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System of Variational Inclusions Involving Cayley Operator And Yosida Approximation Operator With XOR Operations

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Abstract

In this paper, we consider and study a new class of system of variational inclusions called a system of variational inclusions involving Cayley operator and Yosida approximation operator with XOR operation. We have shown that our problem is equivalent to a fixed point equation. Based on fixed point formulation, an iterative algorithm is designed to obtain existence and convergence result for our problem.

Keywords: Cayley operator; Yosida approximation operator; XOR operation; Resolvent operator; Ordered Hilbert space

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1. Introduction

The variational inclusion problem is one of the most important and interesting generalizations of the variational inequality problem. Rockafellar (1976) developed a proximal point algorithm for

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solving a classical variational inclusion problem. Since then, several algorithms have came into picture; see, for example, Agarwal and Verma (2009), Verma (2009a), Verma (2009b), Verma (2009c), Verma (2007), Verma (2006), Lan (2009), Li (2012), and references therein.

Due to the fact that projection methods cannot be used to solve variational inclusion problems, the resolvent operator methods came into the picture to solve them efficiently. It is also known that the monotone operators in abstract spaces can be regularized into single-valued Lipschitzian monotone operators through a process known as Yosida approximation (see Attouch (1984), Attouch et al. (1991), Barbu (1976) and Schaefer (1974)).

The XOR operation $\oplus$ is a binary operation that is commutative as well as associative. The XOR operation depicts interesting facts and observations and forms various real-time applications, that is, data encryption, error detection in digital communication, etc.

Li (2008) introduced and studied nonlinear ordered variational inequalities. Later, many problems related to ordered variational inequalities were studied (see Li (2008); Li (2009); Li (2011a); Li (2012); Li (2011b); Li et al. (2013a); Li et al. (2013b); Li et al. (2014a); Li et al. (2014b)). In the recent past, Ahmad et al. (2018) proposed a new approach by introducing a novel mapping called $H(.,.)$-ordered-compression mapping. They also defined a resolvent operator and explored its properties using the XOR operation. Additionally, they developed an algorithm specifically designed for solving XOR variational inclusion problems. Furthermore, the study of ordered variational inclusions with XOR operations has gained significant attention in various research domains. Very recent examples of this direction of research can be found in Ahmad et al. (2020) and Iqbal et al. (2022).

Motivated by the above discussion, in this paper we introduce a new system of variational inclusions called the system of variational inclusions involving the Cayley operator and the Yosida approximation operator with XOR operation. To solve this problem, we propose an iterative algorithm based on the fixed-point formulation. Through this algorithm, we conduct a comprehensive convergence analysis for the aforementioned problem.

2. Preliminaries

Let $\mathcal{H}$ be a real ordered Hilbert space with the usual norm $\|.\|$ and the inner product $\langle .,. \rangle$. Let $\aleph$ be a cone in $\mathcal{H}$. The partial ordering induced by cone $\aleph$ is denoted by "$\leq$". We denote by $C(\mathcal{H})$, the collection of all compact subsets of $\mathcal{H}$ and by $2^{\mathcal{H}}$, the collection of all nonempty subsets of $\mathcal{H}$. The Hausdorff metric on $C(\mathcal{H})$ is denoted by $D(.,.)$. For any two arbitrary elements x and y of $\mathcal{H}$, we denote $lub\,\{x,y\}$ for the set $\{x,y\}$ by $x \vee y$. Suppose that lub exists for the set $\{x,y\}$. Then, the operation denoted by $\oplus$, called the XOR operation, is defined by $x \oplus y = (x-y) \vee (y-x)$. The elements x and y are said to be comparable to each other if and only if $x \leq y$ or $y \leq x$ and we denote it by $x \propto y$.

From Li (2008), Li (2009), Li (2011a), Li (2012), Li (2011b), Li et al. (2013a), Li et al. (2013b), Li et al. (2014a), and Li et al. (2014b), it can be found that $\mathcal{H}$ is an ordered Hilbert space equipped with partial ordering "$\leq$" induced by the cone $\aleph$. For any elements $x, y, v, u \in \mathcal{H}$, the XOR operation has the following properties:

(i) $x \oplus y = y \oplus x,\ x \oplus x = 0$,
(ii) let τ be a real number, then $(\tau x) \oplus (\tau y) = |\tau| (x \oplus y)$,
(iii) if $x \propto 0$, then $-x \oplus 0 \leq x \leq x \oplus 0$,
(iv) $0 \leq x \oplus y$, if $x \propto y$,
(v) if $x \propto y$, then $x \oplus y = 0$ if and only if $x = y$,
(vi) $(x + y) \oplus (v + u) \geq (x \oplus v) - (y \oplus u) \vee (x \oplus u) - (y \oplus v)$,
(vii) $\|0 \oplus 0\| = \|0\| = 0$,
(viii) $\|x \oplus y\| \leq \|x - y\|$,
(ix) if $x \propto y$, then $\|x \oplus y\| = \|x - y\|$.

Moreover, given a mapping $A : \mathcal{H} \to \mathcal{H}$ and a multivalued mapping mapping $M : \mathcal{H} \to 2^{\mathcal{H}}$, we define several specific types of mappings. A is called ξ-order non-extended mapping, if there exists a constant $\xi > 0$ such that for all $x, y \in \mathcal{H}$, the inequality $\xi(x \oplus y) \leq A(x) \oplus A(y)$ holds, A is a comparison mapping if $x \propto y$, implies $A(x) \propto A(y),\ x \propto A(x)$ and $y \propto A(y)$, for all $x, y \in \mathcal{H}$, A is a strongly comparison mapping, if A is comparison mapping and $A(x) \propto A(y)$ if and only if $x \propto y$, for all $x, y \in \mathcal{H}$. M is a weak-comparison mapping if for $u_x \in M(x),\ x \propto u_x$, and $x \propto y$, there exists $u_y \in M(y)$ such that $u_x \propto u_y$, for all $x, y \in \mathcal{H}$, M is a α_A-weak-non-ordinary difference mapping with respect to A if it is a weak comparison and for each $x, y \in \mathcal{H}$, there exists $\alpha_A > 0$ and $u_x \in M(A(x))$ and $u_y \in M(A(y))$ such that $(u_x \oplus u_y) \oplus \alpha_A(A(x) \oplus A(y)) = 0$, M is called ρ-order different weak-comparison mapping with respect to A, if there exists $\rho > 0$ and for all $x, y \in \mathcal{H}$, there exists $u_x \in M(A(x)),\ u_y \in M(A(y))$ such that $\rho(u_x - u_y) \propto x - y$,. Finally, a weak-comparison mapping M is called (α_A, ρ)-weak ANODD if it is an α_A-weak-non-ordinary difference mapping and ρ-order different weak-comparison mapping associated with A, and it satisfies $[A + \rho M](\mathcal{H}) = \mathcal{H}$, ensuring the image covers the entire space $\mathcal{H}$. Let A be ξ-ordered non-extended mapping and M is α_A-non-ordinary difference mapping with respect to A. The generalized resolvent operator $R^M_{A,\rho} : \mathcal{H} \to \mathcal{H}$ associated with A and M is defined as $R^M_{A,\rho}(x) = [A + \rho M]^{-1}(x),$ for all $x \in \mathcal{H},\ \rho > 0$, the generalized Cayley operator $C^M_{A,\rho} : \mathcal{H} \to \mathcal{H}$ is defined as $C^M_{A,\rho}(x) = \left[2R^M_{A,\rho} - A\right](x),$ for all $x \in \mathcal{H}$ and the generalized Yosida approximation operator $Y^M_{A,\rho} : \mathcal{H} \to \mathcal{H}$ is defined as $Y^M_{A,\rho}(x) = \frac{1}{\rho}\left[A - R^M_{A,\rho}\right](x),$ for all $x \in \mathcal{H}$.

3. Formulation of the Problem and Convergence Analysis

Let $\mathcal{H}$ be a real ordered Hilbert space. Let $G, F : \mathcal{H} \to C(\mathcal{H})$ be the multi-valued mappings and $f, g, q, A, B : \mathcal{H} \to \mathcal{H},\ S, T : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ be the single-valued mappings. Let $M, N : \mathcal{H} \to 2^{\mathcal{H}}$ be the multi-valued mappings. Let $C^M_{A,\rho}$ be the generalized Cayley operator and $Y^M_{A,\rho}$ be the generalized Yosida approximation operator. We consider the following system of variational inclusions involving the Cayley operator and the Yosida approximation operator with the XOR operation.

Find $x, y \in \mathcal{H}, u \in G(x), v \in F(y)$ such that

$$\begin{aligned} &0 \in S\left(Y_{A,\rho}^M(x) \oplus C_{A,\rho}^M(x), v\right) + M\left(f(x)\right), \\ &0 \in T\left(u, y - q(y)\right) \oplus N\left(g(y)\right). \end{aligned} \tag{1}$$

Lemma 3.1.

The elements $x, y \in \mathcal{H}, u \in G(x), v \in F(y)$ are the solution of a system of variational inclusions involving Cayley operator and Yosida approximation operator with XOR operation (1) if and only if the following equations are satisfied:

$$f(x) = R_{A,\rho}^M\left[A\left(f(x)\right) - \rho S\left(Y_{A,\rho}^M(x) \oplus C_{A,\rho}^M(x), v\right)\right], \tag{2}$$

$$g(y) = R_{B,\gamma}^N\left[B\left(g(y)\right) \oplus \gamma T\left(u, y - q(y)\right)\right], \tag{3}$$

where $\rho > 0$ and $\gamma > 0$ are constants.

Proof:

Suppose $x, y \in \mathcal{H}, u \in G(x), v \in F(y)$ satisfy (1). Then,

$$\begin{aligned} & f(x) = R_{A,\rho}^M\left[A\left(f(x)\right) - \rho S\left(Y_{A,\rho}^M(x) \oplus C_{A,\rho}^M(x), v\right)\right], \\ & g(y) = R_{B,\gamma}^N\left[B\left(g(y)\right) \oplus \gamma T\left(u, y - q(y)\right)\right] \\ \Leftrightarrow\quad & f(x) = (A + \rho M)^{-1}\left[A\left(f(x)\right) - \rho S\left(Y_{A,\rho}^M(x) \oplus C_{A,\rho}^M(x), v\right)\right], \\ & g(y) = (B + \gamma N)^{-1}\left[B\left(g(y)\right) \oplus \gamma T\left(u, y - q(y)\right)\right] \\ \Leftrightarrow\quad & A\left(f(x)\right) + \rho M\left(f(x)\right) = A\left(f(x)\right) - \rho S\left(Y_{A,\rho}^M(x) \oplus C_{A,\rho}^M(x), v\right), \\ & B\left(g(y)\right) + \gamma N\left(g(y)\right) = B\left(g(y)\right) \oplus \gamma T\left(u, y - q(y)\right) \\ \Leftrightarrow\quad & 0 \in S\left(Y_{A,\rho}^M(x) \oplus C_{A,\rho}^M(x), v\right) + M\left(f(x)\right), \\ & 0 \in T\left(u, y - q(y)\right) \oplus N\left(g(y)\right). \end{aligned}$$

■

Applying Lemma 3.1, we suggest the following iterative scheme for solving (1).

Algorithm 3.1.

For any given $x_0, y_0 \in \mathcal{H}$, choose $u_0 \in G(x_0), v_0 \in F(y_0)$, and compute the sequences $\{x_n\}, \{y_n\}, \{u_n\}$ and $\{v_n\}$ by the following scheme:

$$x_{n+1} = x_n - f(x_n) + R_{A,\rho}^M\left[A\left(f(x_n)\right) - \rho S\left(Y_{A,\rho}^M(x_n) \oplus C_{A,\rho}^M(x_n), v_n\right)\right], \tag{4}$$

$$y_{n+1} = y_n - g(y_n) + R_{B,\gamma}^N\left[B\left(g(y_n)\right) \oplus \gamma T\left(u_n, y_n - q(y_n)\right)\right]. \tag{5}$$

Let $u_{n+1} \in G(x_{n+1})$ and $v_{n+1} \in F(y_{n+1})$ such that

$$\|u_n - u_{n+1}\| \leq D\left(G(x_n), G(x_{n+1})\right), \tag{6}$$

$$\|v_n - v_{n+1}\| \leq D\left(F(y_n), F(y_{n+1})\right), \tag{7}$$

where $\rho > 0$ and $\gamma > 0$ are constants, and $n = 0, 1, 2, \cdots$.

Theorem 3.1.

Let $\mathcal{H}$ be an ordered Hilbert space. Let the mappings $f, g, q, A, B : \mathcal{H} \to \mathcal{H}$, $S, T : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$, $G, F : \mathcal{H} \to C(\mathcal{H})$ and $M, N : \mathcal{H} \to 2^{\mathcal{H}}$ fulfill the following conditions:

(i) The mappings f, g are strongly monotone mappings with constants δ_f and δ_g, respectively and Lipschitz continuous with constants λ_f and λ_g, respectively.
(ii) The mappings q, A and B are Lipschitz continuous with constants λ_q, λ_A and λ_B, respectively.
(iii) Let A be ξ_1-ordered non-extended mapping and B be ξ_2-ordered non-extended mapping.
(iv) The mappings F and G are D-Lipschitz continuous with constants λ_{D_F} and λ_{D_G}, respectively.
(v) The mapping S is Lipschitz continuous in both arguments with constants λ_{S1} and λ_{S2}, respectively.
(vi) The mapping T is Lipschitz continuous in both arguments with Lipschitz constants λ_{T1} and λ_{T2}, respectively.
(vii) Let M be (α_A, ρ)-weak ANODD mapping and N is (α_B, γ)-weak BNODD mapping.

Let $R^M_{A,\rho}$ and $R^N_{B,\gamma}$ satisfy

$$R^M_{A,\rho}(x) \oplus R^M_{A,\rho}(y) \leq \frac{1}{\xi(\alpha_A \rho - 1)}(x \oplus y), \text{ for all } x, y \in \mathcal{H}, \tag{8}$$

the generalized Cayley operator $C^M_{A,\rho}$ is λ_C-Lipschitz continuous, and the generalized Yosida approximation operator $Y^M_{A,\rho}$ is λ_Y-Lipschitz continuous. Suppose $x_n \propto x_{n+1}$ and $y_n \propto y_{n+1}$, $n = 0, 1, 2, \cdots$, $C^M_{A,\rho}(x) \propto C^M_{A,\rho}(y)$, $Y^M_{A,\rho}(x) \propto Y^M_{A,\rho}(y)$, for all $x, y \in \mathcal{H}$. Suppose that the following conditions hold:

$$0 < K(f) + P_1(\theta)\lambda_A\lambda_f + P_1(\theta)\rho\lambda_{S1}(\lambda_Y + \lambda_C) + P_2(\theta)\gamma\lambda_{T1}\lambda_{D_G} < 1, \tag{9}$$
$$0 < K(g) + P_2(\theta)\lambda_B\lambda_g + P_2(\theta)\gamma\lambda_{T2}(1 + \lambda_q) + P_1(\theta)\rho\lambda_{S2}\lambda_{D_T} < 1, \tag{10}$$

where $K(f) = \sqrt{1 - 2\delta_f + \lambda_f^2}$, $K(g) = \sqrt{1 - 2\delta_g + \lambda_g^2}$, $P_1(\theta) = \frac{1}{\xi_1(\alpha_A\rho - 1)}$, $P_2(\theta) = \frac{1}{\xi_2(\alpha_B\gamma - 1)}$, $\lambda_Y = \frac{1+\lambda_A\xi_1(\alpha_A\rho-1)}{\rho\xi_1(\alpha_A\rho-1)}$, $\lambda_C = \frac{2+\lambda_A\xi_1(\alpha_A\rho-1)}{\xi_1(\alpha_A\rho-1)}$, $\alpha_A > \frac{1}{\rho}$ and $\alpha_B > \frac{1}{\gamma}$. Then, problem (1) has a solution (x, y, u, v), and the sequences $\{x_n\}$, $\{y_n\}$,$\{u_n\}$ and $\{v_n\}$ generated by Algorithm 3.1 converges to x, y, u and v, respectively.

Proof:

Using (4) of Algorithm 3.1, property (iv) of XOR operation, and Lemma 3.6 of Li et al. (2013b), we have

$$\begin{aligned}
0 \leq x_{n+1} \oplus x_n &= \left[x_n - f(x_n) + R^M_{A,\rho}\left[A\left(f(x_n)\right) - \rho S\left(Y^M_{A,\rho}(x_n) \oplus C^M_{A,\rho}(x_n), v_n\right)\right]\right] \\
&\quad \oplus \left[x_{n-1} - f(x_{n-1}) + R^M_{A,\rho}\left[A\left(f(x_{n-1})\right) - \rho S\left(Y^M_{A,\rho}(x_{n-1})\right.\right.\right. \\
&\quad \left.\left.\left.\oplus C^M_{A,\rho}(x_{n-1}), v_{n-1}\right)\right]\right] \\
&= \left[(x_n - f(x_n)) \oplus (x_{n-1} - f(x_{n-1}))\right] \\
&\quad + R^M_{A,\rho}\left[A\left(f(x_n)\right) - \rho S\left(Y^M_{A,\rho}(x_n) \oplus C^M_{A,\rho}(x_n), v_n\right)\right] \\
&\quad \oplus R^M_{A,\rho}\left[A\left(f(x_{n-1})\right) - \rho S\left(Y^M_{A,\rho}(x_{n-1}) \oplus C^M_{A,\rho}(x_{n-1}), v_{n-1}\right)\right] \\
&\leq \left[(x_n - f(x_n)) \oplus (x_{n-1} - f(x_{n-1}))\right] \\
&\quad + P_1(\theta)\left[\left(A\left(f(x_n)\right) - \rho S\left(Y^M_{A,\rho}(x_n) \oplus C^M_{A,\rho}(x_n), v_n\right)\right)\right. \\
&\quad \left.\oplus \left(A\left(f(x_{n-1})\right) - \rho S\left(Y^M_{A,\rho}(x_{n-1}) \oplus C^M_{A,\rho}(x_{n-1}), v_{n-1}\right)\right)\right]. \qquad (11)
\end{aligned}$$

Using property (viii) of XOR operation, from (11), we have

$$\begin{aligned}
\|x_{n+1} \oplus x_n\| &\leq \|(x_n - f(x_n)) - (x_{n-1} - f(x_{n-1}))\| \\
&\quad + P_1(\theta)\left\|\left(A\left(f(x_n)\right) - \rho S\left(Y^M_{A,\rho}(x_n) \oplus C^M_{A,\rho}(x_n), v_n\right)\right)\right. \\
&\quad \left.- \left(A\left(f(x_{n-1})\right) - \rho S\left(Y^M_{A,\rho}(x_{n-1}) \oplus C^M_{A,\rho}(x_{n-1}), v_{n-1}\right)\right)\right\| \\
&\leq \|(x_n - x_{n-1}) - (f(x_n) - f(x_{n-1}))\| + P_1(\theta)\left\|\left(A\left(f(x_n)\right) - A\left(f(x_{n-1})\right)\right)\right. \\
&\quad \left.-\rho\left(S\left(Y^M_{A,\rho}(x_n) \oplus C^M_{A,\rho}(x_n), v_n\right) - S\left(Y^M_{A,\rho}(x_{n-1}) \oplus C^M_{A,\rho}(x_{n-1}), v_{n-1}\right)\right)\right\| \\
&\leq \|(x_n - x_{n-1}) - (f(x_n) - f(x_{n-1}))\| + P_1(\theta)\left\|A\left(f(x_n)\right) - A\left(f(x_{n-1})\right)\right\| \\
&\quad + P_1(\theta)\rho\left\|S\left(Y^M_{A,\rho}(x_n) \oplus C^M_{A,\rho}(x_n), v_n\right) - S\left(Y^M_{A,\rho}(x_{n-1}) \oplus C^M_{A,\rho}(x_{n-1}), v_{n-1}\right)\right\|. \qquad (12)
\end{aligned}$$

Since f is strongly monotone with constant δ_f and Lipschitz continuous with constant λ_f, and using the technique of Ahmad and Usman (2009), we have

$$\|(x_n - x_{n-1}) - (f(x_n) - f(x_{n-1}))\|^2 \leq \left(1 - 2\delta_f + \lambda_f^2\right)\|x_n - x_{n-1}\|^2,$$

which implies $$\|(x_n - x_{n-1}) - (f(x_n) - f(x_{n-1}))\| \leq K(f)\|x_n - x_{n-1}\|, \qquad (13)$$

where $K(f) = \sqrt{1 - 2\delta_f + \lambda_f^2}$.

As $x_{n+1} \propto x_n$ for all n, using property (ix) of XOR operation, (13) and Lipschitz continuity of f and A, from (12), we have

$$\begin{aligned}
\|x_{n+1} - x_n\| &\leq K(f)\|x_n - x_{n-1}\| + P_1(\theta)\lambda_A\lambda_f\|x_n - x_{n-1}\| \\
&\quad + P_1(\theta)\rho\left\|S\left(Y^M_{A,\rho}(x_n) \oplus C^M_{A,\rho}(x_n), v_n\right) - S\left(Y^M_{A,\rho}(x_{n-1}) \oplus C^M_{A,\rho}(x_{n-1}), v_{n-1}\right)\right\|. \qquad (14)
\end{aligned}$$

Using the Lipschitz continuity of S in both arguments and D-Lipschitz continuity of F, we have

$$\begin{aligned}
&\left\|S\left(Y_{A,\rho}^{M}(x_n)\oplus C_{A,\rho}^{M}(x_n),v_n\right)-S\left(Y_{A,\rho}^{M}(x_{n-1})\oplus C_{A,\rho}^{M}(x_{n-1}),v_{n-1}\right)\right\|\\
&=\left\|S\left(Y_{A,\rho}^{M}(x_n)\oplus C_{A,\rho}^{M}(x_n),v_n\right)-S\left(Y_{A,\rho}^{M}(x_n)\oplus C_{A,\rho}^{M}(x_n),v_{n-1}\right)\right.\\
&\quad\left.+S\left(Y_{A,\rho}^{M}(x_n)\oplus C_{A,\rho}^{M}(x_n),v_{n-1}\right)-S\left(Y_{A,\rho}^{M}(x_{n-1})\oplus C_{A,\rho}^{M}(x_{n-1}),v_{n-1}\right)\right\|\\
&\leq\left\|S\left(Y_{A,\rho}^{M}(x_n)\oplus C_{A,\rho}^{M}(x_n),v_n\right)-S\left(Y_{A,\rho}^{M}(x_n)\oplus C_{A,\rho}^{M}(x_n),v_{n-1}\right)\right\|\\
&\quad+\left\|S\left(Y_{A,\rho}^{M}(x_n)\oplus C_{A,\rho}^{M}(x_n),v_{n-1}\right)-S\left(Y_{A,\rho}^{M}(x_{n-1})\oplus C_{A,\rho}^{M}(x_{n-1}),v_{n-1}\right)\right\|\\
&\leq\lambda_{S2}\left\|v_n-v_{n-1}\right\|+\lambda_{S1}\left\|\left(Y_{A,\rho}^{M}(x_n)\oplus C_{A,\rho}^{M}(x_n)\right)-\left(Y_{A,\rho}^{M}(x_{n-1})\oplus C_{A,\rho}^{M}(x_{n-1})\right)\right\|\\
&\leq\lambda_{S2}D\left(F(y_n),F(y_{n-1})\right)+\lambda_{S1}\left\|\left(Y_{A,\rho}^{M}(x_n)\oplus C_{A,\rho}^{M}(x_n)\right)-\left(Y_{A,\rho}^{M}(x_{n-1})\oplus C_{A,\rho}^{M}(x_{n-1})\right)\right\|\\
&\leq\lambda_{S2}\lambda_{D_F}\left\|y_n-y_{n-1}\right\|+\lambda_{S1}\left\|\left(Y_{A,\rho}^{M}(x_n)\oplus C_{A,\rho}^{M}(x_n)\right)-\left(C_{A,\rho}^{M}(x_{n-1})\oplus Y_{A,\rho}^{M}(x_{n-1})\right)\right\|. \qquad (15)
\end{aligned}$$

Now, using property (vi) of XOR operation, we have

$$\begin{aligned}
&\left(Y_{A,\rho}^{M}(x_n)\oplus C_{A,\rho}^{M}(x_n)\right)-\left(C_{A,\rho}^{M}(x_{n-1})\oplus Y_{A,\rho}^{M}(x_{n-1})\right)\vee\left(Y_{A,\rho}^{M}(x_n)\oplus Y_{A,\rho}^{M}(x_{n-1})\right)\\
&-\left(C_{A,\rho}^{M}(x_{n-1})\oplus C_{A,\rho}^{M}(x_n)\right)\leq\left(Y_{A,\rho}^{M}(x_n)+C_{A,\rho}^{M}(x_{n-1})\right)\oplus\left(C_{A,\rho}^{M}(x_n)+Y_{A,\rho}^{M}(x_{n-1})\right). \qquad (16)
\end{aligned}$$

We know that if $lub\,\{x,y\}\leq z$, then $x\leq z$ and $y\leq z$. Thus, from (16), we deduce that

$$\begin{aligned}
&\left(Y_{A,\rho}^{M}(x_n)\oplus C_{A,\rho}^{M}(x_n)\right)-\left(C_{A,\rho}^{M}(x_{n-1})\oplus Y_{A,\rho}^{M}(x_{n-1})\right)\\
&\leq\left(Y_{A,\rho}^{M}(x_n)+C_{A,\rho}^{M}(x_{n-1})\right)\oplus\left(C_{A,\rho}^{M}(x_n)+Y_{A,\rho}^{M}(x_{n-1})\right). \qquad (17)
\end{aligned}$$

From (17) and property (viii) of XOR operation, we have

$$\begin{aligned}
&\left\|\left(Y_{A,\rho}^{M}(x_n)\oplus C_{A,\rho}^{M}(x_n)\right)-\left(C_{A,\rho}^{M}(x_{n-1})\oplus Y_{A,\rho}^{M}(x_{n-1})\right)\right\|\\
&\leq\left\|\left(Y_{A,\rho}^{M}(x_n)+C_{A,\rho}^{M}(x_{n-1})\right)-\left(C_{A,\rho}^{M}(x_n)+Y_{A,\rho}^{M}(x_{n-1})\right)\right\|\\
&\leq\left\|\left(Y_{A,\rho}^{M}(x_n)-Y_{A,\rho}^{M}(x_{n-1})\right)+\left(C_{A,\rho}^{M}(x_{n-1})-C_{A,\rho}^{M}(x_n)\right)\right\|\\
&\leq\left\|Y_{A,\rho}^{M}(x_n)-Y_{A,\rho}^{M}(x_{n-1})\right\|+\left\|C_{A,\rho}^{M}(x_n)-C_{A,\rho}^{M}(x_{n-1})\right\|\\
&\leq\lambda_Y\left\|x_n-x_{n-1}\right\|+\lambda_C\left\|x_n-x_{n-1}\right\|\\
&=\left(\lambda_Y+\lambda_C\right)\left\|x_n-x_{n-1}\right\|. \qquad (18)
\end{aligned}$$

Using (18), (15) becomes

$$\begin{aligned}
&\left\|S\left(Y_{A,\rho}^{M}(x_n)\oplus C_{A,\rho}^{M}(x_n),v_n\right)-S\left(Y_{A,\rho}^{M}(x_{n-1})\oplus C_{A,\rho}^{M}(x_{n-1}),v_{n-1}\right)\right\|\\
&\leq\lambda_{S2}\lambda_{D_F}\left\|y_n-y_{n-1}\right\|+\lambda_{S1}\left(\lambda_Y+\lambda_C\right)\left\|x_n-x_{n-1}\right\|. \qquad (19)
\end{aligned}$$

Combining (14) and (19), we have

$$\begin{aligned}
\left\|x_{n+1}-x_n\right\|\leq&\left[K(f)+P_1(\theta)\lambda_A\lambda_f+P_1(\theta)\rho\lambda_{S1}\left(\lambda_Y+\lambda_C\right)\right]\left\|x_n-x_{n-1}\right\|\\
&+P_1(\theta)\rho\lambda_{S2}\lambda_{D_F}\left\|y_n-y_{n-1}\right\|. \qquad (20)
\end{aligned}$$

Applying (5) of Algorithm 3.1, property (iv) of XOR operation, and Lemma 3.6 of Li et al. (2013b), we have

$$\begin{aligned}0 \leq y_{n+1} \oplus y_n = &\left[y_n - g(y_n) + R^N_{B,\gamma}\left[B\left(g(y_n)\right) \oplus \gamma T\left(u_n, y_n - q(y_n)\right)\right]\right] \oplus \left[y_{n-1} - g(y_{n-1})\right.\\ &\left.+ R^N_{B,\gamma}\left[B\left(g(y_{n-1})\right) \oplus \gamma T\left(u_{n-1}, y_{n-1} - q(y_{n-1})\right)\right]\right]\\ &\leq \left[(y_n - g(y_n)) \oplus (y_{n-1} - g(y_{n-1}))\right] + \left[R^N_{B,\gamma}\left[B\left(g(y_n)\right) \oplus \gamma T\left(u_n, y_n - q(y_n)\right)\right]\right.\\ &\left.\oplus R^N_{B,\gamma}\left[B\left(g(y_{n-1})\right) \oplus \gamma T\left(u_{n-1}, y_{n-1} - q(y_{n-1})\right)\right]\right]\\ &\leq \left[(y_n - g(y_n)) \oplus (y_{n-1} - g(y_{n-1}))\right] + P_2(\theta)\left[\left(B\left(g(y_n)\right) \oplus \gamma T\left(u_n, y_n - q(y_n)\right)\right)\right.\\ &\left.\oplus \left(B\left(g(y_{n-1})\right) \oplus \gamma T\left(u_{n-1}, y_{n-1} - q(y_{n-1})\right)\right)\right]. \qquad (21)\end{aligned}$$

Using property (viii) of XOR operation, from (21), we have

$$\begin{aligned}\|y_{n+1} \oplus y_n\| \leq &\left\|(y_n - g(y_n)) \oplus (y_{n-1} - g(y_{n-1}))\right\| + P_2(\theta)\left\|\left(B\left(g(y_n)\right) \oplus \gamma T\left(u_n, y_n - q(y_n)\right)\right)\right.\\ &\left.\oplus \left(B\left(g(y_{n-1})\right) \oplus \gamma T\left(u_{n-1}, y_{n-1} - q(y_{n-1})\right)\right)\right\|\\ &\leq \left\|(y_n - g(y_n)) - (y_{n-1} - g(y_{n-1}))\right\| + P_2(\theta)\left\|\left(B\left(g(y_n)\right) \oplus \gamma T\left(u_n, y_n - q(y_n)\right)\right)\right.\\ &\left.- \left(B\left(g(y_{n-1})\right) \oplus \gamma T\left(u_{n-1}, y_{n-1} - q(y_{n-1})\right)\right)\right\|\\ &\leq \left\|(y_n - y_{n-1}) - (g(y_n) - g(y_{n-1}))\right\| + P_2(\theta)\left\|\left(B\left(g(y_n)\right) \oplus \gamma T\left(u_n, y_n - q(y_n)\right)\right)\right.\\ &\left.- \left(B\left(g(y_{n-1})\right) \oplus \gamma T\left(u_{n-1}, y_{n-1} - q(y_{n-1})\right)\right)\right\|. \qquad (22)\end{aligned}$$

Since g is strongly monotone with constant δ_g and Lipschitz continuous with constant λ_g using the same technique as for (13), we have

$$\left\|(y_n - y_{n-1}) - (g(y_n) - g(y_{n-1}))\right\|^2 \leq \left(1 - 2\delta_g + \lambda_g^2\right)\|y_n - y_{n-1}\|^2,$$

which implies $$\left\|(y_n - y_{n-1}) - (g(y_n) - g(y_{n-1}))\right\| \leq K(g)\|y_n - y_{n-1}\|, \qquad (23)$$

where $K(g) = \sqrt{1 - 2\delta_g + \lambda_g^2}$.

Since g and B are Lipschitz continuous with constants λ_g and λ_B, respectively, using the same arguments as for (18), we have

$$\begin{aligned}&\left\|\left(B\left(g(y_n)\right) \oplus \gamma T\left(u_n, y_n - q(y_n)\right)\right) - \left(B\left(g(y_{n-1})\right) \oplus \gamma T\left(u_{n-1}, y_{n-1} - q(y_{n-1})\right)\right)\right\|\\ &\leq \left\|\left(B\left(g(y_n)\right) + \gamma T\left(u_{n-1}, y_{n-1} - q(y_{n-1})\right)\right) \oplus \left(\gamma T\left(u_n, y_n - q(y_n)\right) + B\left(g(y_{n-1})\right)\right)\right\|\\ &\leq \left\|\left(B\left(g(y_n)\right) + \gamma T\left(u_{n-1}, y_{n-1} - q(y_{n-1})\right)\right) - \left(\gamma T\left(u_n, y_n - q(y_n)\right) + B\left(g(y_{n-1})\right)\right)\right\|\\ &\leq \left\|B\left(g(y_n)\right) - B\left(g(y_{n-1})\right)\right\| + \gamma\left\|T\left(u_n, y_n - q(y_n)\right) - T\left(u_{n-1}, y_{n-1} - q(y_{n-1})\right)\right\|\\ &\leq \lambda_B\lambda_g\|y_n - y_{n-1}\| + \gamma\left\|T\left(u_n, y_n - q(y_n)\right) - T\left(u_{n-1}, y_{n-1} - q(y_{n-1})\right)\right\|. \qquad (24)\end{aligned}$$

Using the Lipschitz continuity of T in both the arguments, the Lipschitz continuity of q, and D-

Lipschitz continuity of G, we have

$$\begin{aligned}
&\|T(u_n, y_n - q(y_n)) - T(u_{n-1}, y_{n-1} - q(y_{n-1}))\| \\
&\leq \|T(u_n, y_n - q(y_n)) - T(u_{n-1}, y_n - q(y_n)) + T(u_{n-1}, y_n - q(y_n)) \\
&\quad - T(u_{n-1}, y_{n-1} - q(y_{n-1}))\| \\
&\leq \|T(u_n, y_n - q(y_n)) - T(u_{n-1}, y_n - q(y_n))\| + \|T(u_{n-1}, y_n - q(y_n)) \\
&\quad - T(u_{n-1}, y_{n-1} - q(y_{n-1}))\| \\
&\leq \lambda_{T1}\|u_n - u_{n-1}\| + \lambda_{T2}\|(y_n - q(y_n)) - (y_{n-1} - q(y_{n-1}))\| \\
&\leq \lambda_{T1}\|u_n - u_{n-1}\| + \lambda_{T2}[\|y_n - y_{n-1}\| + \|q(y_n) - q(y_{n-1})\|] \\
&\leq \lambda_{T1} D(G(x_n), G(x_{n-1})) + \lambda_{T2}\|y_n - y_{n-1}\| + \lambda_{T2}\lambda_q\|y_n - y_{n-1}\| \\
&\leq \lambda_{T1}\lambda_{D_G}\|x_n - x_{n-1}\| + (\lambda_{T2} + \lambda_{T2}\lambda_q)\|y_n - y_{n-1}\|. \qquad (25)
\end{aligned}$$

Using (25) in (24), we have

$$\begin{aligned}
&\|(B(g(y_n)) \oplus \gamma T(u_n, y_n - q(y_n))) - (B(g(y_{n-1})) \oplus \gamma T(u_{n-1}, y_{n-1} - q(y_{n-1})))\| \\
&\leq (\lambda_B\lambda_g + \gamma\lambda_{T2} + \gamma\lambda_{T2}\lambda_q)\|y_n - y_{n-1}\| + \gamma\lambda_{T1}\lambda_{D_G}\|x_n - x_{n-1}\|. \qquad (26)
\end{aligned}$$

As $y_{n+1} \propto y_n$ for all n, from (22), (23) and (26), we have

$$\begin{aligned}
\|y_{n+1} - y_n\| \leq{}& [K(g) + P_2(\theta)\lambda_B\lambda_g + P_2(\theta)\gamma\lambda_{T2}(1 + \lambda_q)]\|y_n - y_{n-1}\| \\
&+ P_2(\theta)\gamma\lambda_{T1}\lambda_{D_G}\|x_n - x_{n-1}\|. \qquad (27)
\end{aligned}$$

Adding (20) and (27), we have

$$\begin{aligned}
\|x_{n+1} - x_n\| + \|y_{n+1} - y_n\| &\leq [K(f) + P_1(\theta)\lambda_A\lambda_f + P_1(\theta)\rho\lambda_{S1}(\lambda_Y + \lambda_C) \\
&\quad + P_2(\theta)\gamma\lambda_{T1}\lambda_{D_G}]\|x_n - x_{n-1}\| \\
&\quad + [K(g) + P_2(\theta)\lambda_B\lambda_g + P_2(\theta)\gamma\lambda_{T2}(1 + \lambda_q) \\
&\quad + P_1(\theta)\rho\lambda_{S2}\lambda_{D_F}]\|y_n - y_{n-1}\| \\
&\leq \eta(\theta)\Big[\|x_n - x_{n-1}\| + \|y_n - y_{n-1}\|\Big], \qquad (28)
\end{aligned}$$

where $\eta(\theta) = \max\Big\{[K(f) + P_1(\theta)\lambda_A\lambda_f + P_1(\theta)\rho\lambda_{S1}(\lambda_Y + \lambda_C) + P_2(\theta)\gamma\lambda_{T1}\lambda_{D_G}],$
$[K(g) + P_2(\theta)\lambda_B\lambda_g + P_2(\theta)\gamma\lambda_{T2}(1 + \lambda_q) + P_1(\theta)\rho\lambda_{S2}\lambda_{D_F}]\Big\}.$

By (9) and (10), it follows that $0 < \eta(\theta) < 1$, and thus (28) implies that $\{x_n\}$ and $\{y_n\}$ are both Cauchy sequences in $\mathcal{H}$. Therefore, there exists $x, y \in \mathcal{H}$ such that $x_n \to x$ and $y_n \to y$ as $n \to \infty$.

Now we prove that $u_n \to u \in G(x)$ and $v_n \to v \in F(y)$. In fact, it follows from the D-Lipschitz continuity of G, F and from Algorithm 3.1 that

$$\|u_{n+1} - u_n\| \leq D(G(x_{n+1}), G(x_n)) \leq \lambda_{D_G}\|x_{n+1} - x_n\| \qquad (29)$$

$$\|v_{n+1} - v_n\| \leq D(F(y_{n+1}), F(y_n)) \leq \lambda_{D_F}\|y_{n+1} - y_n\|. \qquad (30)$$

From (29) and (30), it is clear that $\{u_n\}$ and $\{v_n\}$ are also Cauchy sequences in $\mathcal{H}$. Thus, there exist $u, v \in \mathcal{H}$ such that $u_n \to u$ and $v_n \to v$ as $n \to \infty$.

Further,

$$\begin{aligned} d(u,G(x)) &\leq \|u-u_n\| + d(u_n,G(x)) \\ &\leq \|u-u_n\| + D(G(x_n),G(x)) \\ &\leq \|u-u_n\| + \lambda_{D_G}\|x_n - x\| \to 0, \text{ as } n\to\infty, \end{aligned}$$

which implies that $d(u,G(x)) = 0$. Since $G(x) \in CB(\mathcal{H})$, it follows that $u \in G(x)$. Similarly, we can show that $v \in F(y)$. By continuity of $f, g, A, B, S, T, G, F, q, R^M_{A,\rho}, Y^M_{A,\rho}, C^M_{A,\rho}, R^N_{B,\gamma}$ and Algorithm 3.1, we have

$$\begin{aligned} f(x) &= R^M_{A,\rho}\left[A(f(x)) - \rho S\left(Y^M_{A,\rho}(x) \oplus C^M_{A,\rho}(x), v\right)\right], \\ g(y) &= R^N_{B,\gamma}\left[B(g(y)) \oplus \gamma T(u, y - q(y))\right]. \end{aligned}$$

It follows from Lemma 3.1 that (x, y, u, v) is a solution of the problem (1). ∎

4. Numerical Example

The following example is presented in support of the main result and shows the convergence by using MATLAB R2021a.

Example 4.1.

Let $\mathcal{H} = \mathbb{R}$ with usual norm and inner product.

(i) Let $S, T : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ be single-valued mappings such that

$$\begin{aligned} S(x,y) &= \frac{3x}{23} + \frac{2y}{19}, \\ T(x,y) &= \frac{2x}{25} + \frac{y}{17}. \end{aligned}$$

Then, for any $x_1, x_2, y \in \mathcal{H}$, we have

$$\begin{aligned} \|S(x_1,y) - S(x_2,y)\| &= \left\|\frac{3x_1}{23} + \frac{2y}{19} - \frac{3x_2}{23} - \frac{2y}{19}\right\| \\ &= \frac{3}{23}\|x_1 - x_2\| \\ &\leq \frac{1}{7}\|x_1 - x_2\|, \end{aligned}$$

that is, S is Lipschitz continuous in the first argument with constant $\lambda_{S1} = \frac{1}{7}$. It is easy to check that S is Lipschitz continuous in the second argument with constant $\lambda_{S2} = \frac{1}{9}$.

Similarly, one can show that T is Lipschitz continuous in both arguments with constants $\lambda_{T1} = \frac{1}{12}$ and $\lambda_{S1} = \frac{1}{15}$, respectively.

(ii) Let $G, F : \mathcal{H} \to C(\mathcal{H})$ be multi-valued mappings such that

$$G(x) = \left\{ \frac{3x}{23} \right\},$$
$$F(x) = \left\{ \frac{2x}{19} \right\}.$$

Now for any $x, y \in \mathcal{H}$, we have

$$\begin{aligned} D\left(G(x), G(y)\right) &\leq \max\left\{ \left\| \frac{3x}{23} - \frac{3y}{23} \right\|, \left\| \frac{3y}{23} - \frac{3x}{23} \right\| \right\} \\ &= \frac{3}{23} \max\left\{ \|x-y\|, \|y-x\| \right\} \\ &\leq \frac{1}{6} \|x-y\|, \end{aligned}$$

that is, G is D-Lipschitz continuous with constant $\lambda_{D_G} = \frac{1}{6}$.

Similarly, it can be shown that F is D-Lipschitz continuous with constant $\lambda_{D_F} = \frac{1}{8}$.

(iii) Let $A, B, q : \mathcal{H} \to \mathcal{H}$ be single-valued mappings such that

$$A(x) = \frac{x}{7},$$
$$B(x) = \frac{x}{5},$$
$$q(x) = \frac{2x}{5}.$$

Then, for any $x, y \in \mathcal{H}$, we have

$$\begin{aligned} \|A(x) - A(y)\| &= \left\| \frac{x}{7} - \frac{y}{7} \right\| \\ &= \frac{1}{7} \|x-y\| \\ &\leq \frac{1}{5} \|x-y\|, \end{aligned}$$

that is, A is Lipschitz continuous with constant $\lambda_A = \frac{1}{5}$. It is easy to check that A is ξ_1-ordered non-extended mapping with the constant $\xi_1 = \frac{1}{7}$.

Similarly, one can show that B is Lipschitz continuous with constant $\lambda_B = \frac{1}{3}$ and ξ_2-ordered non-extended mapping with constant $\xi_2 = \frac{1}{5}$, and q is Lipschitz continuous with constant $\lambda_q = \frac{1}{2}$.

(iv) Let $M, N : \mathcal{H} \to 2^{\mathcal{H}}$ be multi-valued mappings such that

$$M(x) = \{3x\},$$
$$N(x) = \{2x\}.$$

For $\rho = 7$ it is clear that M is (α_A, ρ)-weak ANODD mapping with $\alpha_A = 3$.

For $\gamma = 5$ it is clear that N is (α_B, γ)-weak BNODD mapping with $\alpha_B = 2$.

(v) Let $f, g : \mathcal{H} \to \mathcal{H}$ be single-valued mappings such that

$$f(x) = \frac{x}{3},$$
$$g(x) = \frac{2x}{7}.$$

It can be easily shown that f is Lipschitz continuous with constant $\lambda_f = \frac{1}{2}$ and strongly monotone with constant $\delta_f = \frac{1}{4}$; and g is Lipschitz continuous with constant $\lambda_g = \frac{1}{2}$ and strongly monotone with constant $\delta_g = \frac{1}{3}$.

(vi) In view of the above calculation, we obtained the resolvent operators $R^M_{A,\rho}$ and $R^N_{B,\gamma}$ such that

$$R^M_{A,\rho}(x) = [A + \rho M]^{-1}(x) = \frac{1}{21}x,$$
$$R^N_{B,\gamma}(x) = [B + \gamma N]^{-1}(x) = \frac{5}{51}x.$$

Now, for any $x, y \in \mathcal{H}$, we have

$$\begin{aligned} R^M_{A,\rho}(x) \oplus R^M_{A,\rho}(y) &= \frac{1}{21}x \oplus \frac{1}{21}y \\ &= \frac{1}{21}(x \oplus y) \\ &\leq \frac{7}{20}(x \oplus y), \end{aligned}$$

that is, $R^M_{A,\rho}$ satisfy condition (8) with $P_1(\theta) = \frac{1}{\xi_1(\alpha_A \rho - 1)} = \frac{7}{20}$.

In the same manner, one can show that $R^N_{B,\gamma}$ satisfy condition (8) with $P_2(\theta) = \frac{1}{\xi_2(\alpha_B \gamma - 1)} = \frac{5}{9}$.

(vii) Using the value of $R^M_{A,\rho}$ calculated in step (vi), we obtained the generalized Cayley operator as

$$C^M_{A,\rho}(x) = \left[2R^M_{A,\rho} - A\right](x) = \frac{-1}{21}x.$$

Then, for any $x, y \in \mathcal{H}$, we have

$$\begin{aligned} \left\|C^M_{A,\rho}(x) - C^M_{A,\rho}(y)\right\| &= \frac{1}{21}\|x - y\| \\ &\leq \frac{9}{10}\|x - y\|, \end{aligned}$$

that is, $C^M_{A,\rho}$ is Lipschitz continuous with constant $\lambda_C = \frac{2 + \lambda_A \xi_1(\alpha_A \rho - 1)}{\xi_1(\alpha_A \rho - 1)} = \frac{9}{10}$.

(viii) Using the value of $R^M_{A,\rho}$ calculated in step (vi), we obtained the generalized Yosida approximation operator as

$$Y^M_{A,\rho}(x) = \frac{1}{\rho}\left[A - R^M_{A,\rho}\right](x) = \frac{2}{147}x.$$

Then, for any $x, y \in \mathcal{H}$, we have

$$\begin{aligned} \left\|Y^M_{A,\rho}(x) - Y^M_{A,\rho}(y)\right\| &= \frac{2}{147}\|x - y\| \\ &\leq \frac{11}{140}\|x - y\|, \end{aligned}$$

that is, $Y^M_{A,\rho}$ is Lipschitz continuous with constant $\lambda_Y = \frac{1+\lambda_A \xi_1(\alpha_A \rho - 1)}{\rho \xi_1(\alpha_A \rho - 1)} = \frac{11}{140}$.

(ix) Using all the values of constants calculated in the above steps, conditions (9) and (10) of Theorem 3.1 are fulfilled.

Thus, all the conditions of Theorem 3.1 are satisfied and the problem (1) admits a solution (x, y, u, v). Subsequently, the sequences $\{x_n\}$, $\{y_n\}$,$\{u_n\}$ and $\{v_n\}$ generated by Algorithm 3.1 converge to x, y, u and v, respectively.

Now, from Algorithm 3.1, we have

$$x_{n+1} = x_n - f(x_n) + R^M_{A,\rho}\left[A\left(f(x_n)\right) - \rho S\left(Y^M_{A,\rho}(x_n) \oplus C^M_{A,\rho}(x_n), v_n\right)\right] = \frac{16982788}{25631361} x_n,$$

$$\text{and } y_{n+1} = y_n - g(y_n) + R^N_{B,\gamma}\left[B\left(g(y_n)\right) \oplus \gamma T\left(u_n, y_n - q(y_n)\right)\right] = \frac{104007}{142324} y_n.$$

Table 1 and Table 2 show the numerical values of $\{x_n\}$ and $\{y_n\}$ for different initial values, respectively.

Table 1. Computational results for different initial values of x_0.

No. of iterations	$x_0 = 1.0$ x_n	No. of iterations	$x_0 = 2$ x_n	No. of iterations	$x_0 = 4$ x_n
1	1.0000	1	2.0000	1	4.0000
2	0.6626	2	1.3252	2	2.6503
3	0.4390	3	0.8780	3	1.7560
4	0.2909	4	0.5818	4	1.1635
5	0.1927	5	0.3855	5	0.7709
10	0.0246	10	0.0492	10	0.0984
15	0.0031	15	0.0063	15	0.0126
20	0.0004	20	0.0008	20	0.0016
25	0.0001	25	0.0001	25	0.0002
26	0.0000	26	0.0001	26	0.0001
27	0.0000	27	0.0000	27	0.0001
28	0.0000	28	0.0000	28	0.0001
29	0.0000	29	0.0000	29	0.0000
30	0.0000	30	0.0000	30	0.0000

In Figure 1 and Figure 2, we show the convergence of $\{x_n\}$ and $\{y_n\}$ with different initial values using MATLAB R2021a, respectively. In Figure 3, we plot a combined graph for $\{x_n\}$ and $\{y_n\}$ for the initial value $x_0 = y_0 = 4$ by using MATLAB R2021a.

5. Conclusions

In this paper, we study a new class of system of variational inclusions that involves the Cayley operator, the Yosida approximation operator and the XOR operation. It is established that a system of variational inclusions involving the Cayley operator and the Yosida approximation operator with the XOR operation is equivalent to a fixed-point equation. We propose an iterative algorithm based on this fixed point formulation to obtain an existence and convergence result for a "system of variational inclusions involving the Cayley operator and the Yosida approximation operator with

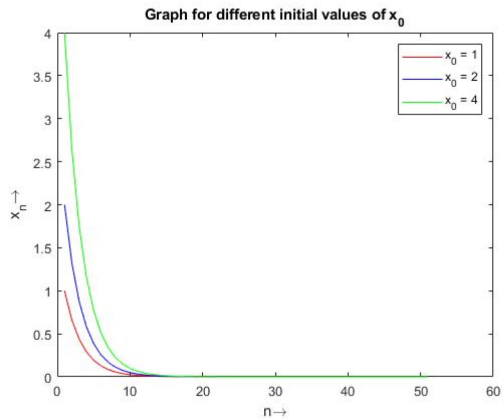

Figure 1. The convergence of $\{x_n\}$ with initial values $x_0 = 1,\ x_0 = 2$ and $x_0 = 4$.

Table 2. Computational results for different initial values of y_0.

No. of iterations	$y_0 = 1.0$ y_n	No. of iterations	$y_0 = 2$ y_n	No. of iterations	$y_0 = 4$ y_n
1	1.0000	1	2.0000	1	4.0000
2	0.7308	2	1.4616	2	2.9231
3	0.5340	3	1.0681	3	2.1361
4	0.3903	4	0.7805	4	1.5610
5	0.2852	5	0.5704	5	1.1408
10	0.0594	10	0.1189	10	0.2377
15	0.0124	15	0.0248	15	0.0495
20	0.0026	20	0.0052	20	0.0103
25	0.0005	25	0.0011	25	0.0022
30	0.0001	26	0.0002	26	0.0004
35	0.0000	27	0.0000	27	0.0001
36	0.0000	28	0.0000	28	0.0001
37	0.0000	29	0.0000	29	0.0000
38	0.0000	30	0.0000	30	0.0000

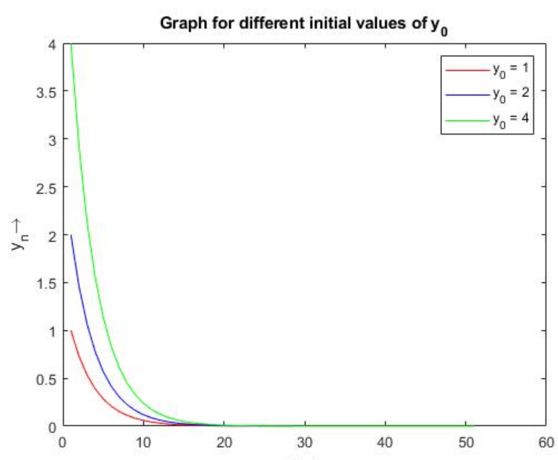

Figure 2. The convergence of $\{y_n\}$ with initial values $y_0 = 1,\ y_0 = 2$ and $y_0 = 4$.

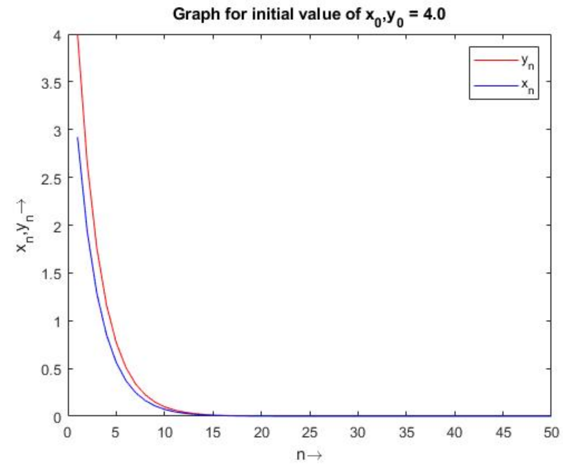

Figure 3. The convergence of $\{x_n\}$ and $\{y_n\}$ with initial values $x_0, y_0 = 4$.

XOR operation." Our results may be extended to higher-dimensional spaces and may be used for practical purposes by other scientists.

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