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Existence and Uniqueness of Solutions of Sobolev Type Second Order Integrodifferential Equation

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Abstract

The primary concern of this article is to establish the existence, uniqueness and continuous dependence on initial data of mild solutions of second order mixed integrodifferential equations of Sobolev type in Banach spaces. For this objective, we employ the idea of strongly continuous cosine family of operators, the modified version of Banach theorem and Grownwall's inequality. The model is demonstrated to elucidate the abstract conclusion.

Keywords: Integrodifferential equation; Modified fixed point theorem; Semigroup theory; Mild solution

MSC 2010 (or 2020) No.: 47H10, 34G20, 45J05, 45N05, 47D09

1. Introduction

Several qualitative properties, namely, existence, uniqueness, and continuous dependence, are established for various kind of differential and integrodifferential equations by Kumar and Kumar (2013), Kumar and Kumar (2014), Jain and Dhakne (2014), Guerfi and Ardjouni (2022) and Pazy (1983).

Second order differential and integrodifferential equations arise in the modelling of various problems associated with the vibration of hinged bars (Kreiger (1950)), the transverse motion of an extensible beam (Ball (1973a), Ball (1973b)) and many other phenomena in the physical and engineering sciences like nuclear physics, mathematical biology, and mechanics of materials and so on. Due to this, existence, uniqueness and continuous dependence on the initial data for second order differential and integrodifferential equations have investigated by many researchers (Akça et al. (2020), Kucche and Dhakne (2015), Xie (2017), Jain and Dhakne (2014), Rezapour et al. (2021)). In several problems it is very useful to treat the abstract differential equations of second order directly in comparison to transform them to first order equations. Second order abstract differential equations are investigated by many researchers (Barbu (1972), Fitzgibbon (1982), Goldstein (1969)). Balachandran et al. (2002) established the existence of solutions of nonlinear extensible beam equations. The approach of strongly continuous cosine families is very important and useful for the investigation of abstract second order equations. For more details, we refer to Travis and Webb (1978), Travis and Webb (1977), and Muslim et al. (2018).

On the other hand, there are several physical phenomena (for example, Kelvin-Voigt model for the non-Newtonian fluid flows (Mohan (2020)), thermodynamics (Chen and Curtin (1968)), sher in second order fluids (Huilgol (1968)) and the propagation of long waves of small Amplitudes (Benjamin et al. (1972))) that are modeled in Sobolev type equations. The authors Ahire et al. (2021) and Kavitha et al. (2021) are investigated the existence of solutions of different kinds of Sobolev type equations.

This paper is devoted to investigate the existence, uniqueness and continuous dependence of a mild solution of equation of (1)-(3) with less restriction by employing the modified version of Banach contraction theorem. Along with this, the concept of strongly continuous cosine family of operators is also applied. Some of outcomes which are investigated in Travis and Webb (1978), Kumar and Kumar (2014), Jain and Dhakne (2014) and Pazy (1983) are generalized and enhanced.

2. Existence and uniqueness of mild solution

Consider the second order mixed integrodifferential equation with Sobolev type is presented as follows:

$$\left(\chi y(t)\right)'' = Ay(t) + H\left(t, y(t), \int_0^t e(t,s,y(s))\,ds, \int_0^c f(t,s,y(s))\,ds\right),\ t \in [0,c], \tag{1}$$

$$y(0) = y_0, \tag{2}$$

$$y'(0) = \zeta \in E, \tag{3}$$

where H, e, f are given functions, which are specified later. Also, A be the infinitesimal generator of a strongly continuous cosine family of bounded linear operators $\{C(t): t \in R\}$ on E. Further, χ is a linear operator with domain and range contained in a Banach spaces W and E, respectively. Consider that $Z = C([0,c], E)$ is the Banach space of all continuous functions from $[0,c]$ into E endowed with supremum norm:

$$\|y\|_Z := sup\left\{\|y(t)\| : 0 \le t \le c\right\}.$$

Sequentially to show our main results, we state certain conditions on the operators A and χ. Suppose we have two Banach space E and W with norm $\|.\|$ and $|.|$ correspondingly. The operators $A : D(A) \subset W \to E$ and $\chi : D(\chi) \subset W \to E$ fulfill the properties which are mentioned below:

(A1) The linear operators A and χ are closed,
(A2) $D(\chi) \subset D(A)$ and χ is bijective, and
(A3) $\chi^{-1} : E \to D(\chi)$ is continuous,

with the help of the properties **(A1), (A2)** and the closed graph theorem mean the boundedness of the linear operator $A\chi^{-1} : E \to E$. Also, we set $\|\chi^{-1}\| \le F_0$ and $\|\chi\| = F_1, \forall 0 \le t \le c$.

Suppose that there exist positive constants $L_0 \ge 1$ and L_1 in such a manner that $\|C(t)\| \le L_0$ and $\|S(t)\| \le L_1$. Here the family $\{C(t) : t \in R\}$ and $\{S(t) : t \in R\}$ is the strongly cosine and sine family of operators. For detailed information about strongly continuous cosine family of operators, see, for instance, Travis and Webb (1978) and Fattorini (1985).

Definition 2.1.

Assume that $H \in (0, c; E)$. The function $y \in Z$ is given as follows:

$$y(t) = \begin{cases} \chi^{-1}C(t)\chi y_0 + \chi^{-1}S(t)\chi\zeta \\ +\int_0^t \chi^{-1}S(t-s)H\left(s, y(s), \int_0^s e(s,\sigma,y(\sigma))d\sigma, \int_0^c f(s,\sigma,y(\sigma))d\sigma\right)ds, \ t \in [0,c], \end{cases} \tag{4}$$

is called a mild solution of the equations (1)-(3).

Our further discussions are based on the modified version of Banach contraction principle (see Siddiqi (1986), p. 196).

To achieve the desired results, we require the assumptions which are stated as follows:

(A4) The function $H : [0,c] \times E \times E \times E \to E$ is continuous in t on $[0,c]$ and there exists a constant $N_0 > 0$ such that

$$\|H(t, x_1, y_1, z_1) - H(t, x_2, y_2, z_2)\| \le N_0\left(\|x_1 - x_2\| + \|y_1 - y_2\| + \|z_1 - z_2\|\right),$$

for $x_i, y_i, z_i \in E$, $i = 1,2$.

(A5) The functions $e, f : [0,c]\times[0,c]\times E \to E$ are continuous in t,s on $[0,c]$ and there exist two positive constants N_1, N_2 such that

$$\|e(t,s,x_1) - e(t,s,x_2)\| \le N_1(\|x_1 - x_2\|),$$
$$\|f(t,s,x_1) - f(t,s,x_2)\| \le N_2(\|x_1 - x_2\|),$$

for $x_i \in E$, $i = 1,2$.

Theorem 2.1.

Assume that the assumptions **(A1) - (A5)** hold. Then, the initial value problem (1)-(3) has a unique mild solution $y \in Z$ on $[0,c]$.

Proof:

Define the operator $\Gamma : Z \to Z$ as follows:

$$(\Gamma y)(t) = \begin{cases} \chi^{-1}C(t)\chi y_0 + \chi^{-1}S(t)\chi\zeta \\ +\int_0^t \chi^{-1}S(t-s)H\left(s, y(s), \int_0^s e(s,\sigma,y(\sigma))d\sigma, \int_0^c f(s,\sigma,y(\sigma))d\sigma\right)ds, t\in[0,c] \end{cases}. \quad (5)$$

Now, we see that the mild solution of (1)-(3) is a fixed point of the operator equation $\Gamma y = y$. We assume that $y, v \in Z$. With the use of the equation (5) and the assumptions, we get:

$$\begin{aligned} &\|(\Gamma y)(t) - (\Gamma v)(t)\| \\ &\le \int_0^t \|\chi^{-1}\|\|S(t-s)\|\Bigg\|H\left(s, y(s), \int_0^s e(s,\sigma,y(\sigma))d\sigma, \int_0^c f(s,\sigma,y(\sigma))d\sigma\right) \\ &\quad -H\left(s, v(s), \int_0^s e(s,\sigma,v(\sigma))d\sigma, \int_0^c f(s,\sigma,v(\sigma))d\sigma\right)\Bigg\|ds \\ &\le L_1F_0\int_0^t N_0\left[\|y(s)-v(s)\|_Z + \int_0^s N_1\|y-v\|_Z\,d\sigma + \int_0^c N_2\|y-v\|_Z\,d\sigma\right]ds \\ &\le L_1N_0F_0\int_0^t\|y(s)-v(s)\|_Z ds + L_1N_0F_0\int_0^t N_1\int_0^s\|y-v\|_Z\,d\sigma ds \\ &\quad + L_1N_0F_0\int_0^t N_2\int_0^c\|y-v\|_Z\,d\sigma ds \end{aligned}$$

$$
\begin{aligned}
&\leq L_1 N_0 F_0 \|y-v\|_Z\, t + L_1 N_0 F_0 N_1 \|y-v\|_Z \frac{t^2}{2} + L_1 N_0 F_0 N_2 \|y-v\|_Z \frac{t^2}{2} \\
&\leq L_1 N_0 F_0 \|y-v\|_Z\, t + L_1 N_0 F_0 N_1 c \|y-v\|_Z \frac{t}{2} + L_1 N_0 F_0 N_2 c \|y-v\|_Z \frac{t}{2} \\
&\leq L_1 N_0 F_0 \|y-v\|_Z\, t + L_1 N_0 F_0 N_1 c \|y-v\|_Z\, t + L_1 N_0 F_0 N_2 c \|y-v\|_Z\, t \\
&\leq L_1 N_0 F_0 \left(1 + N_1 c + N_2 c\right) t \|y-v\|_Z ,
\end{aligned} \tag{6}
$$

$$
\begin{aligned}
&\left\|\left(\Gamma^2 y\right)(t) - \left(\Gamma^2 v\right)(t)\right\| = \left\|\left(\Gamma(\Gamma y)\right)(t) - \left(\Gamma(\Gamma v)\right)(t)\right\| \\
&\quad = \left\|\left(\Gamma(y_1)\right)(t) - \left(\Gamma(v_1)\right)(t)\right\| \\
&\qquad + \int_0^t \|\chi^{-1}\| \|S(t-s)\| \Bigg\| H\left(s, y_1(s), \int_0^s e\left(s,\sigma,y_1(\sigma)\right) d\sigma, \int_0^c f\left(s,\sigma,y_1(\sigma)\right) d\sigma\right) \\
&\qquad - H\left(s, v_1(s), \int_0^s e\left(s,\sigma,v_1(\sigma)\right) d\sigma, \int_0^c f\left(s,\sigma,v_1(\sigma)\right) d\sigma\right)\Bigg\| ds \\
&\quad \leq \int_0^t L_1 N_0 F_0 \Big[\left\|y_1(s) - v_1(s)\right\|_Z \\
&\qquad + \int_0^s N_1 \left\|y_1(\sigma) - v_1(\sigma)\right\|_Z d\sigma + \int_0^c N_2 \left\|y_1(\sigma) - v_1(\sigma)\right\|_Z d\sigma\Big] ds \\
&\quad \leq L_1 N_0 F_0 \int_0^t \left\|(\Gamma y)(s) - (\Gamma v)(s)\right\| ds \\
&\qquad + L_1 N_0 F_0 \int_0^t N_1 \int_0^s \left\|(\Gamma y)(s) - (\Gamma v)(s)\right\| d\sigma ds \\
&\qquad + L_1 N_0 F_0 \int_0^t N_2 \int_0^c \left\|(\Gamma y)(s) - (\Gamma v)(s)\right\| d\sigma ds \\
&\quad \leq L_1 N_0 F_0 \int_0^t \left[L_1 N_0 F_0 \left(1 + N_1 c + N_2 c\right) t\right] \|y-v\|_Z\, ds \\
&\qquad + L_1 N_0 F_0 \int_0^t N_1 \int_0^s \left[L_1 N_0 F_0 \left(1 + N_1 c + N_2 c\right) t\right] \|y-v\|_Z\, d\sigma ds \\
&\qquad + L_1 N_0 F_0 \int_0^t N_2 \int_0^c \left[L_1 N_0 F_0 \left(1 + N_1 c + N_2 c\right) t\right] \|y-v\|_Z\, ds \\
&\quad \leq L_1^{\,2} N_0^{\,2} F_0^{\,2} \left(1 + N_1 c + N_2 c\right) \|y-v\|_Z \left[\frac{t^2}{2!} + N_1 \frac{t^3}{3!} + N_2 \frac{t^3}{3!}\right] \\
&\quad \leq L_1^{\,2} N_0^{\,2} F_0^{\,2} \left(1 + N_1 c + N_2 c\right) \|y-v\|_Z \left[\frac{t^2}{2!} + N_1 c \frac{t^2}{3!} + N_2 c \frac{t^2}{3!}\right] \\
&\quad \leq L_1^{\,2} N_0^{\,2} F_0^{\,2} \left(1 + N_1 c + N_2 c\right) \|y-v\|_Z \left[\frac{t^2}{2!} + N_1 c \frac{t^2}{2!} + N_2 c \frac{t^2}{2!}\right]
\end{aligned}
$$

$$\leq L_1^{\ 2} N_0^{\ 2} F_0^{\ 2} \left(1+N_1 c+N_2 c\right)^2 \left\|y-v\right\|_Z \frac{t^2}{2!}.$$

With the similar way as we have used above, we get

$$\left\|\left(\Gamma^n y\right)(t)-\left(\Gamma^n v\right)(t)\right\| \leq \frac{\left[L_1 N_0 F_0 \left(1+N_1 c+N_2 c\right) t\right]^n}{n!} \left\|y-v\right\|_Z .$$

For n large enough, $\frac{\left[L_1 N_0 F_0 \left(1+N_1 c+N_2 c\right) t\right]^n}{n!} < 1$. Therefore, for a positive integer n, Γ^n is a contraction in Z. From the concept of modified version of Banach contraction principle (see Siddiqi (1986)), the function Γ has a unique fixed point y in Z and any fixed point of Γ is the mild solution of the equation (1)-(3) on $0 \leq t \leq c$ in such a way that $y(t) \in E$ for $0 \leq t \leq c$. So, the proof of the Theorem 2.1 is complete. ■

3. Continuous Dependence on Initial Data

Theorem 3.1.

Suppose that the assumptions **(A1) - (A5)** hold. Consider that y_1, y_2 are the mild solutions of the equation (1)-(3) for $0 \leq t \leq c$ corresponding to the initial conditions $y_1(0) = y_0^*, y'_1(0) = \zeta^*$ and $y_2(0) = y_0^{**}, y'_2(0) = \zeta^{**}$, respectively, and $y_1, y_2 \in Z$, then the inequality which is stated below,

$$\left\|y_1 - y_2\right\|_Z \leq \left[F_0 L_0 F_1 \left\|\left(y_0^* - y_0^{**}\right)\right\| + F_0 L_1 F_1 \left\|\left(\zeta^* - \zeta^{**}\right)\right\|\right] \exp\left(F_0 L_1 N_0 c \left(1+N_1 c+N_2 c\right)\right), \quad (7)$$

is true.

Proof:

Consider y_1, y_2 are the two mild solutions of the problems (1)-(3) on $[0,c]$ with the initial conditions $y_1(0) = y_0^*, y'_1(0) = \zeta^*$ and $y_2(0) = y_0^{**}, y'_2(0) = \zeta^{**}$, respectively, and $y_1, y_2 \in Z$. By applying the equation (4) and assumptions **(A4)** and **(A5)**, we get

$$\begin{aligned}\left\|y_1(t) - y_2(t)\right\| \leq{}& \left\|\chi^{-1} C(t) \chi\right\| \left\|\left(y_0^* - y_0^{**}\right)\right\| + \left\|\chi^{-1} S(t) \chi\right\| \left\|\left(\zeta^* - \zeta^{**}\right)\right\| \\ &+ \int_0^t \left\|\chi^{-1}\right\| \left\|S(t-s)\right\| \left\|\left[H\left(s, y_1(s), \int_0^s e\left(s,\sigma,y_1(\sigma)\right) d\sigma, \int_0^c f\left(s,\sigma,y_1(\sigma)\right) d\sigma\right)\right.\right. \\ &\left.\left. - H\left(s, y_2(s), \int_0^s e\left(s,\sigma,y_2(\sigma)\right) d\sigma, \int_0^c f\left(s,\sigma,y_2(\sigma)\right) d\sigma\right)\right]\right\| ds\end{aligned}$$

$$
\begin{aligned}
&\leq F_0 L_0 F_1 \left\|\left(y_0^* - y_0^{**}\right)\right\| + F_0 L_1 F_1 \left\|\left(\zeta^* - \zeta^{**}\right)\right\| \\
&\quad + \int_0^t F_0 L_1 N_0 \left[\left\|y_1(s) - y_2(s)\right\| + \int_0^s N_1 \left\|y_1(\sigma) - y_2(\sigma) d\sigma\right\| \right. \\
&\qquad\qquad \left. + \int_0^c N_2 \left\|y_1(\sigma) - y_2(\sigma)\right\| d\sigma \right] ds \\
&\leq F_0 L_0 F_1 \left\|\left(y_0^* - y_0^{**}\right)\right\| + F_0 L_1 F_1 \left\|\left(\zeta^* - \zeta^{**}\right)\right\| \\
&\quad + \int_0^t F_0 L_1 N_0 \left[\left\|y_1(s) - y_2(s)\right\| + \int_0^s N_1 \sup_{\sigma \in [0,c]} \left\|y_1(\sigma) - y_2(\sigma)\right\| d\sigma \right. \\
&\quad \left. + \int_0^c N_2 \sup_{\sigma \in [0,c]} \left\|y_1(\sigma) - y_2(\sigma)\right\| d\sigma \right] ds \\
&\leq F_0 L_0 F_1 \left\|\left(y_0^* - y_0^{**}\right)\right\| + F_0 L_1 F_1 \left\|\left(\zeta^* - \zeta^{**}\right)\right\| \\
&\qquad + \int_0^t F_0 L_1 N_0 \left(1 + cN_1 + cN_2\right) \left\|y_1(s) - y_2(s)\right\|_Z ds.
\end{aligned}
$$

Applying Grownwall's inequality (see Gronwall (1919)), we obtain

$$
\left\|y_1 - y_2\right\|_Z \leq \left[F_0 L_0 F_1 \left\|\left(y_0^* - y_0^{**}\right)\right\| + F_0 L_1 F_1 \left\|\left(\zeta^* - \zeta^{**}\right)\right\| \right] \exp\left(F_0 L_1 N_0 c \left(1 + N_1 c + N_2 c\right)\right),
$$

and so, (7) holds. Therefore, the proof is complete. ■

4. Application

To describe the application of our abstract theory, consider the semilinear integrodifferential system as follows:

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left[\frac{\partial}{\partial t}\varpi(\xi,t) - \varpi_{\xi\xi}(\xi,t)\right] = \frac{\partial^2}{\partial \xi^2}\varpi(\xi,t) \\
&\quad + K\left(t, \varpi(\xi,t), \int_0^t \eta\left(t, \varpi(\xi,s)\right), \int_0^c \kappa\left(t, \varpi(\xi,s)\right) ds\right), 0 \leq \xi \leq \pi, t \in [0,c],
\end{aligned} \tag{8}
$$

$$
\varpi(0,t) = \varpi(\pi,t) = 0, 0 \leq t \leq c, \tag{9}
$$

$$
\varpi(\xi,0) = y_0(\xi), 0 \leq \xi \leq \pi, \tag{10}
$$

$$
\frac{\partial}{\partial t}\varpi(\xi,0) = \zeta(\xi), 0 \leq \xi \leq \pi, \tag{11}
$$

where the function $K:[0,c]\times R^3 \to R$, $\eta, \kappa:[0,c]\times[0,c]\times R \to R$ are continuous. It is assumed that the function K, η and κ fulfill the following conditions:

For every $0 \le t \le c$ and $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{z}_1, \tilde{z}_2, \tilde{z}_3 \in R$, there exists a constant

$$\left|K\left(t, \tilde{x}_1, \tilde{x}_2, \tilde{x}_3\right) - K\left(t, \tilde{z}_1, \tilde{z}_2, \tilde{z}_3\right)\right| \le C_0\left(\left|\tilde{x}_1 - \tilde{z}_1\right| + \left|\tilde{x}_2 - \tilde{z}_2\right| + \left|\tilde{x}_3 - \tilde{z}_3\right|\right),$$
$$\left|\eta\left(t, s, \tilde{x}_1\right) - \eta\left(t, s, \tilde{x}_2\right)\right| \le C_1\left(\left|\tilde{x}_1 - \tilde{x}_2\right|\right),$$
$$\left|\kappa\left(t, s, \tilde{x}_1\right) - \kappa\left(t, s, \tilde{x}_2\right)\right| \le C_2\left(\left|\tilde{x}_1 - \tilde{x}_2\right|\right).$$

Let us take $E = L^2[0, \pi]$. Define the operator $A : D(A) \subset E \to E$ and $\chi : D(\chi) \subset E \to E$ by $A(\varpi) = \varpi_{\xi\xi}, , \chi(\varpi) = \varpi - \varpi_{\xi\xi}$ where each domain $D(A)$ and $D(\chi)$ is given by $\{\varpi \in E : \varpi, \varpi_{\xi}$ are absolutely continuous, $\varpi_{\xi\xi} \in E$ and $\varpi(0) = \varpi(\pi) = 0\}$.

Here, explicitly the operator A is the infinitesimal generator of a strongly continuous cosine family $\{C(t)\}_{t \in R}$ on E. Then the operator A and χ have infinite series representation which are given as follows:

$$A\varpi = -\sum_{n=1}^{\infty} n^2 (\varpi, \varpi_n)\varpi_n, \varpi \in D(A),$$
$$\chi\varpi = \sum_{n=1}^{\infty} \left(1 + n^2\right)(\varpi, \varpi_n)\varpi_n, \varpi \in D(\chi),$$

where $\varpi_n(\xi) = \left(\frac{\sqrt{2}}{\pi}\right)\sin n\xi, n = 1, 2, \ldots$ is an orthogonal set of vectors of A. Again, for $\varpi \in E$ it holds the properties which are given as:

$$\chi^{-1}\varpi = \sum_{n=1}^{\infty} \frac{1}{\left(1 + n^2\right)}(\varpi, \varpi_n)\varpi_n,$$
$$-A\chi^{-1}\varpi = \sum_{n=1}^{\infty} \frac{-n^2}{\left(1 + n^2\right)}(\varpi, \varpi_n)\varpi_n,$$
$$C(t)\varpi = \sum_{n=1}^{\infty} \frac{\cos(nt)}{1 + n^2}(\varpi, \varpi_n)\varpi_n$$

and

$$S(t)\varpi = \sum_{n=1}^{\infty} \frac{\sin(nt)}{n\left(1 + n^2\right)}(\varpi, \varpi_n)\varpi_n.$$

As a consequence, $\|C(t)\| = \|S(t)\| \le 1$ and for $t \in R$, $S(t)$ is compact.

The function $H:[0,c]\times E\times E\times E\to E$, $e,f:[0,c]\times[0,c]\times E\to E$ is given as follows:

$$\begin{aligned}
&H(t,u,v,w)(\xi)=K(t,u(\xi),v(\xi),w(\xi)),\\
&e(t,s,u)(\xi)=\eta(t,s,u(\xi)),\\
&f(t,s,u)(\xi)=\kappa(t,s,u(\xi)),
\end{aligned}$$

where $u,v,w\in E$ and $0\le\xi\le\pi$, $t\in[0,c]$. With the choices for the functions H,e,f and the operator A which is taken above, the equations (8)-(11) can be formulated as in the following abstract form in Banach space E:

$$\begin{aligned}
&(\chi y(t))'' = Ay(t)+H\left(t,y(t),\int_0^t e(t,s,y(s))ds,\int_0^c f(t,s,y(s))ds\right), t\in[0,c],\\
&y(0)=y_0,\\
&y'(0)=\zeta\in E.
\end{aligned}$$

Since all the assumptions of Theorem 2.1 are fulfilled, then (8)-(11) contains a solution on $t\in[0,c]$.

5. Conclusion

We summarize our work with the objectives achieved, i.e., we establish some qualitative properties of mild solution of second order mixed integrodifferential equations in Banach spaces. For this purpose, we apply the ideas of modified version of Banach contraction theorem, semigroup theory and the strongly continuous cosine family of operators. Moreover, we also give an example to illustrate the theory.

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