu}$, then

$$\frac{k_1}{\|\mathcal{M}\|_{I_\infty}} \ominus (\|\mathcal{M}\|_{I_1} \oplus [0, v^R]) \preceq \boldsymbol{\nu} \preceq \frac{k_2}{\|\mathcal{M}\|_{I_\infty}} \oplus \|\mathcal{M}\|_{I_1} \oplus [0, -v^L],$$

where

$$k_1 = \sup_{1 \le i \le m, i \ne p} \sum_{j=1}^{n} [0, v^R] |\mathbf{m}_{ij}|, \text{ and } k_2 = \sup_{1 \le i \le m, i \ne p} \sum_{j=1}^{n} [v^L, 0] |\mathbf{m}_{ij}| \quad \textit{for any fixed } p.$$

Proof:

if $0 \in \boldsymbol{\nu}$, then $v^L \le 0$ and $v^R \ge 0$. Multiply by $[0, v^R]$ in (19), we have

$$\frac{k_1}{\|\mathcal{M}\|_{I_\infty}} \preceq [0, v^R], \tag{24}$$

where $k_1 = r\,[0, v^R]$. We know that

$$\boldsymbol{\nu} = \sum_{i=1}^{m} p_i\, \mathbf{m}_{ij}, \qquad \textit{for any } j,$$

$$[v^L, 0] \oplus [0, v^R] \preceq \sum_{i=1}^{m} |\mathbf{m}_{ij}| = \|\mathcal{M}\|_{I_1},$$

$$[0, v^R] \preceq \|\mathcal{M}\|_{I_1} \oplus [0, -v^L]. \tag{25}$$

From (24) and (25),

$$\frac{k_1}{\|\mathcal{M}\|_{I_\infty}} \preceq [0, v^R] \preceq \|\mathcal{M}\|_{I_1} \oplus [0, -v^L]. \tag{26}$$

Multiply by $[v^L, 0]$ in (19),

$$\frac{k_2}{\|\mathcal{M}\|_{I_\infty}} \succeq [v^L, 0], \tag{27}$$

where $k_2 = r[v^L, 0]$,

$$\boldsymbol{\nu} = \sum_{i=1}^{m} p_i\, \mathbf{m}_{ij}, \qquad \textit{for any } j,$$

$$[v^L, 0] \oplus [0, v^R] \succeq \sum_{i=1}^{m} -p_i\, (|\mathbf{m}_{ij}|) \succeq -\sum_{i=1}^{m} |\mathbf{m}_{ij}|,$$

$$[v^L, 0] \oplus [0, v^R] \succeq -\|\mathcal{M}\|_{I_1},$$

$$[v^L, 0] \succeq \ominus \|\mathcal{M}\|_{I_1} \oplus [0, v^R]. \tag{28}$$

From (27) and (28),

$$\ominus(\|\mathcal{M}\|_{I_1} \oplus [0, v^R]) \preceq [v^L, 0] \preceq \frac{k_2}{\|\mathcal{M}\|_{I_\infty}}. \tag{29}$$

Finally from (26) and (29), we have

$$\frac{k_1}{\|\mathcal{M}\|_{I_\infty}} \ominus (\|\mathcal{M}\|_{I_1} \oplus [0, v^R]) \preceq \boldsymbol{\nu} \preceq \frac{k_2}{\|\mathcal{M}\|_{I_\infty}} \oplus (\|\mathcal{M}\|_{I_1} \oplus [0, -v^L]).$$

This ends the proof. ■

Here, we discovered a linkage between the game value and payoff matrix using the definition of I_1-norm, $I_\infty-$norm for the interval matrix and the Lemma 4.1.

Theorem 4.1.

Let $(\mathcal{M}; P, Q)$ be a interval matrix game and $\boldsymbol{\nu}$ be the value of the game. For interval payoff matrix

$$\mathcal{M} = \begin{bmatrix} \mathbf{m}_{11} & \mathbf{m}_{12} \\ \mathbf{m}_{21} & \mathbf{m}_{22} \end{bmatrix},$$

if $|\boldsymbol{\nu}| \geq 1$, then $\frac{|\mathbf{m}_{21}|+|\mathbf{m}_{22}|}{\|\mathcal{M}\|_{I_\infty}} \leq |\boldsymbol{\nu}| \leq \|\mathcal{M}\|_{I_1}$,

if $|\boldsymbol{\nu}| \leq 1$ and $\boldsymbol{\nu} \neq 0$, then $-\frac{(|\mathbf{m}_{21}|+|\mathbf{m}_{22}|)}{\|\mathcal{M}\|_{I_\infty}} \leq |\boldsymbol{\nu}| \leq \|\mathcal{M}\|_{I_1}$,

if $\boldsymbol{\nu} = 0$, then $-\frac{(|\mathbf{m}_{21}|+|\mathbf{m}_{22}|)}{\|\mathcal{M}\|_{I_\infty}} \leq |\boldsymbol{\nu}| \leq \min\left\{\|\mathcal{M}\|_{I_1}, 1 - \frac{(|\mathbf{m}_{21}|+|\mathbf{m}_{22}|)}{\|\mathcal{M}\|_{I_\infty}}\right\}$.

Proof:

Consider two cases.

Case 4.5.

For $\boldsymbol{\nu} \succ 0$,

(1) *if* $\boldsymbol{\nu} \succeq 1$, then by Lemma 4.1,

$$\frac{|\mathbf{m}_{21}| + |\mathbf{m}_{22}|}{\|\mathcal{M}\|_{I_\infty}} \preceq \boldsymbol{\nu} \preceq \|\mathcal{M}\|_{I_1}.$$

(2) *if* $0 \prec \boldsymbol{\nu} \preceq 1$, then by (1), we have

$$\boldsymbol{\nu} \succeq -\frac{(|\mathbf{m}_{21}| + |\mathbf{m}_{22}|)}{\|\mathcal{M}\|_{I_\infty}} \geq -1,$$

$$-\frac{(|\mathbf{m}_{21}| + |\mathbf{m}_{22}|)}{\|\mathcal{M}\|_{I_\infty}} \preceq \boldsymbol{\nu}, \tag{30}$$

From (3) and (30, we obtain

$$-\frac{(|\mathbf{m}_{21}| + |\mathbf{m}_{22}|)}{\|\mathcal{M}\|_{I_\infty}} \preceq \boldsymbol{\nu} \preceq \|\mathcal{M}\|_{I_1}. \tag{31}$$

Case 4.6.

For $\boldsymbol{\nu} \prec 0$,

(1) *if* $\boldsymbol{\nu} \preceq -1$, then from (1)

$$\boldsymbol{\nu} \preceq -1 \leq -\frac{(|\mathbf{m}_{21}| + |\mathbf{m}_{22}|)}{\|\mathcal{M}\|_{I_\infty}},$$

$$-\|\mathcal{M}\|_{I_1} \preceq \boldsymbol{\nu} \preceq \frac{-(|\mathbf{m}_{21}| + |\mathbf{m}_{22}|)}{\|\mathcal{M}\|_{I_\infty}}.$$

Therefore,

$$\frac{|\mathbf{m}_{21}| + |\mathbf{m}_{22}|}{\|\mathcal{M}\|_{I_\infty}} \preceq -\boldsymbol{\nu} \preceq \|\mathcal{M}\|_{I_1}. \tag{32}$$

(2) *if* $-1 \preceq \boldsymbol{\nu} \prec 0$, then form (1)

$$1 \geq \frac{|\mathbf{m}_{21}| + |\mathbf{m}_{22}|}{\|\mathcal{M}\|_{I_\infty}} \succeq \boldsymbol{\nu}. \tag{33}$$

From (6) and (38)

$$-\frac{(|\mathbf{m}_{21}| + |\mathbf{m}_{22}|)}{\|\mathcal{M}\|_{I_\infty}} \preceq -\boldsymbol{\nu} \preceq \|\mathcal{M}\|_{I_1}. \tag{34}$$

From Case 4.5 and Case 4.6, we have

$$\frac{|\mathbf{m}_{21}| + |\mathbf{m}_{22}|}{\|\mathcal{M}\|_{I_\infty}} \leq |\boldsymbol{\nu}| \leq \|\mathcal{M}\|_{I_1}, \quad \textit{when } |\boldsymbol{\nu}| \geq 1.$$

From (31) and (34), we have

$$-\frac{(|\mathbf{m}_{21}| + |\mathbf{m}_{22}|)}{\|\mathcal{M}\|_{I_\infty}} \leq |\boldsymbol{\nu}| \leq \|\mathcal{M}\|_{I_1}, \quad \textit{when } |\boldsymbol{\nu}| \leq 1 \textit{ and } \boldsymbol{\nu} \neq 0.$$

Case 4.7.

Consider $\boldsymbol{\nu} = 0$, then by (1), we have

$$-1 \leq -\frac{(|\mathbf{m}_{21}| + |\mathbf{m}_{22}|)}{\|\mathcal{M}\|_{I_\infty}}.$$

Therefore,

$$\boldsymbol{\nu} = 0 \leq 1 - \frac{(|\mathbf{m}_{21}| + |\mathbf{m}_{22}|)}{\|\mathcal{M}\|_{I_\infty}}. \tag{35}$$

From (3) and (35), we have

$$-\frac{(|\mathbf{m}_{21}| + |\mathbf{m}_{22}|)}{\|\mathcal{M}\|_{I_\infty}} \leq |\boldsymbol{\nu}| \leq \min\left\{\|\mathcal{M}\|_{I_1}, 1 - \frac{(|\mathbf{m}_{21}| + |\mathbf{m}_{22}|)}{\|\mathcal{M}\|_{I_\infty}}\right\}.$$

■

Example 4.2.

Consider the following interval payoff matrix

$$\mathcal{M} = \begin{bmatrix} [2,3] & [4,5] \\ [3,6] & [3,4] \end{bmatrix},$$

For the given payoff matrix, $u_1 = 8,\ u_2 = 10,\ t_1 = 9,\ t_2 = 9$. Since, $\|\mathcal{M}\|_{I_\infty} = \max\{u_1, u_2\} = 10$ and $\|\mathcal{M}\|_{I_1} = \max\{t_1, t_2\} = 9$. We have,

$$\frac{|\mathbf{m}_{11}| + |\mathbf{m}_{12}|}{\|\mathcal{M}\|_{I_\infty}} \preceq |\boldsymbol{\nu}| \preceq \|\mathcal{M}\|_{I_1},$$

$$0.8 \preceq \boldsymbol{\nu} \preceq 9.$$

Here, we conclude that the value of the interval matrix game lies between 0.8 to 9. Also, we obtain the value of the game is $\boldsymbol{\nu} = [3,4]$ using interval linear programming method due to Bhurjee and Panda (2017), which contains in $[0.8, 9]$.

Theorem 4.2.

Let $(\mathcal{M}; P, Q)$ be an interval matrix game and $\boldsymbol{\nu}$ be the value of the game. For the interval payoff matrix

$$\mathcal{M} = \begin{bmatrix} \mathbf{m}_{11} & \mathbf{m}_{12} & \cdots & \mathbf{m}_{1n} \\ \mathbf{m}_{21} & \mathbf{m}_{22} & \cdots & \mathbf{m}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{m}_{m1} & \mathbf{m}_{m2} & \cdots & \mathbf{m}_{mn} \end{bmatrix},$$

if $|\boldsymbol{\nu}| \geq 1$, then $\frac{k}{\|\mathcal{M}\|_{I_\infty}} \leq |\boldsymbol{\nu}| \leq \|\mathcal{M}\|_{I_1}$,

if $|\boldsymbol{\nu}| \leq 1$ and $\boldsymbol{\nu} \neq 0$, then $\frac{-k}{\|\mathcal{M}\|_{I_\infty}} \leq |\boldsymbol{\nu}| \leq \|\mathcal{M}\|_{I_1}$,

if $\boldsymbol{\nu} = 0$, then $\frac{k}{\|\mathcal{M}\|_{I_\infty}} \leq |\boldsymbol{\nu}| \leq \min\left\{\|\mathcal{M}\|_{I_1}, 1 - \frac{k}{\|\mathcal{M}\|_{I_\infty}}\right\}$,

where

$$k = \sup_{1 \leq i \leq m, i \neq p} \sum_{j=1}^{n} |\mathbf{m}_{ij}| \quad \text{and} \quad \|\mathcal{M}\|_{I_\infty} = \sum_{j=1}^{n} |[m_{pj}^L, m_{pj}^R]|, \qquad \textit{for any fixed } p.$$

Proof:

We consider three cases.

Case 4.8.

For $\boldsymbol{\nu} \succ 0$,

(1) *if* $\boldsymbol{\nu} \succeq 1$, then by Lemma 4.3,

$$\frac{k}{\|\mathcal{M}\|_{I_\infty}} \preceq \boldsymbol{\nu} \preceq \|\mathcal{M}\|_{I_1}.$$

(2) *if* $0 \prec \boldsymbol{\nu} \preceq 1$, then by Lemma 4.3, we have

$$\frac{-k}{\|\mathcal{M}\|_{I_\infty}} \preceq \boldsymbol{\nu}. \tag{36}$$

From (21) and (36), we have

$$\frac{-k}{\|\mathcal{M}\|_{I_\infty}} \preceq \boldsymbol{\nu} \preceq \|\mathcal{M}\|_{I_1}.$$

Case 4.9.

For $\boldsymbol{\nu} \prec 0$,

(1) *if* $\boldsymbol{\nu} \preceq -1$, then

$$\boldsymbol{\nu} \preceq -1 \leq \frac{-k}{\|\mathcal{M}\|_{I_\infty}}.$$

From (23),

$$-\|\mathcal{M}\|_{I_1} \preceq \boldsymbol{\nu} \preceq \frac{-k}{\|\mathcal{M}\|_{I_\infty}}.$$

Then,

$$\frac{k}{\|\mathcal{M}\|_{I_\infty}} \preceq -\boldsymbol{\nu} \preceq \|\mathcal{M}\|_{I_1}.$$

(2) *if* $-1 \preceq \boldsymbol{\nu} \prec 0$, then by Lemma 4.3, we have

$$1 \geq \frac{k}{\|\mathcal{M}\|_{I_\infty}} \succeq \boldsymbol{\nu}.$$

Then, by (23),

$$\frac{k}{\|\mathcal{M}\|_{I_\infty}} \succeq \boldsymbol{\nu} \succeq -\|\mathcal{M}\|_{I_1}.$$

Then, we have

$$\frac{-k}{\|\mathcal{M}\|_{I_\infty}} \preceq -\boldsymbol{\nu} \preceq \|\mathcal{M}\|_{I_1}.$$

From Case 4.8 and Case 4.9, we obtain

$$\frac{k}{\|\mathcal{M}\|_{I_\infty}} \leq |\boldsymbol{\nu}| \leq \|\mathcal{M}\|_{I_1}, \quad \textit{when } |\boldsymbol{\nu}| \geq 1,$$

and

$$\frac{-k}{\|\mathcal{M}\|_{I_\infty}} \leq |\boldsymbol{\nu}| \leq \|\mathcal{M}\|_{I_1}, \quad \textit{when } |\boldsymbol{\nu}| \leq 1 \textit{ and } \boldsymbol{\nu} \neq 0.$$

Case 4.10.

Consider $\boldsymbol{\nu} = 0$. Then, by Lemma 4.3, we have

$$\frac{k}{\|\mathcal{M}\|_{I_\infty}} \leq 1,$$

where

$$k = \sup_{1 \leq i \leq m, i \neq p} \sum_{j=1}^{n} |\mathbf{m}_{ij}|.$$

Therefore, we have

$$0 = \boldsymbol{\nu} \preceq 1 - \frac{k}{\|\mathcal{M}\|_{I_\infty}}.$$

From (21), we obtain,

$$\frac{-k}{\|\mathcal{M}\|_{I_\infty}} \leq \boldsymbol{\nu} \leq \min\Big\{\|\mathcal{M}\|_{I_1}, 1 - \frac{k}{\|\mathcal{M}\|_{I_\infty}}\Big\}.$$

Example 4.3.

Consider the following interval payoff matrix

$$\mathcal{M} = \begin{bmatrix} [0.12, 0.17] & [0.11, 0.16] & [0.075, 0.12] & [0.068, 0.13] \\ [0.18, 0.22] & [0.12, 0.15] & [0.072, 0.14] & [-0.05, 0.15] \\ [0.043, 0.043] & [0.043, 0.043] & [0.043, 0.043] & [0.043, 0.043] \end{bmatrix},$$

For payoff matrix $\mathcal{M}$, $u_1 = 0.58$, $u_2 = 0.66$, $u_3 = 0.129$, $t_1 = 0.433$, $t_2 = 0.353$, $t_3 = 0.303$, $t_4 = 0.323$. Therefore, $\|\mathcal{M}\|_{I_\infty} = \max\{u_1, u_2, u_3\} = 0.66$ and $\|\mathcal{M}\|_{I_1} = \max\{t_1, t_2, t_3, t_4\} = 0.433$. Then, by Theorem 4.2, we have

$$-0.878 \leq |\boldsymbol{\nu}| \leq 0.433.$$

Thus, we can say that the value of the interval matrix game lies between -0.878 to 0.433. Also, we find the value of the game as $\boldsymbol{\nu} = [0.068, 0.12]$ using interval linear programming method due to Bhurjee and Panda (2017), which contains in $[-0.878, 0.433]$.

Note 4.1.

Theorem 4.1 and Theorem 4.2 may not be solved as matrix game models, as they can be solved simply through known methods. However, these theorems play an important role in obtaining the lower and upper bounds of the value of larger scale matrix game models with interval payoffs without solving any linear programming problems or systems of linear equations or inequalities.

5. Bounds for Optimal Mixed Strategy

Consider $\|\mathcal{M}\|_{I_\infty} = \sum_{j=1}^{n} |\mathbf{m}_{lj}|$ for $\mathcal{M} \in \mathbb{R}^{m \times n}$. If we eliminate the l^{th} row of the matrix $\mathcal{M}$, then we get the matrix $\mathcal{N} \in \mathbb{R}^{(m-1) \times n}$, which is called a row-wise generated interval matrix of

$\mathcal{M}$. Likewise, consider $\|\mathcal{M}\|_{I_1} = \sum_{i=1}^{m} |\mathbf{m}_{ip}|$ for $\mathcal{M} \in \mathbb{R}^{m\times n}$. If we eliminate the p^{th} column of the matrix $\mathcal{M}$, then we get the matrix $\mathcal{N} \in \mathbb{R}^{m\times(n-1)}$, which is called a ***column-wise generated interval matrix of*** $\mathcal{M}$

Here, p_{sup} and p_{inf} represent the most and least elements of the set P of mixed strategies, respectively. In order to optimize the game value, we find the bounds of the strategy set in terms of p_{sup} and p_{inf}. First, we prove a result that will help us determine the lower and upper bounds for p_{sup} and p_{inf} using the min-max theorem.

Theorem 5.1.

Let $\mathcal{M} \in R^{2\times 2}$ be an interval payoff matrix for a two-person zero-sum interval game with positive entries and $\mathcal{N}$ is the column-wise generated matrix of $\mathcal{M}$. Then,

$$p_{\text{sup}} \geq L \text{ and } p_{\text{inf}} \leq U,$$

where

$$L = \sup\left\{1 - \frac{v^R}{\|\mathcal{M}\|_{I_1}}, \frac{v^L}{\|\mathcal{N}\|_{I_1}}\right\} \quad \text{and} \quad U = \inf\left\{1 - \frac{v^L}{\|\mathcal{N}\|_{I_1}}, \frac{v^R}{\|\mathcal{M}\|_{I_1}}\right\}.$$

Proof:

Consider $\mathcal{M}$ be a given payoff matrix

$$\mathcal{M} = \begin{bmatrix} \mathbf{m}_{11} \; \mathbf{m}_{12} \\ \mathbf{m}_{21} \; \mathbf{m}_{22} \end{bmatrix}.$$

Let $\|\mathcal{M}\|_{I_1} = \mathbf{m}_{11} + \mathbf{m}_{21}$, then $\mathbf{m}_{11} + \mathbf{m}_{21} \succeq \mathbf{m}_{12} + \mathbf{m}_{22}$. Then, we have

$$\boldsymbol{\nu} = \mathbf{m}_{11}\, p + (1-p)\, \mathbf{m}_{21} = \mathbf{m}_{12}\, p + (1-p)\, \mathbf{m}_{22}.$$

Then,

$$(\mathbf{m}_{11} + \mathbf{m}_{21})\, p_{\text{sup}} \succeq (\mathbf{m}_{12} + \mathbf{m}_{22})\, p_{\text{sup}} \quad \text{and} \quad (\mathbf{m}_{11} + \mathbf{m}_{21})\, p_{\text{inf}} \succeq (\mathbf{m}_{12} + \mathbf{m}_{22})\, p_{\text{inf}},$$

$$(\mathbf{m}_{11} + \mathbf{m}_{21})\, p_{\text{inf}} \preceq \boldsymbol{\nu} \preceq (\mathbf{m}_{12} + \mathbf{m}_{22})\, p_{\text{sup}}.$$

Therefore, we have

$$p_{\text{inf}} \preceq \frac{\boldsymbol{\nu}}{\mathbf{m}_{11} + \mathbf{m}_{21}} = \frac{\boldsymbol{\nu}}{\|\mathcal{M}\|_{I_1}} \preceq \frac{v^R}{\|\mathcal{M}\|_{I_1}}, \tag{37}$$

and

$$p_{\text{sup}} \succeq \frac{\boldsymbol{\nu}}{\mathbf{m}_{12} + \mathbf{m}_{22}} = \frac{\boldsymbol{\nu}}{\|\mathcal{N}\|_{I_1}} \succeq \frac{v^L}{\|\mathcal{N}\|_{I_1}}, \tag{38}$$

where $\mathcal{N}$ is the column-wise generated matrix. In the case of 2×2 matrix game, the strategy set has a ordered pair (p_1, p_2) such that $p_1 + p_2 = 1$. Hence,

$$p_{\text{sup}} + p_{\text{inf}} = 1. \tag{39}$$

From (37) and (39),

$$p_{\text{sup}} + p_{\text{inf}} \leq \frac{v^R}{\|\mathcal{M}\|_{I_1}} + p_{\text{sup}}.$$

Then, we obtain

$$1 - \frac{v^R}{\|\mathcal{M}\|_{I_1}} \leq p_{\sup}.$$

Therefore,

$$p_{\sup} \geq L,$$

where

$$L = \sup\left\{1 - \frac{v^R}{\|\mathcal{M}\|_{I_1}}, \frac{v^L}{\|\mathcal{N}\|_{I_1}}\right\}.$$

Similarly, we have

$$p_{\inf} \leq U,$$

where

$$U = \inf\left\{1 - \frac{v^L}{\|\mathcal{N}\|_{I_1}}, \frac{v^R}{\|\mathcal{M}\|_{I_1}}\right\}.$$

This ends the proof. ■

The following theorem is used to represent the relationships between least and most elements of the mixed strategies set.

Theorem 5.2.

Let $\mathcal{M} \in R^{m\times n}$ be the interval payoff game model. Then,

$$\frac{1-p_{\inf}}{m-1} \leq p_{\sup} \leq 1-(m-1)\,p_{\inf} \text{ and } 1-(m-1)\,p_{\sup} \leq p_{\inf} \leq \frac{1-p_{\sup}}{m-1}.$$

Proof:

For player-I,

$$p_1 + \cdots + p_{\inf} + \cdots + p_{\sup} + \cdots + p_m = 1.$$

Then

$$p_1 + \cdots + p_{\inf} + \cdots + p_m = 1 - p_{\sup} \quad \text{or} \quad p_1 + \cdots + p_{\sup} + \cdots + p_m = 1 - p_{\inf},$$

$$(m-1)\,p_{\inf} \leq 1 - p_{\sup} \quad \text{or} \quad (m-1)\,p_{\sup} \leq 1 - p_{\inf}.$$

Thus, we have

$$p_{\sup} \leq 1-(m-1)\,p_{\inf} \quad \text{and} \quad p_{\inf} \leq \frac{1-p_{\sup}}{m-1},$$

or

$$\frac{1-p_{\inf}}{m-1} \leq p_{\sup} \quad \text{and} \quad 1-(m-1)\,p_{\sup} \leq p_{\inf}.$$

■

Theorem 5.3.

Let $\mathcal{M} \in R^{m\times n}$ as the interval payoff game model. The most and least elements of the mixed strategy set are $p_{\sup}$ and $p_{\inf}$, respectively. Then,

$$p_{\sup} \geq L \quad \text{and} \quad p_{\inf} \leq U,$$

where

$$L = \sup\left\{\frac{1-\frac{|\boldsymbol{\nu}|}{\|\mathcal{M}\|_{I_1}}}{m-1}, \frac{|\boldsymbol{\nu}|}{\|\mathcal{N}\|_{I_1}}\right\} \text{ and } U = \inf\left\{\frac{1-\frac{|\boldsymbol{\nu}|}{\|\mathcal{N}\|_{I_1}}}{m-1}, \frac{|\boldsymbol{\nu}|}{\|\mathcal{M}\|_{I_1}}\right\}.$$

Proof:

Consider

$$\|\mathcal{M}\|_{I_1} = |\mathrm{m}_{1k}| + \cdots + |\mathrm{m}_{mk}| \quad \text{and} \quad \|\mathcal{N}\|_{I_1} = |\mathrm{m}_{1l}| + \cdots + |\mathrm{m}_{ml}|, \quad \textit{for fixed } k, l \leq n.$$

Then,

$$\boldsymbol{\nu} = p_1\mathrm{m}_{1l} + \cdots + p_m\mathrm{m}_{ml} \preceq p_{\sup}\left(|\mathrm{m}_{1l}| + \cdots + |\mathrm{m}_{ml}|\right) = p_{\sup}\|\mathcal{N}\|_{I_1}.$$

Thus, we have

$$p_{\sup} \geq \frac{|\boldsymbol{\nu}|}{\|\mathcal{N}\|_{I_1}}.$$

Consequently,

$$\boldsymbol{\nu} = p_1\mathrm{m}_{1k} + \cdots + p_m\mathrm{m}_{mk} \succeq p_{\inf}\left(|\mathrm{m}_{1k}| + \cdots + |\mathrm{m}_{mk}|\right) = p_{\inf}\|\mathcal{M}\|_{I_1}.$$

This gives

$$p_{\inf} \leq \frac{|\boldsymbol{\nu}|}{\|\mathcal{M}\|_{I_1}}.$$

We have $p_1 + p_2 + \cdots + p_{\sup} + \cdots + p_m = 1$. Then,

$$\frac{|\boldsymbol{\nu}|}{\|\mathcal{N}\|_1} + (m-1)p_{\inf} \leq 1 \Rightarrow p_{\inf} \leq \frac{1-\frac{|\boldsymbol{\nu}|}{\|\mathcal{N}\|_{I_1}}}{m-1}.$$

Therefore,

$$p_{\inf} \leq U,$$

where

$$U = \inf\left\{\frac{1-\frac{|\boldsymbol{\nu}|}{\|\mathcal{N}\|_{I_1}}}{m-1}, \frac{|\boldsymbol{\nu}|}{|\ \mathcal{M}\|_{I_1}}\right\}.$$

Similarly, we have

$$p_{\sup} \geq L,$$

where

$$L = \sup\left\{\frac{1-\frac{|\boldsymbol{\nu}|}{\|\mathcal{M}\|_{I_1}}}{m-1}, \frac{|\boldsymbol{\nu}|}{\|\mathcal{N}\|_{I_1}}\right\}.$$

This ends the proof. ∎

Example 5.1.

This example illustrates the Theorem 5.3. We find the interval payoff matrices $\boldsymbol{\Delta}$ and $\boldsymbol{\Delta}^{\mathbf{T}}$ due to Li et al. (2012) for companies C_1 and C_2, consider as players-I and player-II, respectively. The game value is given as $[141.3, 158.1]$ in the paper. The interval payoff matrix

$$\boldsymbol{\Delta} = \begin{bmatrix} [175,190] & [120,158] \\ [80,100] & [180,190] \end{bmatrix}.$$

We explore the game from the perspective of the C_1 company. Here, the most and least element for the set of mixed strategies are provided as $p_{\sup} = 0.645$ and $p_{\inf} = 0.355$, respectively. Furthermore

$$\|\boldsymbol{\Delta}\|_{I_1} = 348, \quad \|\boldsymbol{\Delta}\|_{I_\infty} = 348, \quad \|\bar{\boldsymbol{\Delta}}\|_{I_1} = 290, \quad \text{and} \quad \|\boldsymbol{\Delta}^*\|_{I_\infty} = 290,$$

where $\bar{\boldsymbol{\Delta}}$ is the column-wise generated matrix and $\boldsymbol{\Delta}^*$ is the row-wise generated matrix of $\boldsymbol{\Delta}$. Using Theorem 4.1, we calculate the bound for the approximated value of the interval game $\boldsymbol{\nu}$. Therefore, we have

$$0.833 \leq |\boldsymbol{\nu}| \leq 348.$$

We find the bounds for $p^*_{\sup}$ and $p^*_{\inf}$ using Theorem 5.3 in the case $m = 2$. We have $p^*_{\sup} \geq L$ and $p^*_{\inf} \leq U$. where

$$L = \sup\left\{\frac{1-\frac{158.1}{348}}{2-1}, \frac{158.1}{290}\right\} = \sup\{0.5456, 0.5452\},$$

and

$$U = \inf\left\{\frac{1-\frac{158.1}{290}}{2-1}, \frac{158.1}{348}\right\} = \inf\{0.4548, 0.4544\}.$$

Thus $p^*_{\sup} \geq 0.5456$ and $p^*_{\inf} \leq 0.4544$. Hence, we show that the above inequalities hold for the given $p_{\sup}$ and $p_{\inf}$.

As Example 5.1, we consider the strategy as $p_1 = 0.5456$ and $p_2 = 0.4544$. In this case, we use the first column of the game model to calculate the approximate game value

$$\boldsymbol{\nu}_{app} = 0.5456\,[175, 190] + 0.4544\,[80, 100] = [130.83, 149.1],$$

for the matrix game. Furthermore, by selecting another column for the payoff matrix, we may obtain a more accurate game value of the game model.

Note 5.1.

The approximate game value $\boldsymbol{\nu}_{app}$ always lies within the optimum value of the game value, and the strategy may be selected using the inequalities $p_{\sup}$ and $p_{\inf}$. We must also follow the strategy theory principle when selecting the mixed strategies. To put it another way, the sum of the strategies must be equal to one. Therefore, the set of possible mixed strategies (MS) for 2×2 payoffs matrix game is given by the following set:

$$MS = \{(p_1, p_2) \in P | p_1 \leq p_{\inf}, p_1 + p_2 = 1\} \cap \{(p_1, p_2) \in P | p_1 \geq p_{\sup}, p_1 + p_2 = 1\}.$$

6. Conclusions

A two-player zero-sum game model with interval payoffs is considered and defines the saddle point as well as the value of the game model. Several theoretical results have been established to discuss the range of the value of the game using interval analysis. Furthermore, we drive lemmas and theorems to relate norms of 2×2 as well as the generalized case for the $m \times n$ interval payoff matrix and the value of the game models. Also, we focus on largest and lowest elements of the mixed strategy sets by deriving various results. Using these results, we can find the lower and upper bounds of the set of strategies for the game problem. We also use several data samples to analyze and verify the consistency of our methodology. In light of the developed theoretical work in the paper, we can extend it to determine optimal game values for two-player non-zero sum interval payoffs game problems.

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