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Analysis of Some Unified Integral Equations of Fredholm Type Associated with Multivariable Incomplete H and I-Functions

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Abstract

In this research paper, we examine various effective methods for addressing the problem of solving Fredholm-type integral equations. Our investigation commences by applying the principles of fractional calculus theory. We employ series representations and products of multivariable incomplete H-functions and multivariable incomplete I-functions to solve these integrals. The outcomes derived from our analysis possess a general nature and hold the potential to yield numerous results.

Keywords: Fredholm type integral equations; Riemann-Liouville fractional integral; Weyl fractional integral; Multivariable incomplete H-functions; Multivariable incomplete I-functions

MSC 2010 No.: 45B05, Primary 26A33, 33C99

1. Introduction

Integral equations are of ordinary use in various mechanics and mathematical physics fields. Specifically, integral equations closely tied to differential equations are known as Fredholm Integral Equations. In mathematics, the Fredholm integral equations play an important role, giving rise to Fredholm theory, which delves into the study of Fredholm kernels and Fredholm operators Bushman (1962) and Higgins (1964). The theory, named after Erik Ivar Fredholm, primarily focuses on solving the Fredholm integral equation Love (1967, 1975). These integral equations originate from boundary value problems associated with differential equations and can be effectively tackled using a range of simplified methods Sharma et al. (2024A, 2024B).

Fredholm Type Integral Equations are when the limits of integration are fixed as defined in Kumar and Ayant (2021). Fredholm integral equations have natural applications in signal processing Rushforth (1987), as well as in linear forward modelling and inverse problems Tarantola (2005). They also play a significant role in addressing challenges such as the spectral concentration problem. Moreover, they find utility in solving fractional Volterra-Fredholm integro-differential equations Hamoud and Ghadle (2018) with mixed boundary conditions, employing the hybrid orthonormal Bernstein and block-pulse functions wavelet method Ali and Hadhoud (2019). Furthermore, researchers have investigated a novel Neumann series method for effectively solving a wide range of local fractional Fredholm and Volterra integral equations Hamoud and Ghadle (2018).

In signal processing, Fredholm equations are used to solve problems such as spectral concentration linear forward modeling and inverse problems. In physics, Fredholm equations relate experimental spectra to various underlying distributions, such as the mass distribution of polymers or the distribution of relaxation times Yager (1936). In fluid mechanics, Fredholm equations are used to model hydrodynamic interactions near finite-sized elastic interfaces Ider and Gekle (2016).

A specific application of Fredholm equations in computer graphics is the generation of photo-realistic images Collins (1998). In this context, Fredholm equations model light transport from virtual light sources to the image plane. This is known as the rendering equation.

Over the past few years, numerous authors, including Buchman Buchman (1962), Higgins (1964), Love (1967 and 1975), Prabhakar and Kashyap (1980), Srivastava and Buchman (1977), Srivastava and Raina (1992), Chaurasia and Patni (2001), Chaurasia and Kumar (2012) and Chaurasia and Singh (2014) and Singh et al. (2023), have extensively examined a wide range of Fredholm-type integral equations and other integral equations. These equations involve diverse polynomials or special functions as their kernels. The collective efforts of these researchers have significantly contributed to advancing our understanding of such integral equations. We will now introduce several fundamental definitions that will be used to derive the main results of the theorems.

Series Representation of Multivariable Incomplete H-functions

We introduced the series representation of multivariable incomplete H-functions as follows:

$$\Gamma^{0,n:m_1,n_1,m_2,n_2;\ldots;m_r,n_r}_{p,q;p_1,q_1;\ldots;p_r,q_r}[z_r] = \sum_{\mu_k=1}^{m_k}\sum_{\nu_k=1}^{\infty}\phi_1\phi_2\frac{(-1)^{\sum_{k=1}^r \nu_k}\prod_{k=1}^r (z_k)^{Z_k}}{\prod_{k=1}^r \delta^{(k)}_{\mu_k}\nu_k!}, \tag{1}$$

and

$$\gamma^{0,n:m_1,n_1,m_2,n_2;\ldots;m_r,n_r}_{p,q;p_1,q_1;\ldots;p_r,q_r}[z_r] = \sum_{\mu_k=1}^{m_k}\sum_{\nu_k=1}^{\infty}\phi'_1\phi_2\frac{(-1)^{\sum_{k=1}^r \nu_k}\prod_{k=1}^r (z_k)^{Z_k}}{\prod_{k=1}^r \delta^{(k)}_{\mu_k}\nu_k!}, \tag{2}$$

where,

$$\phi_1 = \frac{\Gamma(1-a_1+\sum_{k=1}^r \alpha_1^{(k)}Z_k,t)\prod_{j=2}^n\Gamma(1-a_j+\sum_{k=1}^r\alpha_j^{(k)}Z_k)}{\prod_{j=n+1}^p\Gamma\left(a_j-\sum_{k=1}^r\alpha_j^{(k)}Z_k\right)\prod_{j=1}^q\Gamma\left(1-b_j+\sum_{k=1}^r\beta_j^{(k)}Z_k\right)},$$

$$\phi'_1 = \frac{\gamma(1-a_1+\sum_{k=1}^r \alpha_1^{(k)}Z_k,t)\prod_{j=2}^n\Gamma(1-a_j+\sum_{k=1}^r\alpha_j^{(k)}Z_k)}{\prod_{j=n+1}^p\Gamma\left(a_j-\sum_{k=1}^r\alpha_j^{(k)}Z_k\right)\prod_{j=1}^q\Gamma\left(1-b_j+\sum_{k=1}^r\beta_j^{(k)}Z_k\right)},$$

$$\phi_2 = \frac{\prod_{j=1,j\neq\mu_k}^{m_k}\Gamma(d_j^{(k)}-\delta_j^{(k)}Z_k)\prod_{j=1}^{n_k}\Gamma(1-c_j^{(k)}+\zeta_j^{(k)}Z_k)}{\prod_{j=m_k+1}^{q_k}\Gamma\left(1-d_j^{(k)}+\delta_j^{(k)}Z_k\right)\prod_{j=n_k+1}^{p_k}\Gamma\left(c_j^{(k)}-\zeta_j^{(k)}Z_k\right)},$$

$$Z_k = \frac{d_{\mu_k}^{(k)}+\nu_k}{\delta_{\mu_k}^{(k)}}, \quad k=1,\ldots,r.$$

This series expansion holds under the state of conditions

$$\begin{cases}\text{for } j\neq\mu_k,\ \delta_{\mu_k}^{(k)}\left[d_j^{(k)}+d_k\right]\neq\delta_j^{(k)}\left[d_{\mu_k}^{(k)}+\nu_k\right],\\ \text{for } j=\mu_k,\ \mu_k=1,\ldots,m_r;\ \ d_k,\ \nu_k=0,1,\ldots,\end{cases} \tag{3}$$

$$z_k\neq 0,\ \nabla_k=\sum_{j=1}^p\alpha_j^{(k)}-\sum_{j=1}^q\beta_j^{(k)}+\sum_{j=1}^{p_k}\zeta_j^{(k)}-\sum_{j=1}^{q_k}\delta_j^{(k)}<0\ \forall\ k=1,\ldots,r. \tag{4}$$

The series representations of the multivariable incomplete H-functions defined in Equations (1) and (2) can be simplified to the multivariable H-function by either substituting $t=0$ or by adding the expressions of Equations (1) and (2). This result was presented by Olkha and Chaurasia (1985), where they provided the series representation for the multivariable H-function.

Series Representation of Multivariable Incomplete I-functions

We introduced the series representation of multivariable incomplete I-functions is as follows:

$$^{(\Gamma)}I^{0,N:M_1,N_1;\ldots;M_r,N_r}_{P_i,Q_i,R;P_{i^{(1)}},Q_{i^{(1)}},R_{(1)};\ldots;P_{i^{(r)}},Q_{i^{(r)}},R_{(r)}}[z_r] = \sum_{\lambda_k=1}^{M_k}\sum_{\Lambda_k=1}^{\infty}\varphi_1\varphi_2\frac{(-1)^{\sum_{k=1}^r\Lambda_k}\prod_{k=1}^r(u_k)^{U_k}}{\prod_{k=1}^r\mathfrak{d}_{\lambda_k}^{(k)}\Lambda_k!}, \tag{5}$$

and

$$^{(\gamma)}I^{0,N:M_1,N_1;\ldots;M_r,N_r}_{P_i,Q_i,R;P_{i^{(1)}},Q_{i^{(1)}},R_{(1)};\ldots;P_{i^{(r)}},Q_{i^{(r)}},R_{(r)}}[z_r] = \sum_{\lambda_k=1}^{M_k}\sum_{\Lambda_k=1}^{\infty}\varphi'_1\varphi_2\frac{(-1)^{\sum_{k=1}^r\Lambda_k}\prod_{k=1}^r(u_k)^{U_k}}{\prod_{k=1}^r\mathfrak{d}_{\lambda_k}^{(k)}\Lambda_k!}, \tag{6}$$

where,

$$\varphi_1 = \frac{\Gamma(1-e_1+\sum_{k=1}^{r} E_1^{(k)}U_1,t)\prod_{j=2}^{N}\Gamma(1-e_j+\sum_{k=1}^{r} E_j^{(k)}U_k)}{\sum_{i=1}^{r}\left[\prod_{j=N+1}^{P_i}\Gamma\left(e_{ji}-\sum_{k=1}^{r} E_{ji}^{(k)}U_k\right)\prod_{j=1}^{Q_i}\Gamma\left(1-f_{ji}+\sum_{k=1}^{r} F_{ji}^{(k)}U_k\right)\right]},$$

$$\varphi_1' = \frac{\gamma(1-e_1+\sum_{k=1}^{r} E_1^{(k)}U_1,t)\prod_{j=2}^{N}\Gamma(1-e_j+\sum_{k=1}^{r} E_j^{(k)}U_k)}{\sum_{i=1}^{r}\left[\prod_{j=N+1}^{P_i}\Gamma\left(e_{ji}-\sum_{k=1}^{r} E_{ji}^{(k)}U_k\right)\prod_{j=1}^{Q_i}\Gamma\left(1-f_{ji}+\sum_{k=1}^{r} F_{ji}^{(k)}U_k\right)\right]},$$

$$\varphi_2 = \frac{\prod_{j=1,j\neq\lambda_k}^{M_k}\Gamma(f_j^{(k)}-F_j^{(k)}U_k)\prod_{j=1}^{N_k}\Gamma(1-e_j^{(k)}+E_j^{(k)}U_k)}{\sum_{i^{(k)}=1}^{R^{(k)}}\left[\prod_{j=M_k+1}^{Q_{i^{(k)}}}\Gamma\left(1-f_{ji^{(k)}}^{(k)}+F_{ji^{(k)}}^{(k)}U_k\right)\prod_{j=N_k+1}^{P_{k^{(k)}}}\Gamma\left(e_{ji^{(k)}}^{(k)}-E_{ji^{(k)}}^{(k)}U_k\right)\right]},$$

$$U_k = \frac{f_{\lambda_k}^{(k)}+\Lambda_k}{\mathfrak{d}_{\lambda_k}^{(k)}}, \quad k=1,\ldots,r.$$

This series expansion holds under the state of conditions

$$\mathfrak{d}_{\lambda_k}^{(k)}\left[f_j^{(k)}+f_k\right] \neq \mathfrak{d}_j^{(k)}\left[f_{\lambda_k}^{(k)}+\Lambda_k\right]$$
$$\text{for } j=\lambda_k,\ \lambda_k=1,\ldots,M_r;\ f_k,\ \Lambda_k=0,1,\ldots;\ z_i\neq 0, \tag{7}$$

$$u_k \neq 0,\ \nabla_k' = \sum_{j=1}^{P} E_j^{(k)} - \sum_{j=1}^{Q} F_j^{(k)} + \sum_{j=1}^{P_k} E_{ji}^{(k)} - \sum_{j=1}^{Q_k} F_{ji}^{(k)} < 0\ \forall\ k=1,\ldots,r. \tag{8}$$

The series representations of the multivariable incomplete I-functions given in Equations (5) and (6) can be simplified to the multivariable incomplete H-functions by substituting $R_{(k)} = 1$ $(k = 1,\ldots,r)$ expressed in Equations (1) and (2).

Fractional Integrals

Fractional Integral conditions are defined by Chaurasia and Kumar (2012). To establish a correspondence with the space of well-behaved functions defined on the entire real line $(-\infty,\ \infty)$, refer to the work of Lighthill (1958).

The Riemann-Liouville fractional integral (fractional order α) was established in Laurent (1884) and the Weyl fractional integral (of order β) was introduced in Laurent (1884).

The main findings of this work are delineated in the second part with specific cases also examined in the third part of the paper.

2. Main Results

This paper presents the solutions obtained for the Fredholm integral equations which associated multivariable incomplete H-functions given in Equations (9) and (10) and multivariable incomplete

I-functions given in Equations (11) and (12) stated below:

$$\int_0^\infty y^{-\eta}\Gamma^{0,n:m_1,n_1,m_2,n_2;\ldots;m_r,n_r}_{p,q;p_1,q_1;\ldots;p_r,q_r}\left[\begin{matrix} z_1(x/y)^g \\ \vdots \\ z_r(x/y)^g \end{matrix}\left|\begin{matrix}\left(a_1;\alpha_1^{(1)},\ldots,\alpha_1^{(r)},t\right),\left(a_j;\alpha_j^{(1)},\ldots,\alpha_j^{(r)}\right)_{2,p}: \\ \left(b_j;\beta_j^{(1)},\ldots,\beta_j^{(r)}\right)_{m+1,q_i}:\end{matrix}\right.\right.$$

$$\left.\begin{matrix}\left(c_j^{(1)},\zeta_j^{(1)}\right)_{1,p_1};\ldots;\left(c_j^{(r)},\zeta_j^{(r)}\right)_{1,p_r} \\ \left(d_j^{(1)},\delta_j^{(1)}\right)_{1,q_1};\ldots;\left(d_j^{(r)},\delta_j^{(r)}\right)_{1,q_r}\end{matrix}\right]\Gamma^{0,n':m'_1,n'_1,m'_2,n'_2;\ldots;m'_r,n'_r}_{p',q';p'_1,q'_1;\ldots;p'_r,q'_r}\left[\begin{matrix} z'_1(x/y)^h \\ \vdots \\ z'_r(x/y)^h\end{matrix}\right|$$

$$\left.\begin{matrix}\left(a'_1;\alpha'^{(1)}_1,\ldots,\alpha'^{(r)}_1,t\right),\left(a'_j;\alpha'^{(1)}_j,\ldots,\alpha'^{(r)}_j\right)_{2,p}:\left(c'^{(1)}_j,\zeta'^{(1)}_j\right)_{1,p_1};\ldots;\left(c'^{(r)}_j,\zeta'^{(r)}_j\right)_{1,p_r} \\ \left(b'_j;\beta'^{(1)}_j,\ldots,\beta'^{(r)}_j\right)_{m+1,q_i}:\left(d'^{(1)}_j,\delta'^{(1)}_j\right)_{1,q_1};\ldots;\left(d'^{(r)}_j,\delta'^{(r)}_j\right)_{1,q_r}\end{matrix}\right]$$

$$\times f(y)dy = w(x)\ (0<x<\infty). \tag{9}$$

For the lower form of multivariable incomplete H-function which is associated lower form of gamma function as follows:

$$\int_0^\infty y^{-\eta}\gamma^{0,n:m_1,n_1,m_2,n_2;\ldots;m_r,n_r}_{p,q;p_1,q_1;\ldots;p_r,q_r}\left[\begin{matrix} z_1(x/y)^g \\ \vdots \\ z_r(x/y)^g \end{matrix}\left|\begin{matrix}\left(a_1;\alpha_1^{(1)},\ldots,\alpha_1^{(r)},t\right),\left(a_j;\alpha_j^{(1)},\ldots,\alpha_j^{(r)}\right)_{2,p}: \\ \left(b_j;\beta_j^{(1)},\ldots,\beta_j^{(r)}\right)_{m+1,q_i}:\end{matrix}\right.\right.$$

$$\left.\begin{matrix}\left(c_j^{(1)},\zeta_j^{(1)}\right)_{1,p_1};\ldots;\left(c_j^{(r)},\zeta_j^{(r)}\right)_{1,p_r} \\ \left(d_j^{(1)},\delta_j^{(1)}\right)_{1,q_1};\ldots;\left(d_j^{(r)},\delta_j^{(r)}\right)_{1,q_r}\end{matrix}\right]\gamma^{0,n':m'_1,n'_1,m'_2,n'_2;\ldots;m'_r,n'_r}_{p',q';p'_1,q'_1;\ldots;p'_r,q'_r}\left[\begin{matrix} z'_1(x/y)^h \\ \vdots \\ z'_r(x/y)^h\end{matrix}\right|$$

$$\left.\begin{matrix}\left(a'_1;\alpha'^{(1)}_1,\ldots,\alpha'^{(r)}_1,t\right),\left(a'_j;\alpha'^{(1)}_j,\ldots,\alpha'^{(r)}_j\right)_{2,p}:\left(c'^{(1)}_j,\zeta'^{(1)}_j\right)_{1,p_1};\ldots;\left(c'^{(r)}_j,\zeta'^{(r)}_j\right)_{1,p_r} \\ \left(b'_j;\beta'^{(1)}_j,\ldots,\beta'^{(r)}_j\right)_{m+1,q_i}:\left(d'^{(1)}_j,\delta'^{(1)}_j\right)_{1,q_1};\ldots;\left(d'^{(r)}_j,\delta'^{(r)}_j\right)_{1,q_r}\end{matrix}\right]$$

$$\times f(y)dy = w(x)\ (0<x<\infty). \tag{10}$$

For the upper form of multivariable incomplete I-function, we can write it by

$$\int_0^\infty y^{-\eta(\Gamma)} I^{0,N:M_1,N_1;\ldots;M_r,N_r}_{P_i,Q_i,R;P_{i^{(1)}},Q_{i^{(1)}},R_{(1)};\ldots;P_{i^{(r)}},Q_{i^{(r)}},R_{(r)}}\left[\begin{matrix} u_1(x/y)^g \\ \vdots \\ u_r(x/y)^g\end{matrix}\left|\begin{matrix}\left(e_1,E_1^{(1)},\ldots,E_1^{(r)},t\right), \\ \left(f_{ji},F_{ji}^{(1)},\ldots,F_{ji}^{(r)}\right)_{M+1,Q_l},\end{matrix}\right.\right.$$

$$\begin{matrix}\left(e_j,E_j^{(1)},\ldots,E_j^{(r)}\right)_{2,N},\left(e_{ji},E_{ji}^{(1)},E_{ji}^{(2)}\right)_{N+1,P_i},(e_j^{(1)},E_j^{(1)})_{1,N_1},\left(e^{(1)}_{ji^{(1)}},E^{(1)}_{ji^{(1)}}\right)_{N_1+1,P_i^{(1)}}, \\ \left(f_j^{(1)},F_j^{(1)}\right)_{1,M_1},\left(f^{(1)}_{ji^{(1)}},F^{(1)}_{ji^{(1)}}\right)_{M_1+1,Q_i^{(1)}},\ldots,\left(f_j^{(r)},F_j^{(r)}\right)_{1,M_r},\end{matrix}$$

$$\left.\begin{matrix}\ldots,(e_j^{(r)},E_j^{(r)})_{1,N_r},\left(e^{(r)}_{ji^{(r)}},E^{(r)}_{ji^{(r)}}\right)_{N_r+1,P_i^{(r)}} \\ \left(f^{(r)}_{ji^{(r)}},F^{(r)}_{ji^{(r)}}\right)_{M_r+1,Q_i^{(r)}}\end{matrix}\right]$$

$$\times^{(\Gamma)} I^{0,N':M'_1,N'_1;\ldots;M'_r,N'_r}_{P'_i,Q'_i,R';P'_{i(1)},Q'_{i(1)},R'_{(1)};\ldots;P'_{i(r)},Q'_{i(r)},R'_{(r)}} \left[\begin{matrix} u'_1(x/y)^h \\ \vdots \\ u'_r(x/y)^h \end{matrix} \middle| \begin{matrix} \left(e'_1, E'^{(1)}_1, \ldots, E'^{(r)}_1, t\right), \\ \left(f'_{ji}, F'^{(1)}_{ji}, \ldots, F'^{(r)}_{ji}\right)_{M+1,Q_l}, \end{matrix} \right.$$

$$\begin{matrix} \left(e'_j, E'^{(1)}_j, \ldots, E'^{(r)}_j\right)_{2,N}, \left(e'_{ji}, E'^{(1)}_{ji}, \ldots, E'^{(r)}_{ji}\right)_{N+1,P_i}, (e'^{(1)}_j, E'^{(1)}_j)_{1,N_1}, \\ \left(f'^{(1)}_j, F'^{(1)}_j\right)_{1,M_1}, \left(f'^{(1)}_{ji^{(1)}}, F'^{(1)}_{ji^{(1)}}\right)_{M_1+1,Q^{(1)}_i}, \ldots, \left(f'^{(r)}_j, F'^{(r)}_j\right)_{1,M_r}, \end{matrix}$$

$$\left. \begin{matrix} \left(e'^{(1)}_{ji^{(1)}}, E'^{(1)}_{ji^{(1)}}\right)_{N_1+1,P^{(1)}_i}, \ldots, (e'^{(r)}_j, E'^{(r)}_j)_{1,N_r}, \left(e'^{(r)}_{ji^{(r)}}, E'^{(r)}_{ji^{(r)}}\right)_{N_r+1,P^{(r)}_i} \\ \left(f'^{(r)}_{ji^{(r)}}, F'^{(r)}_{ji^{(r)}}\right)_{M_r+1,Q^{(r)}_i} \end{matrix} \right] f(y)dy$$

$$= w(x) \ (0 < x < \infty). \tag{11}$$

Similarly, we can write for lower multivariable incomplete I-function as follows:

$$\int_0^\infty y^{-\eta(\gamma)} I^{0,N:M_1,N_1;\ldots;M_r,N_r}_{P_i,Q_i,R;P_{i(1)},Q_{i(1)},R_{(1)};\ldots;P_{i(r)},Q_{i(r)},R_{(r)}} \left[\begin{matrix} u_1(x/y)^g \\ \vdots \\ u_r(x/y)^g \end{matrix} \middle| \begin{matrix} \left(e_1, E^{(1)}_1, \ldots, E^{(r)}_1, t\right), \\ \left(f_{ji}, F^{(1)}_{ji}, \ldots, F^{(r)}_{ji}\right)_{M+1,Q_l}, \end{matrix} \right.$$

$$\begin{matrix} \left(e_j, E^{(1)}_j, \ldots, E^{(r)}_j\right)_{2,N}, \left(e_{ji}, E^{(1)}_{ji}, E^{(2)}_{ji}\right)_{N+1,P_i}, (e^{(1)}_j, E^{(1)}_j)_{1,N_1}, \left(e^{(1)}_{ji^{(1)}}, E^{(1)}_{ji^{(1)}}\right)_{N_1+1,P^{(1)}_i} \\ \left(f^{(1)}_j, F^{(1)}_j\right)_{1,M_1}, \left(f^{(1)}_{ji^{(1)}}, F^{(1)}_{ji^{(1)}}\right)_{M_1+1,Q^{(1)}_i}, \ldots, \left(f^{(r)}_j, F^{(r)}_j\right)_{1,M_r}, \end{matrix}$$

$$\left. \begin{matrix} , \ldots, (e^{(r)}_j, E^{(r)}_j)_{1,N_r}, \left(e^{(r)}_{ji^{(r)}}, E^{(r)}_{ji^{(r)}}\right)_{N_r+1,P^{(r)}_i} \\ \left(f^{(r)}_{ji^{(r)}}, F^{(r)}_{ji^{(r)}}\right)_{M_r+1,Q^{(r)}_i} \end{matrix} \right]$$

$$\times^{(\gamma)} I^{0,N':M'_1,N'_1;\ldots;M'_r,N'_r}_{P'_i,Q'_i,R';P'_{i(1)},Q'_{i(1)},R'_{(1)};\ldots;P'_{i(r)},Q'_{i(r)},R'_{(r)}} \left[\begin{matrix} u'_1(x/y)^h \\ \vdots \\ u'_r(x/y)^h \end{matrix} \middle| \begin{matrix} \left(e'_1, E'^{(1)}_1, \ldots, E'^{(r)}_1, t\right), \\ \left(f'_{ji}, F'^{(1)}_{ji}, \ldots, F'^{(r)}_{ji}\right)_{M+1,Q_l}, \end{matrix} \right.$$

$$\begin{matrix} \left(e'_j, E'^{(1)}_j, \ldots, E'^{(r)}_j\right)_{2,N}, \left(e'_{ji}, E'^{(1)}_{ji}, \ldots, E'^{(r)}_{ji}\right)_{N+1,P_i}, (e'^{(1)}_j, E'^{(1)}_j)_{1,N_1}, \\ \left(f'^{(1)}_j, F'^{(1)}_j\right)_{1,M_1}, \left(f'^{(1)}_{ji^{(1)}}, F'^{(1)}_{ji^{(1)}}\right)_{M_1+1,Q^{(1)}_i}, \ldots, \left(f'^{(r)}_j, F'^{(r)}_j\right)_{1,M_r}, \end{matrix}$$

$$\left. \begin{matrix} \left(e'^{(1)}_{ji^{(1)}}, E'^{(1)}_{ji^{(1)}}\right)_{N_1+1,P^{(1)}_i}, \ldots, (e'^{(r)}_j, E'^{(r)}_j)_{1,N_r}, \left(e'^{(r)}_{ji^{(r)}}, E'^{(r)}_{ji^{(r)}}\right)_{N_r+1,P^{(r)}_i} \\ \left(f'^{(r)}_{ji^{(r)}}, F'^{(r)}_{ji^{(r)}}\right)_{M_r+1,Q^{(r)}_i} \end{matrix} \right] f(y)dy$$

$$= w(x) \ (0 < x < \infty). \tag{12}$$

2.1. Solution of the Integral Equations Involving Product of Multivariable Incomplete H-functions

Lemma 2.1.

Let multivariable incomplete H-functions and the series representation satisfy the required conditions, then we can express:

$$W^{\eta-\ell}\left[y^{-\eta}\Gamma^{0,n:m_1,n_1,m_2,n_2;\ldots;m_r,n_r}_{p,q;p_1,q_1;\ldots;p_r,q_r}\left[\begin{matrix}z_1(x/y)^g\\ \vdots\\ z_r(x/y)^g\end{matrix}\middle|\begin{matrix}\left(a_1;\alpha_1^{(1)},\ldots,\alpha_1^{(r)},t\right),\left(a_j;\alpha_j^{(1)},\ldots,\alpha_j^{(r)}\right)_{2,p}:\\ \left(b_j;\beta_j^{(1)},\ldots,\beta_j^{(r)}\right)_{m+1,q_i}:\end{matrix}\right.\right.$$

$$\left.\begin{matrix}\left(c_j^{(1)},\zeta_j^{(1)}\right)_{1,p_1};\ldots;\left(c_j^{(r)},\zeta_j^{(r)}\right)_{1,p_r}\\ \left(d_j^{(1)},\delta_j^{(1)}\right)_{1,q_1};\ldots;\left(d_j^{(r)},\delta_j^{(r)}\right)_{1,q_r}\end{matrix}\right]\Gamma^{0,n':m'_1,n'_1,m'_2,n'_2;\ldots;m'_r,n'_r}_{p',q';p'_1,q'_1;\ldots;p'_r,q'_r}\left[\begin{matrix}z'_1(x/y)^h\\ \vdots\\ z'_r(x/y)^h\end{matrix}\right|$$

$$\left.\left.\begin{matrix}\left(a'_1;\alpha'^{(1)}_1,\ldots,\alpha'^{(r)}_1,t\right),\left(a'_j;\alpha'^{(1)}_j,\ldots,\alpha'^{(r)}_j\right)_{2,p}:\left(c'^{(1)}_j,\zeta'^{(1)}_j\right)_{1,p_1};\ldots;\left(c'^{(r)}_j,\zeta'^{(r)}_j\right)_{1,p_r}\\ \left(b'_j;\beta'^{(1)}_j,\ldots,\beta'^{(r)}_j\right)_{m+1,q_i}:\left(d'^{(1)}_j,\delta'^{(1)}_j\right)_{1,q_1};\ldots;\left(d'^{(r)}_j,\delta'^{(r)}_j\right)_{1,q_r}\end{matrix}\right]\right]$$

$$=y^{-\ell}\sum_{\mu_k=1}^{m_k}\sum_{\nu_k=1}^{\infty}\phi_1\phi_2\frac{(-1)^{\sum_{k=1}^r\nu_k}\prod_{k=1}^r(z_k)^{Z_k}}{\prod_{k=1}^r\delta^{(k)}_{\mu_k}\nu_k!}\left(\frac{x}{y}\right)^{g\sum_{k=1}^rZ_k}$$

$$\times\Gamma^{0,n'+1:m'_1,n'_1,m'_2,n'_2;\ldots;m'_r,n'_r}_{p'+1,q'+1;p'_1,q'_1;\ldots;p'_r,q'_r}\left[\begin{matrix}z'_1(x/y)^h\\ \vdots\\ z'_r(x/y)^h\end{matrix}\middle|\begin{matrix}\left(1-\ell-g\sum_{k=1}^rZ_k:h,\ldots,h\right),\\ \left(b'_j;\beta'^{(1)}_j,\ldots,\beta'^{(r)}_j\right)_{m+1,q_i}:\end{matrix}\right.$$

$$\left.\begin{matrix}\left(a'_1;\alpha'^{(1)}_1,\ldots,\alpha'^{(r)}_1,t\right),\left(a'_j;\alpha'^{(1)}_j,\ldots,\alpha'^{(r)}_j\right)_{2,p}:\left(c'^{(1)}_j,\zeta'^{(1)}_j\right)_{1,p_1};\ldots;\left(c'^{(r)}_j,\zeta'^{(r)}_j\right)_{1,p_r}\\ \left(1-\eta-g\sum_{k=1}^rZ_k:h,\ldots,h\right),\left(d'^{(1)}_j,\delta'^{(1)}_j\right)_{1,q_1};\ldots;\left(d'^{(r)}_j,\delta'^{(r)}_j\right)_{1,q_r}\end{matrix}\right].\quad(13)$$

Proof:

To prove Lemma 2.1, our first step involves utilizing the definition of the Weyl fractional integral. We proceed by expressing one of the multivariable incomplete H-functions in series form, while the other is expressed in the Mellin-Barnes type. Subsequently, we justify the interchange of summations and integration based on the specified conditions. Further, we evaluate the t-integral and then reinterpret the resultant Mellin-Barnes contour integral in terms of the upper form of the multivariable incomplete H-function. By following this approach, we easily arrive at the desired result. ■

Theorem 2.1.

By employing the sufficient conditions outlined in Lemma 2.1, we can present a theorem in the following manner:

$$\int_0^\infty y^{-\ell} \sum_{\mu_k=1}^{m_k} \sum_{\nu_k=1}^{\infty} \phi_1\phi_2 \frac{(-1)^{\sum_{k=1}^r \nu_k} \prod_{k=1}^r (z_k)^{Z_k}}{\prod_{k=1}^r \delta_{\mu_k}^{(k)} \nu_k!} \left(\frac{x}{y}\right)^{g\sum_{k=1}^r Z_k}$$

$$\times \Gamma^{0,n'+1:m'_1,n'_1,m'_2,n'_2;\ldots;m'_r,n'_r}_{p'+1,q'+1;p'_1,q'_1;\ldots;p'_r,q'_r} \left[\begin{matrix} z'_1(x/y)^h \\ \vdots \\ z'_r(x/y)^h \end{matrix} \middle| \begin{matrix} (1-\ell-g\sum_{k=1}^r Z_k : h,\ldots,h), \\ \left(b'_j;\beta'^{(1)}_j,\ldots,\beta'^{(r)}_j\right)_{m+1,q_i} : \end{matrix}\right.$$

$$\left.\begin{matrix} \left(a'_1;\alpha'^{(1)}_1,\ldots,\alpha'^{(r)}_1,t\right), \left(a'_j;\alpha'^{(1)}_j,\ldots,\alpha'^{(r)}_j\right)_{2,p} : \left(c'^{(1)}_j,\zeta'^{(1)}_j\right)_{1,p_1};\ldots;\left(c'^{(r)}_j,\zeta'^{(r)}_j\right)_{1,p_r} \\ (1-\eta-g\sum_{k=1}^r Z_k : h,\ldots,h), \left(d'^{(1)}_j,\delta'^{(1)}_j\right)_{1,q_1};\ldots;\left(d'^{(r)}_j,\delta'^{(r)}_j\right)_{1,q_r} \end{matrix}\right]$$

$$\times f(y)dy = \int_0^\infty y^{-\ell} \Gamma^{0,n':m'_1,n'_1,m'_2,n'_2;\ldots;m'_r,n'_r}_{p',q';p'_1,q'_1;\ldots;p'_r,q'_r} \left[\begin{matrix} z'_1(x/y)^h \\ \vdots \\ z'_r(x/y)^h \end{matrix} \middle| \begin{matrix} \left(a'_1;\alpha'^{(1)}_1,\ldots,\alpha'^{(r)}_1,t\right), \\ \left(b'_j;\beta'^{(1)}_j,\ldots,\beta'^{(r)}_j\right)_{m+1,q_i} : \end{matrix}\right]$$

$$\left.\begin{matrix} \left(a'_1;\alpha'^{(1)}_1,\ldots,\alpha'^{(r)}_1,t\right), \left(a'_j;\alpha'^{(1)}_j,\ldots,\alpha'^{(r)}_j\right)_{2,p} : \left(c'^{(1)}_j,\zeta'^{(1)}_j\right)_{1,p_1};\ldots;\left(c'^{(r)}_j,\zeta'^{(r)}_j\right)_{1,p_r} \\ \left(b'_j;\beta'^{(1)}_j,\ldots,\beta'^{(r)}_j\right)_{m+1,q_i} : \left(d'^{(1)}_j,\delta'^{(1)}_j\right)_{1,q_1};\ldots;\left(d'^{(r)}_j,\delta'^{(r)}_j\right)_{1,q_r} \end{matrix}\right]$$

$$\times \Gamma^{0,n:m_1,n_1,m_2,n_2;\ldots;m_r,n_r}_{p,q;p_1,q_1;\ldots;p_r,q_r} \left[\begin{matrix} z_1(x/y)^g \\ \vdots \\ z_r(x/y)^g \end{matrix} \middle| \begin{matrix} \left(a_1;\alpha^{(1)}_1,\ldots,\alpha^{(r)}_1,t\right), \left(a_j;\alpha^{(1)}_j,\ldots,\alpha^{(r)}_j\right)_{2,p} : \\ \left(b_j;\beta^{(1)}_j,\ldots,\beta^{(r)}_j\right)_{m+1,q_i} : \end{matrix}\right.$$

$$\left.\begin{matrix} \left(c^{(1)}_j,\zeta^{(1)}_j\right)_{1,p_1};\ldots;\left(c^{(r)}_j,\zeta^{(r)}_j\right)_{1,p_r} \\ \left(d^{(1)}_j,\delta^{(1)}_j\right)_{1,q_1};\ldots;\left(d^{(r)}_j,\delta^{(r)}_j\right)_{1,q_r} \end{matrix}\right] D^{\eta-\ell}[f(y)]dy, \tag{14}$$

in addition, under the conditions, $f \in \int$ defined in Chaurasia and Kumar (2012) and $x > 0$.

Proof:

Assume $\mathcal{K}$ represents the initial component of the equation (14). By applying Lemma 2.1 defined in (13) and the Weyl fractional integration, we obtain the following result:

$$\mathcal{K} = \int_0^\infty \frac{f(y)}{\Gamma(\eta-\ell)} \left\{ \int_0^\infty (\rho-y)^{\eta-\ell-1} \rho^{-\eta} \Gamma^{0,n':m'_1,n'_1,m'_2,n'_2;\ldots;m'_r,n'_r}_{p',q';p'_1,q'_1;\ldots;p'_r,q'_r} [z'_r(x/\rho)^h] \right.$$
$$\left. \Gamma^{0,n:m_1,n_1,m_2,n_2;\ldots;m_r,n_r}_{p,q;p_1,q_1;\ldots;p_r,q_r} [z_r(x/\rho)^g] d\rho \right\} dy$$

$$= \int_0^\infty \rho^{-\eta} \Gamma^{0,n':m'_1,n'_1,m'_2,n'_2;\ldots;m'_r,n'_r}_{p',q';p'_1,q'_1;\ldots;p'_r,q'_r} [z'_r(x/\rho)^h] \Gamma^{0,n:m_1,n_1,m_2,n_2;\ldots;m_r,n_r}_{p,q;p_1,q_1;\ldots;p_r,q_r} [z_r(x/\rho)^g] \times$$
$$\left\{ \int_0^\infty \frac{f(y)(\rho-y)^{\eta-\ell-1}}{\Gamma(\eta-\ell)} dy \right\} d\rho.$$

Assuming the permissibility of changing the order of integration, similar to the proof of the Lemma 2.1, we can now employ the Riemann-Liouville fractional integral to derive the following expression,

$$\mathcal{K}=\int_0^\infty \rho^{-\eta}\Gamma^{0,n':m'_1,n'_1,m'_2,n'_2;\ldots;m'_r,n'_r}_{p',q';p'_1,q'_1;\ldots;p'_r,q'_r}[z'_r(x/\rho)^h]\Gamma^{0,n:m_1,n_1,m_2,n_2;\ldots;m_r,n_r}_{p,q;p_1,q_1;\ldots;p_r,q_r}[z_r(x/\rho)^g]_0D_\infty^{\eta-\ell}[f(\rho)]d\rho$$

This expression precisely corresponds to the right-hand side of Equation (14). Thus, this concludes the proof of Theorem 2.1. ■

2.2. Solution of the Integral Equations Involving Product of Multivariable Incomplete I-functions

Lemma 2.2.

Provided that the necessary conditions are met by the multivariable incomplete I-functions and the series representation, we can present the following expression:

$$W^{\eta-\ell}\Bigg[y^{-\eta(\Gamma)}I^{0,N:M_1,N_1;\ldots;M_r,N_r}_{P_i,Q_i,R;P_{i^{(1)}},Q_{i^{(1)}},R_{(1)};\ldots;P_{i^{(r)}},Q_{i^{(r)}},R_{(r)}}\left[\begin{matrix}u_1(x/y)^g\\ \vdots\\ u_r(x/y)^g\end{matrix}\,\middle|\,\begin{matrix}\left(e_1,E_1^{(1)},\ldots,E_1^{(r)},t\right),\\ \left(f_{ji},F_{ji}^{(1)},\ldots,F_{ji}^{(r)}\right)_{M+1,Q_l},\end{matrix}\right.$$

$$\begin{matrix}\left(e_j,E_j^{(1)},\ldots,E_j^{(r)}\right)_{2,N},\left(e_{ji},E_{ji}^{(1)},E_{ji}^{(2)}\right)_{N+1,P_i},(e_j^{(1)},E_j^{(1)})_{1,N_1},\left(e_{ji^{(1)}}^{(1)},E_{ji^{(1)}}^{(1)}\right)_{N_1+1,P_i^{(1)}},\\ \left(f_j^{(1)},F_j^{(1)}\right)_{1,M_1},\left(f_{ji^{(1)}}^{(1)},F_{ji^{(1)}}^{(1)}\right)_{M_1+1,Q_i^{(1)}},\ldots,\left(f_j^{(r)},F_j^{(r)}\right)_{1,M_r},\end{matrix}$$

$$\left.\begin{matrix}\ldots,(e_j^{(r)},E_j^{(r)})_{1,N_r},\left(e_{ji^{(r)}}^{(r)},E_{ji^{(r)}}^{(r)}\right)_{N_r+1,P_i^{(r)}}\\ \left(f_{ji^{(r)}}^{(r)},F_{ji^{(r)}}^{(r)}\right)_{M_r+1,Q_i^{(r)}}\end{matrix}\right]{}^{(\Gamma)}I^{0,N':M'_1,N'_1;\ldots;M'_r,N'_r}_{P'_i,Q'_i,R';P'_{i^{(1)}},Q'_{i^{(1)}},R'_{(1)};\ldots;P'_{i^{(r)}},Q'_{i^{(r)}},R'_{(r)}}$$

$$\left[\begin{matrix}u'_1(x/y)^h\\ \vdots\\ u'_r(x/y)^h\end{matrix}\,\middle|\,\begin{matrix}\left(e'_1,E'^{(1)}_1,\ldots,E'^{(r)}_1,t\right),\left(e'_j,E'^{(1)}_j,\ldots,E'^{(r)}_j\right)_{2,N},\left(e'_{ji},E'^{(1)}_{ji},\ldots,E'^{(r)}_{ji}\right)_{N+1,P_i},\\ \left(f'_{ji},F'^{(1)}_{ji},\ldots,F'^{(r)}_{ji}\right)_{M+1,Q_l},\left(f'^{(1)}_j,F'^{(1)}_j\right)_{1,M_1},\end{matrix}\right.$$

$$\left.\begin{matrix}(e'^{(1)}_j,E'^{(1)}_j)_{1,N_1},\left(e'^{(1)}_{ji^{(1)}},E'^{(1)}_{ji^{(1)}}\right)_{N_1+1,P_i^{(1)}},\ldots,(e'^{(r)}_j,E'^{(r)}_j)_{1,N_r},\left(e'^{(r)}_{ji^{(r)}},E'^{(r)}_{ji^{(r)}}\right)_{N_r+1,P_i^{(r)}}\\ \left(f'^{(1)}_{ji^{(1)}},F'^{(1)}_{ji^{(1)}}\right)_{M_1+1,Q_i^{(1)}},\ldots,\left(f'^{(r)}_j,F'^{(r)}_j\right)_{1,M_r},\left(f'^{(r)}_{ji^{(r)}},F'^{(r)}_{ji^{(r)}}\right)_{M_r+1,Q_i^{(r)}}\end{matrix}\right]\Bigg]$$

$$=y^{-\ell}\sum_{\lambda_k=1}^{M_k}\sum_{\Lambda_k=1}^{\infty}\varphi_1\varphi_2\frac{(-1)^{\sum_{k=1}^r\Lambda_k}\prod_{k=1}^r(u_k)^{U_k}}{\prod_{k=1}^r\mathfrak{d}_{\lambda_k}^{(k)}\Lambda_k!}\left(\frac{x}{y}\right)^{h\sum_{k=1}^rU_k}$$

$${}^{(\Gamma)}I^{0,N'+1:M'_1,N'_1;\ldots;M'_r,N'_r}_{P'_i+1,Q'_i+1,R';P'_{i^{(1)}},Q'_{i^{(1)}},R'_{(1)};\ldots;P'_{i^{(r)}},Q'_{i^{(r)}},R'_{(r)}}\left[\begin{matrix}u'_1(x/y)^h\\ \vdots\\ u'_r(x/y)^h\end{matrix}\,\middle|\,\begin{matrix}(1-\ell-g\sum_{k=1}^rU_k:h,\ldots,h),\\ \left(f'_{ji},F'^{(1)}_{ji},\ldots,F'^{(r)}_{ji}\right)_{M+1,Q_l},\end{matrix}\right.$$

$$\left(e'_1, E'^{(1)}_1, \ldots, E'^{(r)}_1, t\right), \left(e'_j, E'^{(1)}_j, \ldots, E'^{(r)}_j\right)_{2,N}, \left(e'_{ji}, E'^{(1)}_{ji}, \ldots, E'^{(r)}_{ji}\right)_{N+1,P_i},$$
$$(1-\eta-g\textstyle\sum_{k=1}^r U_k : h, \ldots, h), \left(f'_{ji}, F'^{(1)}_{ji}, \ldots, F'^{(r)}_{ji}\right)_{M+1,Q_l}, \left(f'^{(1)}_j, F'^{(1)}_j\right)_{1,M_1},$$

$$\left.\begin{matrix}(e'^{(1)}_j, E'^{(1)}_j)_{1,N_1}, \left(e'^{(1)}_{ji^{(1)}}, E'^{(1)}_{ji^{(1)}}\right)_{N_1+1,P_i^{(1)}}, \ldots, (e'^{(r)}_j, E'^{(r)}_j)_{1,N_r}, \left(e'^{(r)}_{ji^{(r)}}, E'^{(r)}_{ji^{(r)}}\right)_{N_r+1,P_i^{(r)}} \\ \left(f'^{(1)}_{ji^{(1)}}, F'^{(1)}_{ji^{(1)}}\right)_{M_1+1,Q_i^{(1)}}, \ldots, \left(f'^{(r)}_j, F'^{(r)}_j\right)_{1,M_r}, \left(f'^{(r)}_{ji^{(r)}}, F'^{(r)}_{ji^{(r)}}\right)_{M_r+1,Q_i^{(r)}}\end{matrix}\right]. \quad (15)$$

Proof:

The proof of Lemma 2.2 follows the same steps as Lemma 2.1. ∎

Theorem 2.2.

By utilizing the satisfactory conditions of Lemma 2.2, we can present a theorem as follows:

$$= y^{-\ell} \sum_{\lambda_k=1}^{M_k} \sum_{\Lambda_k=1}^{\infty} \varphi_1 \varphi_2 \frac{(-1)^{\sum_{k=1}^r \Lambda_k} \prod_{k=1}^r (u_k)^{U_k}}{\prod_{k=1}^r \mathfrak{d}^{(k)}_{\lambda_k} \Lambda_k!} \left(\frac{x}{y}\right)^{h\sum_{k=1}^r U_k}$$

$$\times^{(\Gamma)} I^{0,N'+1:M'_1,N'_1;\ldots;M'_r,N'_r}_{P'_i+1,Q'_i+1,R';P'_{i^{(1)}},Q'_{i^{(1)}},R'_{(1)};\ldots;P'_{i^{(r)}},Q'_{i^{(r)}},R'_{(r)}} \left[\begin{matrix} u'_1 (x/y)^h \\ \vdots \\ u'_r (x/y)^h \end{matrix} \,\middle|\, \begin{matrix}(1-\ell-g\sum_{k=1}^r U_k : h, \ldots, h), \\ \left(f'_{ji}, F'^{(1)}_{ji}, \ldots, F'^{(r)}_{ji}\right)_{M+1,Q_l},\end{matrix}\right.$$

$$\left(e'_1, E'^{(1)}_1, \ldots, E'^{(r)}_1, t\right), \left(e'_j, E'^{(1)}_j, \ldots, E'^{(r)}_j\right)_{2,N}, \left(e'_{ji}, E'^{(1)}_{ji}, \ldots, E'^{(r)}_{ji}\right)_{N+1,P_i},$$
$$(1-\eta-g\textstyle\sum_{k=1}^r U_k : h, \ldots, h), \left(f'_{ji}, F'^{(1)}_{ji}, \ldots, F'^{(r)}_{ji}\right)_{M+1,Q_l}, \left(f'^{(1)}_j, F'^{(1)}_j\right)_{1,M_1},$$

$$\left.\begin{matrix}(e'^{(1)}_j, E'^{(1)}_j)_{1,N_1}, \left(e'^{(1)}_{ji^{(1)}}, E'^{(1)}_{ji^{(1)}}\right)_{N_1+1,P_i^{(1)}}, \ldots, (e'^{(r)}_j, E'^{(r)}_j)_{1,N_r}, \left(e'^{(r)}_{ji^{(r)}}, E'^{(r)}_{ji^{(r)}}\right)_{N_r+1,P_i^{(r)}} \\ \left(f'^{(1)}_{ji^{(1)}}, F'^{(1)}_{ji^{(1)}}\right)_{M_1+1,Q_i^{(1)}}, \ldots, \left(f'^{(r)}_j, F'^{(r)}_j\right)_{1,M_r}, \left(f'^{(r)}_{ji^{(r)}}, F'^{(r)}_{ji^{(r)}}\right)_{M_r+1,Q_i^{(r)}}\end{matrix}\right]$$

$$f(y)dy = \int_0^\infty y^{-\ell (\Gamma)} I^{0,N':M'_1,N'_1;\ldots;M'_r,N'_r}_{P'_i,Q'_i,R';P'_{i^{(1)}},Q'_{i^{(1)}},R'_{(1)};\ldots;P'_{i^{(r)}},Q'_{i^{(r)}},R'_{(r)}} \left[\begin{matrix} u'_1 (x/y)^h \\ \vdots \\ u'_r (x/y)^h \end{matrix} \,\middle|\right.$$

$$\left(e'_1, E'^{(1)}_1, \ldots, E'^{(r)}_1, t\right), \left(e'_j, E'^{(1)}_j, \ldots, E'^{(r)}_j\right)_{2,N}, \left(e'_{ji}, E'^{(1)}_{ji}, \ldots, E'^{(r)}_{ji}\right)_{N+1,P_i},$$
$$\left(f'_{ji}, F'^{(1)}_{ji}, \ldots, F'^{(r)}_{ji}\right)_{M+1,Q_l}, \left(f'^{(1)}_j, F'^{(1)}_j\right)_{1,M_1},$$

$$\left.\begin{matrix}(e'^{(1)}_j, E'^{(1)}_j)_{1,N_1}, \left(e'^{(1)}_{ji^{(1)}}, E'^{(1)}_{ji^{(1)}}\right)_{N_1+1,P_i^{(1)}}, \ldots, (e'^{(r)}_j, E'^{(r)}_j)_{1,N_r}, \left(e'^{(r)}_{ji^{(r)}}, E'^{(r)}_{ji^{(r)}}\right)_{N_r+1,P_i^{(r)}} \\ \left(f'^{(1)}_{ji^{(1)}}, F'^{(1)}_{ji^{(1)}}\right)_{M_1+1,Q_i^{(1)}}, \ldots, \left(f'^{(r)}_j, F'^{(r)}_j\right)_{1,M_r}, \left(f'^{(r)}_{ji^{(r)}}, F'^{(r)}_{ji^{(r)}}\right)_{M_r+1,Q_i^{(r)}}\end{matrix}\right]$$

$$\times {}^{(\Gamma)}I^{0,N:M_1,N_1;\ldots;M_r,N_r}_{P_i,Q_i,R;P_{i^{(1)}},Q_{i^{(1)}},R_{(1)};\ldots;P_{i^{(r)}},Q_{i^{(r)}},R_{(r)}}\left[\begin{matrix} u_1(x/y)^g \\ \vdots \\ u_r(x/y)^g \end{matrix}\,\middle|\, \begin{matrix} \left(e_1, E_1^{(1)}, \ldots, E_1^{(r)}, t\right), \\ \left(f_{ji}, F_{ji}^{(1)}, \ldots, F_{ji}^{(r)}\right)_{M+1,Q_l}, \end{matrix}\right.$$

$$\left(e_j, E_j^{(1)}, \ldots, E_j^{(r)}\right)_{2,N}, \left(e_{ji}, E_{ji}^{(1)}, E_{ji}^{(2)}\right)_{N+1,P_i}, (e_j^{(1)}, E_j^{(1)})_{1,N_1}, \left(e_{ji^{(1)}}^{(1)}, E_{ji^{(1)}}^{(1)}\right)_{N_1+1,P_i^{(1)}},$$
$$\left(f_j^{(1)}, F_j^{(1)}\right)_{1,M_1}, \left(f_{ji^{(1)}}^{(1)}, F_{ji^{(1)}}^{(1)}\right)_{M_1+1,Q_i^{(1)}}, \ldots, \left(f_j^{(r)}, F_j^{(r)}\right)_{1,M_r},$$

$$\left.\begin{matrix} \ldots, (e_j^{(r)}, E_j^{(r)})_{1,N_r}, \left(e_{ji^{(r)}}^{(r)}, E_{ji^{(r)}}^{(r)}\right)_{N_r+1,P_i^{(r)}} \\ \left(f_{ji^{(r)}}^{(r)}, F_{ji^{(r)}}^{(r)}\right)_{M_r+1,Q_i^{(r)}} \end{matrix}\right], \tag{16}$$

provided, $f \in \int$ defined in Chaurasia and Kumar (2012) and $x > 0$.

Proof:

The proof of Theorem 2.2 relies on the utilization of Lemma 2.2 and the Weyl fractional integration following a similar approach as presented in the proof of Theorem 2.1. ■

3. Particular Cases

(1) By setting $g = t = 0$, the outcomes in Equations (13) and (14) can be simplified to the established findings acquired by Chaurasia and Patni (2001).
(2) When we set $g = t = R = 0$ and $r = 1$, the outcomes of Equations (15) and (16) can be simplified to the results obtained by Chaurasia and Patni (2001), with a slight modification.
(3) By specializing the parameters in Equations (13), (14), (15) and (16), we reach the results obtained by Srivastava and Raina (1992).
(4) By putting suitable parameters in Equations (13), (14), (15) and (16), the results can easily convert the results obtained by Chaurasia and Kumar (2012) and Chaurasia and Singh (2014).

4. Conclusion

The significance of our findings lies in their broad applicability across diverse domains. Through the utilization of series representations and products of multivariable incomplete H-functions and multivariable incomplete I-functions, along with appropriate adjustments of parameters and variables, we can identify the multitude of integral equations and solutions encompassing various useful functions. Moreover, by manipulating the parameters within these results, we can derive an excess of integral equations and solutions that cater to different special cases. As a consequence, our research outputs contribute a vast collection of results involving quantum mechanics, electromagnetism, fluid dynamics, and aerospace engineering like beam deflection heat transfer, wave propagation and stochastic systems. These applications demonstrate the importance and versatility of integral equations of Fredholm type in solving real-world problems.

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