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Approximate Solution of Fuzzy Volterra Integro-differential Equations Using Numerical Techniques

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Abstract

To determine the approximate solution to fuzzy Volterra integro-differential equations, the Adomian Decomposition Method (ADM), Modified Adomian Decomposition Method (MADM), Variational Iteration Method (VIM), and Homotopy Perturbation Method (HPM) are proposed in this study. We present two examples to support the methodology, and the results are presented in tables to demonstrate the method's efficiency and correctness. Wolfram Mathematica 11.3 is used to perform the computations.

Keywords: Adomian Decomposition Method; Modified Adomian Decomposition Method; Variational Iteration Method; Homotopy Perturbation Method; Fuzzy Volterra integro-differential equations

MSC 2010 No.: 65R20, 65R99

1. Introduction

In recent years, the concepts of fuzzy integral equations that have attracted increased interest, particularly about fuzzy control, have advanced significantly. The fuzzy differential and integral equations are an important part of fuzzy analysis theory and have powerful theoretical and practical applications in control theory. Our main goal is to demonstrate that properly constructing the Adomian Decomposition Method (ADM), Modified Adomian Decomposition Method (MADM), Variational Iteration Method (VIM), and Homotopy Perturbation Method (HPM) produces extremely accurate results while requiring less computational work. To assure the efficiency and performance of the methodology, they are stated through stability and convergence analysis in Narayana Moorthy and Sathiyapriya (2016). Zadeh (1965) was the first to establish the concept of fuzzy numbers, arithmetic operations and the fuzzy mapping function. In addition, Dubois and Prade (1978) proposed the concept of fuzzy function integration. Fuzzy integro-differential equations are a natural technique to model dynamical system uncertainty. Many numerical methods have been used in the past to solve fuzzy linear Fredholm integro-differential equations (FLFIDE), such as the Homotopy Perturbation Method (Khaleel (2010)), Laplace Transformation Method, Homotopy Analysis Method (Hussain and Ali (2013)), Differential Transform Method (Behiry and Mohamed (2012)), Fixed Point Theorems (Rahimi et al. (2011)), Adomian Decomposition Method (Hamoud and Ghadle (2018); Ansari and Ahmad (2023b); Ansari and Ahmad (2023a); Ansari and Ahmad (2023c); Babolian et al. (2004); Babolian et al. (2005)), Direct Computation Method (Ansari et al. (2023)), fuzzy Laplace transformation method (Das and Talukdar (2014)), Expansion Method (Allahviranloo et al. (2014)), Rationalized Haar functions method (Maleknejad and Mirzaee (2006)), Laplace Adomian Decomposition Method (Ahmad and Singh (2022)), Series Solution Method (Ansari et al. (2024)) and Variational Iteration Method (Hashemi and Abbasbandy (2011); Ansari and Ahmad (2023b)). In addition, some mathematicians have studied fuzzy integral and integro-differential equations numerically (Hamoud and Ghadle (2018); Shao and Zhang (2014); Bede et al. (2007); Bede and Gal (2005)) extended the class of differentiable by introducing a more general definition of the derivative for fuzzy mappings, while Park and Jeong (Park and Jeong (1999); Park et al. (2000)) investigated the existence of solutions to fuzzy integral equations.

On the other hand, certain numerical methods have explored integro-differential equations in a crisp sense, as seen in Berenguer et al. (2004), Zhao and Corless (2006), Pour-Mahmoud et al. (2005), and Maleknejad and Mirzaee (2006). As we all know, fuzzy integro-differential and integral equations are fundamental components of fuzzy analysis theory that play a vital role in numerical analysis. We investigate the parametric solution of fuzzy integro-differential equations using the Adomian Decomposition Method, Variational Iteration Method, Homotopy Perturbation Method, Modified Adomian Decomposition Method, and the Homotopy Analysis Method (HAM) (Hussain and Ali (2013); Hamoud and Ghadle (2018)), an analytic approach to obtaining series solutions of many types of linear and nonlinear integro-differential equations. Allahviranloo et al. (2011) presented a new method for solving fuzzy integro-differential equations under generalized differentiability, and in Effati and Pakdaman (2010) and Mosleh and Otadi (2012) used artificial neural networks to solve fuzzy ordinary differential equations and fuzzy Fredholm integro-differential equations. In addition, Allahviranloo et al. (2009) demonstrated the existence and uniqueness of solutions to second-order fuzzy differential equations, and Chalco and Roman-Flores (2008) declared additional fuzzy differential equation solutions. As a result, Zhang et al. (2009) defined the fuzzy derivative, and several generalizations of it have been examined in Rodríguez-López (2008), Rodríguez-Muñize and López-Diaz (2003), and Stefanini (2007).

Numerical examples are given to demonstrate the numerical technique's convergence and validity in Narayana Moorthy and Sathiyapriya (2016). Also, graphical representations of the precise and acquired approximation fuzzy answers help to clarify the approach's accuracy in Narayana Moorthy and Sathiyapriya (2016). The Adomian Decomposition Method has many advantages: it solves the problem directly without the need for linearization, perturbation, or any other transformation; it converges very rapidly and is highly accurate. According to Srivastava and Awasthi (2014) the applications of the fractional differentiation for the mathematical modeling of real world physical problems such as the earthquake modeling, the traffic flow model with fractional derivatives, measurement of viscoelastic material properties, etc., have been widespread in this modern era. Before the nineteenth century, no analytical solution method was available for such type of equations even for the linear fractional differential equations in Srivastava et al. (2014a).

The following is how the paper is structured. The linear Volterra fuzzy integro-differential equations are addressed in Section 2. Section 3, "Description of methods," covers the essential notion of the methods: ADM, MADM, VIM, and HPM. A "Numerical illustrations" part is also presented, then followed by a "Conclusion" section.

2. Fuzzy Volterra Integro-differential Equations

Fuzzy numbers are a generalization of classical real numbers, and we can say that a fuzzy number is a fuzzy subset of the real line with some additional properties. The concept of fuzzy numbers is essential for fuzzy analysis, fuzzy integral equations, fuzzy differential equations, and it is a very useful tool in a variety of fuzzy set applications. Ahmed and Kirtiwant (2018) provides a basic definition of fuzzy numbers. The basic notations of fuzzy arithmetic number and fuzzy linear systems are then reviewed in Abbasbandy and Hashemi (2012). Puri and Ralescu (1983) proved that a strongly measurable and integrally bounded mapping $F : I \to \mathrm{E}^1$ is integrable, i.e., $\int_I F(t)dt \in \mathrm{E}^1$.

In this study, $\Re$ is the set of all real numbers, $\tilde{v}$ denotes a fuzzy number, and E^1 denotes the set of fuzzy numbers. It was proved by Puri and Ralescu (1983) that a strongly measurable and integrally bounded mapping $F : I \to \mathrm{E}^1$ is integrable, i.e., $\int_I F(t)dt \in \mathrm{E}^1$.

When a physical system is described in a differential sense, the result is an integral equation or a fuzzy integrodifferential equation, and so the solution of fuzzy Volterra integro-differential equations (FVIDE) plays an important role in science and engineering. In a parametric example, a fuzzy Volterra integro-differential equation (FVIDE) is transformed to its corresponding crisp case. Nonlinear fuzzy integro-differential equations are commonly hard to solve analytically, with few exact solutions available. As a result, they have attracted the interest of several authors. We present various approaches for solving fuzzy Volterra integro-differential equations, which are motivated by the studies discussed in Section 1.

Consider the second type of fuzzy Volterra integrodifferential equation of the form

$$\tilde{u}^{'}(z) = \tilde{f}(z, \tilde{u}(z)) + \tilde{\lambda} \int_a^z \tilde{K}(z,s)\tilde{u}(s)ds, \quad z \in (a,b], \tag{2.1}$$

with the initial condition

$$\tilde{u}(a) = \tilde{u}_0,$$

where $\hat{f} : (a,b] \to \mathrm{E}^1$ and $\tilde{K} : (a \times b] \times (a \times b] \times \mathrm{E}^1 \to \mathrm{E}^1$ are continuous. Moreover, $\tilde{K}$ is uniformly continuous to z.

Also, $\tilde{\lambda} > 0$ is a fuzzy parameter, $\tilde{K}$ is an arbitrary function called the kernel of the integro-differential equation (2.1) defined over the square $0 \leq a \leq z$, $s \leq z$ and $\tilde{f}(z, \tilde{u}(z))$ is a given fuzzy function of $z \in (a,b]$ with $\tilde{u}(s)$ is the unknown fuzzy function to be determined. Here, $\tilde{u}^{'}$ is the fuzzy derivative of $\tilde{u}$; this equation may only possess a fuzzy solution.

If $u(z) = (\underline{u}(z,r), \overline{u}(z,r))$ is a fuzzy solution of Equation (2.1), therefore, by Sadigh Behzadi (2011), we have the equivalent system

$$\begin{aligned} \underline{u}^{'}(z) &= \underline{f}(z, \underline{u}(z)) + \lambda \int_a^z \underline{K(z,s)u(s)}ds, \quad \underline{u}(a) = \underline{u}_0, \\ \overline{u}^{'}(z) &= \overline{f}(z, \overline{u}(z)) + \lambda \int_a^z \overline{K(z,s)u(s)}ds, \quad \overline{u}(a) = \overline{u}_0. \end{aligned} \tag{2.2}$$

This has a unique solution $(\underline{u}, \overline{u})$ that is a fuzzy solution, i.e., for each t, the pair $(\underline{u}(t,r), \overline{u}(t,r))$ is a fuzzy number, so each solution of Equation (2.1) is also a solution of system of Equation (2.2), and vice versa.

Through defuzzification by Saberirad et al. (2019), Equation (2.2) adopts the features of the standard second-kind linear fuzzy Volterra integro-differential equation in the crisp domain. As a result, Equation (2.2) with fuzzy parametric forms corresponds to the following system:

$$\begin{aligned} \underline{u}^{'}(z,r) &= \underline{f}(z, \underline{u}(z,r)) + \lambda \int_a^z \underline{K(z,s)u(s,r)}ds, \quad \underline{u}(a) = \underline{u}_0(r), r \in [0,1], \\ \overline{u}^{'}(z,r) &= \overline{f}(z, \overline{u}(z,r)) + \lambda \int_a^z \overline{K(z,s)u(s,r)}ds, \quad \overline{u}(a) = \overline{u}_0(r), r \in [0,1], \end{aligned} \tag{2.3}$$

or

$$\begin{aligned} \underline{u}(z,r) &= \underline{u}_0 + \int_a^z \underline{f}(s, \underline{u}(s,r))ds + \int_a^z \int_a^s \underline{K(z,s)u(s,r)}drds, \\ \overline{u}(z,r) &= \overline{u}_0 + \int_a^z \overline{f}(s, \overline{u}(s,r))ds + \int_a^z \int_a^s \overline{K(z,s)u(s,r)}drds. \end{aligned} \tag{2.4}$$

Suppose $K(z,s)$ is continuous in $a < z \leq b$ and fix s. $K(z,s)$ changes its sign in finite points as z_i where $u_i \in [a, z_1]$. For example, let $K(z,s)$ be non-negative over $[a, z_1]$ and negative over $[z_1, b]$.

Therefore, we have

$$\underline{u}^{'}(z,r) = \underline{f}(z,\underline{u}(z,r)) + \lambda \int_a^{z_1} K(z,s)\underline{u(s,r)}ds + \lambda \int_{z_1}^{z} K(z,s)\overline{u(s,r)}ds, \quad \underline{u}(a) = \underline{u}_0(r),$$

$$\overline{u}^{'}(z,r) = \overline{f}(z,\overline{u}(z,r)) + \lambda \int_a^{z_1} K(z,s)\overline{u(s,r)}ds + \lambda \int_{z_1}^{z} K(z,s)\underline{u(s,r)}ds, \quad \overline{u}(a) = \overline{u}_0(r),$$

where $[\tilde{u}_0]_r = [\underline{u}(z_0,r), \overline{u}(z_0,r)], r \in [0,1]$. Then, Equation (2.4) gives

$$\underline{K(z,s)u(s,r)} = \begin{cases} K(z,s)\underline{u}(s,r), & K(z,s) \geq 0, \\ K(z,s)\overline{u}(s,r), & K(z,s) < 0, \end{cases}$$

$$\overline{K(z,s)u(s,r)} = \begin{cases} K(z,s)\overline{u}(s,r), & K(z,s) \geq 0, \\ K(z,s)\underline{u}(s,r), & K(z,s) < 0. \end{cases}$$

However, in most circumstances, an analytical solution to Equation (2.3) or Equation (2.4) cannot be established, and a numerical method must be considered.

3. Description of the methods

We will discuss briefly some dependable methods for understanding such equations, such as the Adomian Decomposition Method (ADM), Modified Adomian Decomposition Method (MADM), Variational Iteration Method (VIM), and Homotopy Perturbation Method (HPM).

3.1. Analysis of Adomian Decomposition Method (ADM)

First, demonstrate how to approximate the answers of Equation (2.1) and Equation (2.3) using an Adomian Decomposition Technique. Adomian presents the solution as an infinite series that typically converges to a solution.

Consider the following fuzzy Volterra integrodifferential equation of the form

$$\underline{u}(z,r) = \underline{u}_0 + \int_a^z \underline{f}(s,\underline{u}(s,r))ds + \int_a^z \int_a^s \underline{K(z,s)u(s,r)}drds, \tag{3.1}$$

$$\overline{u}(z,r) = \overline{u}_0 + \int_a^z \overline{f}(s,\overline{u}(s,r))ds + \int_a^z \int_a^s \overline{K(z,s)u(s,r)}drds. \tag{3.2}$$

The ADM is based on an infinite series solution for the unknown functions $[\underline{u}, \overline{u}]$, which is provided by

$$\begin{aligned} \underline{u}(z,r) &= \sum_{i=0}^{\infty} \underline{u}_i(z,r), \\ \overline{u}(z,r) &= \sum_{i=0}^{\infty} \overline{u}_i(z,r). \end{aligned} \tag{3.3}$$

The $\hat{A}_m = [\underline{A}_m, \overline{A}_m], m \geq 0$ are the so-called Adomian polynomial, and $f_1(z,r) = u_0 + \int_a^z f(s,u(s))ds$. We get

$$\begin{aligned} \underline{u}_0 &= \underline{f}_1(z,r), \\ \underline{u}_1 &= \int_a^z \int_a^s \underline{K(z,s)A_0} drds, \\ \underline{u}_2 &= \int_a^z \int_a^s \underline{K(z,s)A_1} drds, \\ &\vdots \\ \underline{u}_{m+1} &= \int_a^z \int_a^s \underline{K(z,s)A_m} drds, \quad m > 0, \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} \overline{u}_0 &= \overline{f}_1(z,r), \\ \overline{u}_1 &= \int_a^z \int_a^s \overline{K(z,s)A_0} drds, \\ \overline{u}_2 &= \int_a^z \int_a^s \overline{K(z,s)A_1} drds, \\ &\vdots \\ \overline{u}_{m+1} &= \int_a^z \int_a^s \overline{K(z,s)A_m} drds, \quad m > 0. \end{aligned} \tag{3.5}$$

We approximate $\tilde{u}(z,r) = [\underline{u}(z,r), \overline{u}(z,r)]$ by

$$\underline{\phi}_m = \sum_{i=0}^{m-1} \underline{u}_i(z,r) \quad \text{and} \quad \overline{\phi}_m = \sum_{i=0}^{m-1} \overline{u}_i(z,r),$$

where

$$\lim_{m\to\infty} \underline{\phi}_m = \underline{u}(z,r) \quad \text{and} \quad \lim_{m\to\infty} \overline{\phi}_m = \overline{u}(z,r).$$

The advantages of the Adomian decomposition method are:

- In a few repetitions, the Adomian decomposition approach gives rapidly convergent series solutions for both linear and nonlinear deterministic and stochastic equations.
- The number of variables is not a problem in applications because the solution series does not rely on it directly.
- This method has the advantage of directly solving the problem without the need for any kind of linearization, perturbation, massive computing, or modification of the governing differential equations.

3.2. Analysis of Modified Adomian Decomposition Method (MADM)

The Modified Adomian Decomposition Method is a sophisticated technique that can solve higher-order fuzzy integro-differential equations. We developed a modified version of the Adomian Decomposition Method in this part for examining the system of fuzzy Volterra integro-differential equations (Ansari and Ahmad (2023b)). The Modified Adomian Decomposition Method (Ansari and Ahmad (2023b); Ansari and Ahmad (2023a)) delivers the precise or approximate answer by computing only two terms from the decomposition series. This method is based on the idea that the function $f(z,r)$ may be separated into two parts, $f_1(z,r)$ and $f_2(z,r)$.

Based on the assumption, we established

$$\underline{u}(z,r) = \underline{f}_1(z,r) + \underline{f}_2(z,r) + \int_a^z \int_a^s \underline{K(z,s)u(s,r)} drds, \tag{3.6}$$

$$\overline{u}(z,r) = \overline{f}_1(z,r) + \overline{f}_2(z,r) + \int_a^z \int_a^s \overline{K(z,s)u(s,r)} drds. \tag{3.7}$$

Hence, we can write the following iteration formula,

$$\begin{aligned}
\underline{u}_0 &= \underline{f}_1(z,r), \\
\underline{u}_1 &= \underline{f}_2(z,r) + \int_a^z \int_a^s \underline{K(z,s)A_0} drds, \\
&\vdots \\
\underline{u}_{m+1} &= \int_a^z \int_a^s \underline{K(z,s)A_m} drds, \qquad m > 0,
\end{aligned}$$

and

$$\begin{aligned}
\overline{u}_0 &= \overline{f}_1(z,r), \\
\overline{u}_1 &= \overline{f}_2(z,r) + \int_a^z \int_a^s \overline{K(z,s)A_0} drds, \\
&\vdots \\
\overline{u}_{m+1} &= \int_a^z \int_a^s \overline{K(z,s)A_m} drds, \qquad m > 0.
\end{aligned}$$

We approximate $\tilde{u}(z,r) = [\underline{u}(z,r), \overline{u}(z,r)]$ by

$$\underline{\phi}_m = \sum_{i=0}^{m-1} \underline{u}_i(z,r) \quad \text{and} \quad \overline{\phi}_m = \sum_{i=0}^{m-1} \overline{u}_i(z,r),$$

where

$$\lim_{m\to\infty} \underline{\phi}_m = \underline{u}(z,r) \quad \text{and} \quad \lim_{m\to\infty} \overline{\phi}_m = \overline{u}(z,r).$$

3.3. Analysis of Variational Iteration Method (VIM)

Consider Equation (2.1) as restricted variation. We have the iteration sequence

$$u_{m+1} = u_m + \int_0^y \lambda \left[u_m^{'}(s) - \tilde{f}(s, u(s,r)) - \int_a^s K(s,z) u^{'}(z) dz \right] ds. \tag{3.8}$$

Taking the variation for the independent variable u_m and noticing that $\delta u_m(0) = 0$, we get

$$\delta u_{m+1} = \delta u_m + \lambda(s) \delta u_m|_{s=y} - \int_0^y \lambda^{'}(s) \delta u_m ds = 0.$$

Then, we apply the stationary conditions listed below:

$$1 + \lambda(s)|_{s=y} = 0, \quad \lambda^{'}(s)|_{s=y} = 0.$$

As a result, a general Lagrange multiplier can be easily found $\lambda = -1$, and, we get the following iteration formula from Equation (3.8),

$$u_{m+1} = u_m - \int_0^y [u_m^{'}(s) - \tilde{f}(s, u(s,r)) - \int_a^s K(s,z) u^{'}(z) dz] ds.$$

Hence, we approximate $\tilde{u}_{(m+1)}(y,r) = [\underline{u}_{(m+1)}(y,r), \overline{u}_{(m+1)}(z,r)]$ and write the following iteration formula,

$$\underline{u}_{m+1}(y,r) = \underline{u}_m(y,r) - \int_0^y \left[\underline{u}_m^{'}(s,r) - \underline{f}(s, \underline{u}(s,r)) - \int_a^s K(s,z) \underline{u}(z,r) dz \right] ds,$$

$$\overline{u}_{m+1}(y,r) = \overline{u}_m(y,r) - \int_0^y \left[\overline{u}_m^{'}(s,r) - \overline{f}(s, \overline{u}(s,r)) - \int_a^s K(s,z) \overline{u}(z,r) dz \right] ds.$$

3.4. Analysis of Homotopy Perturbation Method (HPM)

Now, we consider the fuzzy integrodifferential equations of the second kind

$$(1-q)[\tilde{u}^{'}(z) - \tilde{f}(z)] + q \left[\tilde{u}^{'}(z) - \tilde{f}(z, \tilde{u}^{'}(z)) - \int_a^z K(z,s) \tilde{u}(z) dz \right] = 0. \tag{3.9}$$

Then,

$$\tilde{u}^{'}(z) = \tilde{f}(z, \tilde{u}^{'}(z)) + q \int_a^z K(z,s) \tilde{u}(z) dz, \tag{3.10}$$

substituting Equation (3.9) in Equation (3.10) and equating the terms with identical powers of q, we have

$$q^0 : u_0 = \underline{f}_1(z,r),$$
$$q^1 : u_1 = \int_a^z \int_a^s K(z,s) \underline{u}_0(s,r) dr ds,$$
$$q^2 : u_2 = \int_a^z \int_a^s K(z,s) \underline{u}_1(s,r) dr ds,$$
$$q^3 : u_3 = \int_a^z \int_a^s K(z,s) \underline{u}_2(s,r) dr ds,$$
$$\vdots$$

and

$$\begin{aligned}
q^0 &: u_0 = \overline{f}_1(z,r),\\
q^1 &: u_1 = \int_a^z \int_a^s K(z,s)\overline{u}_0(s,r)drds,\\
q^2 &: u_2 = \int_a^z \int_a^s K(z,s)\overline{u}_1(s,r)drds,\\
q^3 &: u_3 = \int_a^z \int_a^s K(z,s)\overline{u}_2(s,r)drds,\\
&\vdots
\end{aligned}$$

4. Numerical Results

In this section, we use ADM, MADM, VIM, and HPM for solving fuzzy Volterra integro-differential equations.

Left bound of Error: $\underline{\mathrm{E}}(z,r) = |\underline{u}_{\mathrm{E}}(z,r) - \underline{u}_{\mathrm{A}}(z,r)|$
Right bound of Error: $\overline{\mathrm{E}}(z,r) = |\overline{u}_{\mathrm{E}}(z,r) - \overline{u}_{\mathrm{A}}(z,r)|$

The efficiency of the method is shown in terms of error which is estimated by using Wolfram Mathematica 11.3.

Example 4.1.

We take the following fuzzy Volterra integrodifferential equation

$$u^{'}(z,r) = [2(r-2)\sin(z), 2(2-3r)\sin(z)] - \int_0^z u(s,r)ds,$$

$$u(0) = [3r-2, 2-r], \quad 0 \le s \le z \le 1.$$

The equivalent system is

$$\underline{u}^{'}(z,r) = 2(r-2)\sin(z) - \int_0^z \underline{u}(s,r)ds, \quad \underline{u}(0,r) = 3r-2,$$

$$\overline{u}^{'}(z,r) = 2(2-3r)\sin(z) - \int_0^z \overline{u}(s,r)ds, \quad \overline{u}(0,r) = 2-r.$$

The exact solutions are

$$\underline{u}(z,r) = (3r-2)(\cos(z) - z\sin(z)), \quad \overline{u}(z,r) = (2-r)(\cos(z) - z\sin(z)).$$

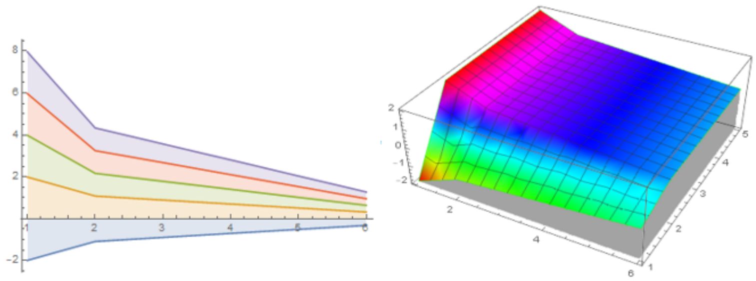

Figure 1. Left Bound Solution $\underline{E}(z = 0.5, r)$ of Example 4.1.

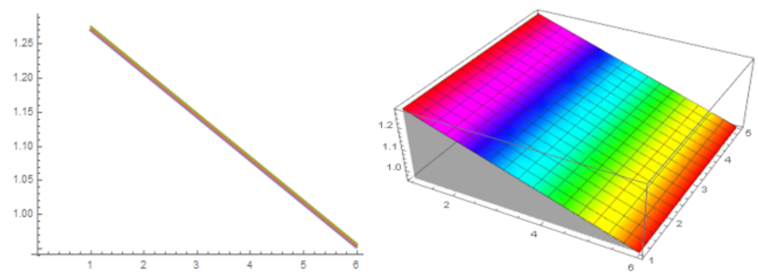

Figure 2. Right Bound Solution $\overline{E}(z = 0.5, r)$ of Example 4.1.

Table 1. Left Bound of Error $\underline{E}(z = 0.5, r)$ of Example 4.1.

r	$\underline{E}_{Exact}$	$\underline{E}_{ADM}$	$\underline{E}_{MADM}$	$\underline{E}_{VIM}$	$\underline{E}_{HPM}$
0.0	-1.9911973106	$9.5818400e^{-9}$	$9.5712800e^{-9}$	$4.6945290e^{-8}$	$9.5945650e^{-9}$
0.1	-1.084378647	$9.2626300e^{-9}$	$8.3558600e^{-9}$	$4.4626840e^{-8}$	$9.4846940e^{-9}$
0.2	-0.893017710	$8.9139700e^{-9}$	$6.8949700e^{-9}$	$3.5628400e^{-8}$	$9.5828600e^{-9}$
0.3	-0.701656772	$6.5953500e^{-9}$	$5.5654600e^{-9}$	$2.1719940e^{-8}$	$8.5319740e^{-9}$
0.4	-0.510295834	$5.2368410e^{-9}$	$4.2457860e^{-9}$	$2.6926410e^{-8}$	$5.5847410e^{-9}$
0.5	-0.318934896	$4.8982900e^{-9}$	$2.7945600e^{-9}$	$2.2642680e^{-8}$	$2.3463860e^{-8}$

Example 4.2.

We take the following fuzzy Volterra integrodifferential equation

$$u^{'}(z,r) = [(r^5+2r)z^3, (6-3r^3)z^3]\frac{1}{12z}(36-5z^4) + \int_0^z (z^2+s^2)u(s,r)ds,$$

$$u(0) = [0,0], \quad 0 \leq s \leq z \leq 1.$$

The equivalent system is

$$\underline{u}^{'}(z,r) = \frac{rz^2}{12}(r^4+2)(36-5z^4) + \int_0^z (z^2+s^2)\underline{u}(s,r)ds, \quad \underline{u}(0,r) = 0,$$

$$\overline{u}^{'}(z,r) = \frac{z^2}{12}(6-3r^3)(36-5z^4) + \int_0^z (z^2+s^2)\overline{u}(s,r)ds, \quad \overline{u}(0,r) = 0.$$

Table 2. Right Bound of Error $\overline{E}(z = 0.5, r)$ of Example 4.1.

r	$\underline{E}_{Exact}$	$\underline{E}_{ADM}$	$\underline{E}_{MADM}$	$\underline{E}_{VIM}$	$\underline{E}_{HPM}$
0.0	1.275739585	$7.9218010e^{-9}$	$9.6916400e^{-9}$	$6.7945290e^{-8}$	$9.8844690e^{-9}$
0.1	1.211952606	$7.6429510e^{-9}$	$9.2487500e^{-9}$	$6.8684670e^{-8}$	$8.6784670e^{-9}$
0.2	1.148165627	$5.8917600e^{-9}$	$8.5369400e^{-9}$	$6.2724250e^{-8}$	$8.4324150e^{-9}$
0.3	1.084378647	$5.4528600e^{-9}$	$7.7594200e^{-9}$	$5.3463500e^{-8}$	$7.2063820e^{-9}$
0.4	1.020591668	$4.6784180e^{-9}$	$7.2479500e^{-9}$	$4.7902930e^{-8}$	$7.9802930e^{-9}$
0.5	0.956804689	$4.8993500e^{-9}$	$6.6753800e^{-9}$	$4.8942250e^{-8}$	$6.7542250e^{-8}$

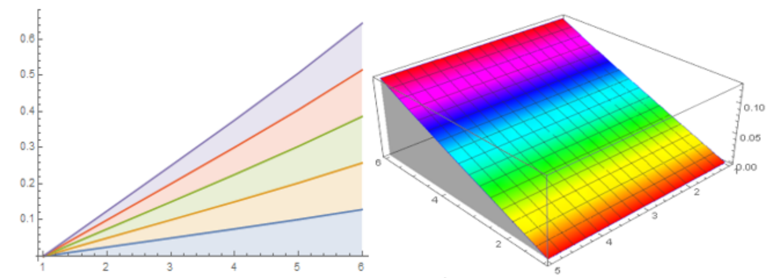

Figure 3. Left Bound Solution $\underline{E}(z = 0.5, r)$ of Example 4.2.

The exact solutions are

$$\underline{u}(z, r) = (r^5 + 2r)z^3, \quad \overline{u}(z, r) = (6 - 3r^3)z^3.$$

Table 3. Left Bound of Error $\underline{E}(z = 0.5, r)$ of Example 4.2.

r	$\underline{E}_{Exact}$	$\underline{E}_{ADM}$	$\underline{E}_{MADM}$	$\underline{E}_{VIM}$	$\underline{E}_{HPM}$
0.0	0.000000000	0.0000000000	0.0000000000	0.000000000	0.000000000
0.1	0.025001250	$7.9568100e^{-11}$	$4.8237130e^{-10}$	$7.66750e^{-11}$	$2.251250e^{-6}$
0.2	0.050040000	$7.9928550e^{-10}$	$9.6546620e^{-10}$	$1.53470e^{-10}$	$5.028800e^{-6}$
0.3	0.075303750	$5.9989880e^{-10}$	$1.4529020e^{-9}$	$2.30950e^{-10}$	$7.567600e^{-6}$
0.4	0.101280000	$5.0335000e^{-10}$	$1.9540860e^{-9}$	$3.10610e^{-10}$	$1.017800e^{-5}$
0.5	0.128906250	$4.1337300e^{-10}$	$2.4871030e^{-9}$	$3.95340e^{-10}$	$1.295300e^{-5}$

Table 4. Right Bound of Error $\overline{E}(z = 0.5, r)$ of Example 4.2.

r	$\underline{E}_{Exact}$	$\underline{E}_{ADM}$	$\underline{E}_{MADM}$	$\underline{E}_{VIM}$	$\underline{E}_{HPM}$
0.0	0.750000000	$3.9868920e^{-9}$	$1.4470420e^{-8}$	$2.30010e^{-9}$	$7.53710e^{-5}$
0.1	0.749625000	$3.9854010e^{-9}$	$1.4463180e^{-8}$	$2.29900e^{-9}$	$7.53330e^{-5}$
0.2	0.747000000	$3.9749450e^{-9}$	$1.4412540e^{-8}$	$2.29090e^{-9}$	$7.50700e^{-5}$
0.3	0.739875000	$3.9465710e^{-9}$	$1.4275060e^{-8}$	$2.26910e^{-9}$	$7.43540e^{-5}$
0.4	0.726000000	$2.8913100e^{-9}$	$1.4007360e^{-8}$	$2.22650e^{-9}$	$7.29590e^{-5}$
0.5	0.703125000	$2.8002140e^{-9}$	$1.3566010e^{-8}$	$2.15640e^{-9}$	$7.06600e^{-5}$

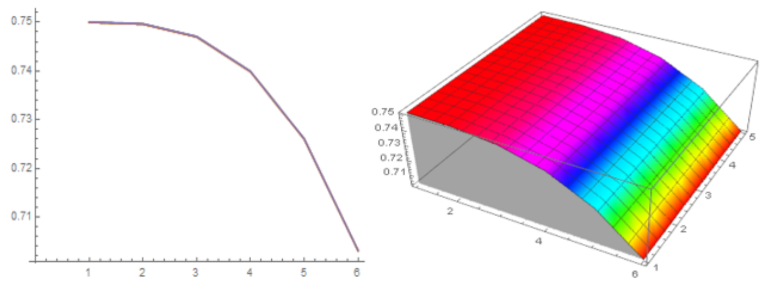

Figure 4. Right Bound Solution $\overline{E}(z = 0.5, r)$ of Example 4.2.

5. Results and Discussions

The utilization of Wolfram Mathematica 11.3 as a practical tool for solving integro-differential equations using the ADM, MADM, VIM, and HPM methods yields the following results, among others.

We implemented the technique suggested by HPM, ADM, MADM, and VIM for obtaining the approximate solution of linear fuzzy integro-differential equations along with a fuzzy parametric form with suitable initial conditions and source functions. According to the tables, the fuzzy MADM's left bound of errors at $z = 0.5$ is competitive with the fuzzy ADM, HPM, and VIM. However, when it came to the remaining errors, fuzzy MADM outperformed fuzzy ADM. Furthermore, for $z = 0.5$, the errors for the right bound of fuzzy HPM are shown to be one decimal point better than the fuzzy VIM. Graphs were created to represent the acquired approximate fuzzy solution.

6. Conclusion

With the use of this software tool (Wolfram Mathematica 11.3), it was a demonstrated that fuzzy MADM outperformed fuzzy ADM. Numerical results, tables, and graphical illustrations demonstrated that the suggested coupling of HPM and ADM is efficient and accurate for use with these fuzzy problems. Meanwhile, the results obtained by MADM are very close to the results obtained by VIM. When approximation results are discovered using the Modified Adomian decomposition approach and compared against exact solutions based on existing results. It can also be noted that the convergence is almost close. When comparing MADM to HPM, the MADM outperforms the HPM.

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