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# The Distinguishing Number of Some Special Kind of Graphs

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## Abstract

In the present study, the distinguishing number of some different graphs is examined where different graphs like the coconut tree graph, firecracker graph, jellyfish graph, triangular book graph, and banana tree graph have been taken into account. The major goal of the proposed study is to understand the distinguishing number of different graphs for better insights. It is evident from the results that the distinguishing numbers and automorphism groups of the above-mentioned graphs have been carried out successfully.

**Keywords:** Distinguishing number; Automorphism of graph; Coconut tree graph; Firecracker graph, Jellyfish graph; Triangular book graph; Banana tree graph

**MSC 2010 No.:** 05C60, 05C78

## 1. Introduction

The distinguishing number of graphs is a concept of Graph Theory that measures how well a graph can be distinguished based on its vertex labeling. Albertson and Collins (1996) introduced the distinguishing number, inspired by Frank Rubin's key puzzle. The distinguishing number of graph $G$ is the minimum number of labels to the vertices of graph $G$; those are only preserved by a trivial automorphism of graph $G$ (Albertson and Collins (1996)). The distinguishing number of graphs can be used as a tool to test whether two graphs are isomorphic or not. If the distinguishing numbers of two graphs are different, it means that the graphs are non-isomorphic.

A graph $G$ that has only trivial automorphism always has $1$ distinguishing number. For a graph $G$ with $n$ vertices, the distinguishing number $G$ is $n$ if and only if $G$ is a complete graph with $n$ vertices. Further, the distinguishing number for a path with $n$ vertices is $2$ if $n$ is greater than or equal to $3$. For a cycle with $n$ vertices, the distinguishing number is $3$ if $n = 3, 4, 5$, and $2$ if $n$ is greater than or equal to $6$. For a complete bipartite graph with $n$ vertices, the distinguishing number is $n + 1$ if $n$ is greater than or equal to $4$ (Alikhani and Soltani (2016)).

Albertson (2005) proposed that the distinguishing number of $G^m$ is $2$ if $m$ is greater than or equal to $5$, where $G$ is a connected prime graph. For a connected prime graph with at least $5$ vertices, the distinguishing number of $G^3$ is $2$. Further, Albertson conjectured that $G$ need not be a prime graph (Albertson (2005)). In the continuation of distinguishing numbers, Klavžar and Zhu (2007) proved that the distinguishing number of the square and higher powers of a connected graph with $n$ vertices is $2$ with respect to the Cartesian product, where $n$ is greater than or equal to $3$ (Klavžar and Zhu (2007)).

Bogstad and Cowen (2004) studied the distinguishing number for the $n$-dimensional hypercube graph. It has been proved that the distinguishing number of the $n$-dimensional hypercube graph is $3$ if $n = 2, 3$, and $2$ if $n$ is greater than or equal to $4$. Further, the distinguishing number of square of the $n$-dimensional hypercube has been determined (Bogstad and Cowen (2004)).

Fisher and Isaak (2008) studied the distinguishing number of the Cartesian product of complete graphs (Fisher and Isaak (2008)). The distinguishing number of friendship graph, book graph, and corona product of two graphs have been incorporated. Further, based on the corona product of two graphs, some interesting results for distinguishing number of corona product of two graphs have been illustrated (Alikhani and Soltani (2016)).

Alikhani and Soltani (2019) studied the distinguishing number of join of two graphs, strong product of two graphs, and Kronecker product of complete graphs. They determined the distinguishing number and the distinguishing index of some important families of graphs which are useful in chemistry. The distinguishing number of a graph $Q(m, n)$ and Dutch windmill graph have been carried out. Further, some results on distinguishing number of circuit graph, bouquet of graphs, chain of graphs, link of graphs, spiro-chain, and polyphenylenes have been provided (Alikhani and Soltani (2019)).

Alikhani and Soltani (2020) studied the distinguishing number and the distinguishing index of line graph of connected graphs. Further, some results for the distinguishing number and the distinguishing index of the graphoidal graph of a simple connected graph have been carried out (Alikhani and Soltani (2020)).

Boutin et al. (2020) studied the distinguishing number of the traditional and generalized Mycielskian graphs. Further, the conjecture of Alikhani and Soltani has been proved, which states "If the number of isolated vertices in generalized Mycielskian graph of a graph $G$ is at most the distinguishing number of $G$, then the distinguishing number of generalized Mycielskian graph of $G$ is at most the distinguishing number of $G$, where $G$ is different from $K_1$ and $K_2$" (Boutin et al. (2020)).

Amouzegar (2021) studied the distinguishing number of the hierarchical products of graphs. The distinguishing number of the hierarchical products of two complete graphs has been determined, and the upper bound for the distinguishing number of the hierarchical product of two connected graphs has been carried out. Further, the distinguishing number for the hierarchical product of an arbitrary graph with some specific graph like a hierarchical power of $K_2$, a complete graph, or a tree, has been determined (Amouzegar (2021)).

The paint cost of $d$-distinguishing labeling and the frugal distinguishing number of a graph have been introduced by Boutin (2021). Some upper bounds for the distinguishing number and the distinguishing index of a graph have been determined based on its domination number (Boutin (2021)).

Rahadi et al. (2022) determined the distinguishing number of the generalized theta graph. Further, the relation between distinguishing number and the partition dimension of a graph has been carried out successfully (Rahadi et al. (2022)).

Shekarriz et al. (2023) introduced the concept of distinguishing threshold for a graph. It has been proved that for any $m$ other than 2, there are infinitely many graphs whose distinguishing thresholds are $m$. Further, the distinguishing thresholds for generalized Johnson graphs have been determined after studying their automorphisms and the distinguishing numbers (Shekarriz et al. (2023)).

## 2. Main Results

**Theorem 2.1.**

The distinguishing number of a coconut tree graph $CT(m,n)$ is $n$.

***Proof:***

Let $CT(m,n)$ be a coconut tree graph constructed from a path $P_m$ by connecting its end vertex to $n$-pendant edges. Let $v_1, v_2, \dots, v_m$ be the consecutive vertices of $P_m$ in $CT(m,n)$, and $u_1, u_2, \dots, u_n$ be the pendant vertices of $n$-pendant edges which are adjacent to the end vertex $v_m$ of $P_m$ in $CT(m,n)$. We have $\deg(v_m) = n+1$ and $\deg(u_i) = 1$, for all $1 \leq i \leq n$. Also, the neighbor of all $u_i$ is $v_m$ in $CT(m,n)$. So, the vertices $u_1, u_2, \dots, u_n$ can mapped to each other by an automorphism $\phi$ of $CT(m,n)$, and the vertices $v_1, v_2, \dots, v_m$ are fixed by $\phi$. Thus, $|Aut\,(CT(m,n))| = n!$ and $Aut\,(CT(m,n)) \cong S_n$.

Since $u_1, u_2, \dots, u_n$ can be mapped to each other by any non-trivial automorphism $\phi$ of $CT(m,n)$, so we have to assign the vertices $u_1, u_2, \dots, u_n$ with different labels. We assign the vertices $v_1, v_2, \dots, v_m$ with label $1$, and assign the vertices $u_1, u_2, \dots, u_n$ with labels $1, 2, \dots, n$, respectively. Thus, the assignment of above mentioned labeling is distinguishing because the vertices $u_1, u_2, \ldots, u_m$ are not fixed by any non-trivial automorphism of $CT(m,n)$. Hence, $D\left[CT(m,n)\right] = n$. ■

**Example 2.1.**

Figure 1 demonstrates the distinguishing number of coconut tree $CT(5,5)$. The left figure shows the vertex labeling while the right figure shows their corresponding distinguishing labeling for coconut tree $CT(5,5)$.

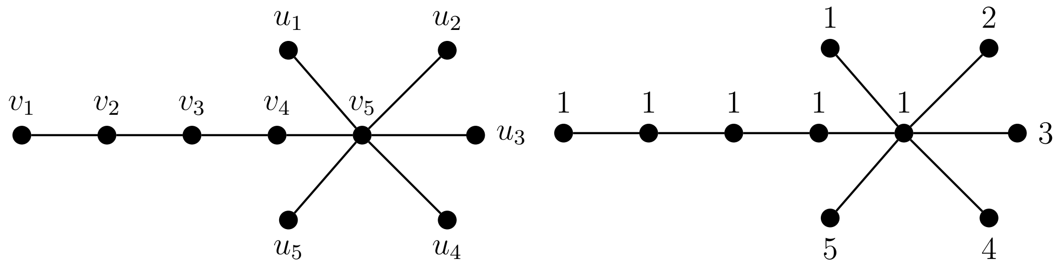


**Figure 1.** The vertex labeling and distinguishing labeling for coconut tree $CT(5,5)$

**Theorem 2.2.**

The distinguishing number of a firecracker $F(m,n)$ is $n-1$.

***Proof:***

Let $F(m,n)$ be a firecracker graph constructed by the series of interconnected $m$-copies of star graph $K_{1,n}$ by connecting one leaf from each. Let $v_1, v_2, \ldots, v_m$ be the consecutive vertices of $P_m$ in $F(m,n)$. Let $u_1, u_2, \ldots, u_m$ be the central vertices of $m$-copies of star graphs $K_{1,n}$ which are adjacent to the vertices $v_1, v_2, \ldots, v_m$, respectively, in $F(m,n)$. For all $1 \leq i \leq m$, let $v_i = u_i^n$, and $u_i^1, u_i^2, \ldots, u_i^{n-1}$ be the rest vertices of star graph $K_{1,n}$ which are adjacent to $u_i$ in $F(m,n)$. We have $\deg\left(u_i\right) = n$ and $\deg\left(u_i^j\right) = 1$, for all $1 \leq i \leq m$ and $1 \leq j \leq n-1$. So, if $\phi$ is an automorphism of $F(m,n)$, then we have the following possible cases for $\phi$:

(i) The vertices $u_i^1, u_i^2, \ldots, u_i^{n-1}$ can be mapped to each other by $\phi$, and the vertices $v_1, v_2, \ldots, v_m$, $u_1, u_2, \ldots, u_m$ are fixed by $\phi$, for all $1 \leq i \leq m$.

(ii) The vertices $u_i^1, u_i^2, \ldots, u_i^{n-1}$ can be mapped to the vertices $u_{m-i+1}^1, u_{m-i+1}^2, \ldots, u_{m-i+1}^{n-1}$ by $\phi$, and $\phi\left(v_i\right) = v_{m-i+1}$ and $\phi\left(u_i\right) = u_{m-i+1}$, for all $1 \leq i \leq m$.

Thus, $\left|Aut\left(F(m,n)\right)\right| = 2\left((n-1)!\right)^m$ and $Aut\left(F(m,n)\right) \cong \mathbb{Z}_2 \times S_{n-1}^m$. Since $u_i^1, u_i^2, \ldots, u_i^{n-1}$ can be mapped to each other by any non-trivial automorphism $\phi$ of $F(m,n)$, so we have to assign the vertices $u_i^1, u_i^2, \ldots, u_i^{n-1}$ with different labels. We assign the vertices $u_i^1, u_i^2, \ldots, u_i^{n-1}$ with labels $1, 2, \ldots, n-1$, respectively. Further, assign the vertices $v_1, v_2, \ldots, v_{\lfloor \frac{m+1}{2} \rfloor}, u_1, u_2, \ldots, u_{\lfloor \frac{m+1}{2} \rfloor}$ with label 1, and assign the vertices $v_{\lfloor \frac{m+1}{2} \rfloor+1}, v_{\lfloor \frac{m+1}{2} \rfloor+2}, \ldots, v_m, u_{\lfloor \frac{m+1}{2} \rfloor+1}, u_{\lfloor \frac{m+1}{2} \rfloor+2}, \ldots, u_m$ with label 2.

Thus, the assignment of above-mentioned labeling is distinguishing because all the labeled vertices are not fixed by any non-trivial automorphism $\phi$ of $F(m,n)$. Hence, $D\left[F(m,n)\right] = n-1$. ■

**Example 2.2.**

Figure 2 demonstrates the distinguishing number of firecracker $F(4,4)$. The left figure shows the vertex labeling while the right figure shows their corresponding distinguishing labeling for firecracker $F(4,4)$.

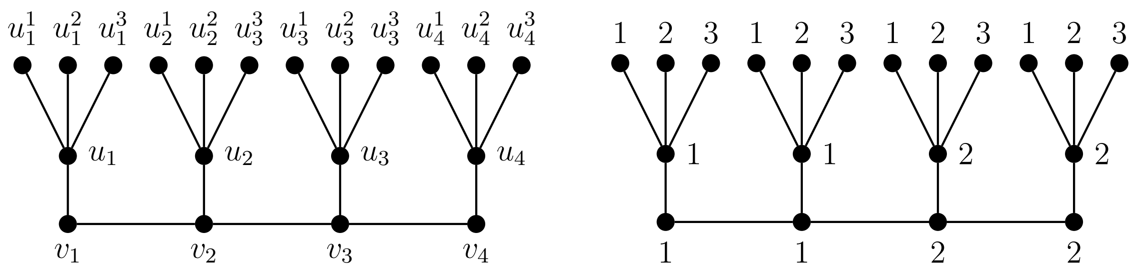


**Figure 2.** The distinguishing labeling for firecracker $F(4,4)$

**Theorem 2.3.**

For a triangular book graph $B_{3,n}$,

(1) $D\left[B_{3,n}\right] = 3$ when $n = 1$.
(2) $D\left[B_{3,n}\right] = n$ when $n > 1$.

***Proof:***

**Case 1:** $n = 1$. In this case, $B_{3,n}$ becomes a cycle $C_3$. Since $D\left[C_3\right] = 3$, so $D\left[B_{3,n}\right] = 3$.

**Case 2:** $n > 1$. In this case, let $B_{3,n}$ be a triangular book graph with the vertex set $V\left(B_{3,n}\right) = \{u, v, u_i : 1 \le i \le n\}$ and the edge set $E\left(B_{3,n}\right) = \{uv, uu_i, vu_i : 1 \le i \le n\}$. We have $\deg\left(u\right) = n+1, \deg\left(v\right) = n+1$ and $\deg\left(u_i\right) = 2$, for all $1 \le i \le n$. Also, the neighbors of all $u_i$ are $u$ and $v$ in $B_{3,n}$. So, if $\phi$ is an automorphism of $B_{3,n}$, then we have the following possible cases for $\phi$:

(i) The vertices $u_1, u_2, \ldots, u_n$ can be mapped to each other by $\phi$, and the vertices $u, v$ are fixed by $\phi$.

(ii) The vertices $u_1, u_2, \ldots, u_n$ can be mapped to each other by $\phi$, along with $\phi\left(u\right) = v$ and $\phi\left(v\right) = u$.

Thus, $\left|Aut\left(B_{3,n}\right)\right| = 2n!$ and $Aut\left(B_{3,n}\right) \cong \mathbb{Z}_2 \times S_n$. Since $u_1, u_2, \ldots, u_n$ can be mapped to each other by any non-trivial automorphism $\phi$ of $B_{3,n}$, we have to assign these vertices with different labels. We assign the vertices $u_1, u_2, \ldots, u_n$ with labels $1, 2, \ldots, n$, respectively. Further, assign the vertices $u, v$ with labels $1, 2$, respectively.

Thus, the assignment of the above-mentioned labeling is distinguishing because all the labeled vertices are not fixed by any non-trivial automorphism $\phi$ of $B_{3,n}$. Hence, $D\left[B_{3,n}\right] = n$. ■

**Example 2.3.**

Figure 3 demonstrates the distinguishing number of triangular book graph $B_{3,1}$ when $n = 1$. Further, Figure 4 demonstrates the distinguishing number of triangular book graph $B_{3,3}$ when $n > 1$. In both figures, the left figure shows the vertex labeling while the right figure shows their corresponding distinguishing labeling for triangular book graph.

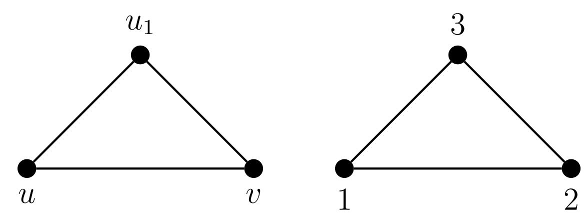


**Figure 3.** The vertex labeling and distinguishing labeling for triangular book graph $B_{3,1}$ when $n = 1$

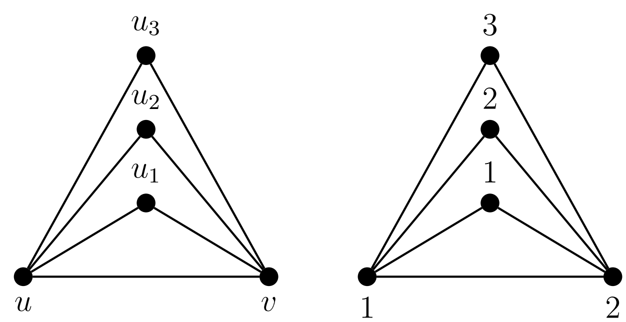


**Figure 4.** The vertex labeling and distinguishing labeling for triangular book graph $B_{3,3}$ when $n > 1$

**Theorem 2.4.**

For a jellyfish graph $J(m, n)$,

(1) $D\left[J(n, n)\right] = 2$ when $n = 1$; and $D\left[J(n, n)\right] = n$ when $n > 1$, where $m = n$.
(2) $D\left[J(m, n)\right] = \max\{m, n\}$, where $m \neq n$.

***Proof:***

Let $J(m, n)$ be a jellyfish graph with the vertex set $V\left(J(m, n)\right) = \{u, v, x, y\} \cup \{u_i, v_j : 1 \leq i \leq m, 1 \leq j \leq n\}$ and the edge set $E\left(J(m, n)\right) = \{xy, ux, uy, vx, vy\} \cup \{uu_i, vv_i : 1 \leq i \leq m, 1 \leq j \leq n\}$.

**Case 1:** $m = n$. There are two cases as follows:

**(a)** If $n = 1$, then $|Aut\left(J(1, 1)\right)| = 4$ and $Aut\left(J(1, 1)\right) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.
We assign the vertices $u, x, u_1$ with label 1, and assign the vertices $v, y, v_1$ with label 2. Thus, the

assignment of the above-mentioned labeling is distinguishing because all the labeled vertices are not fixed by any non-trivial automorphism $\phi$ of $J(1,1)$. Hence, $D\left[J(1,1)\right] = 2$.

**(b)** If $n > 1$, then we have $\deg(u_i) = \deg(v_i) = 1, \deg(u) = \deg(v) = n+2$ and $\deg(x) = \deg(y) = 3$, for all $1 \leq i \leq n$. So, if $\phi$ is an automorphism of $J(n,n)$, then we have the following possible cases for $\phi$:

(i) The vertices $u_1, u_2, \ldots, u_n$ can be mapped to each other by $\phi$, also the vertices $v_1, v_2, \ldots, v_n$ can be mapped to each other by $\phi$, and the vertices $u, v$ are fixed by $\phi$.

(ii) The set of vertices $u_1, u_2, \ldots, u_n$ can be mapped to the set of vertices $v_1, v_2, \ldots, v_n$ by $\phi$, along with $\phi(u) = v$ and $\phi(v) = u$.

Also, in the above-mentioned possibilities, $\phi$ can fix the vertices $x$ and $y$, or $\phi(x) = y$ and $\phi(y) = x$. Thus, $|Aut\,(J(n,n))| = 4(n!)^2$ and $Aut\,(J(n,n)) \cong \mathbb{Z}_2^2 \times S_n^2$. Since $u_1, u_2, \ldots, u_n$ can be mapped to each other by any non-trivial automorphism $\phi$ of $J(n,n)$, we have to assign the vertices $u_1, u_2, \ldots, u_n$ with different labels. We assign the vertices $u_1, u_2, \ldots, u_n$ with labels $1, 2, \ldots, n$, respectively. Similarly, $v_1, v_2, \ldots, v_n$ can mapped to each other by any non-trivial automorphism $\phi$ of $J(n,n)$, so we have to assign the vertices $v_1, v_2, \ldots, v_n$ with different labels. We assign the vertices $v_1, v_2, \ldots, v_n$ with labels $1, 2, \ldots, n$, respectively. Further, assign the vertices $u, x$ with label $1$, and assign the vertices $v, y$ with label $2$.

Thus, the assignment of the above-mentioned labeling is distinguishing because all the labeled vertices are not fixed by any non-trivial automorphism $\phi$ of $J(n,n)$. Hence, $D\left[J(n,n)\right] = n$.

**Case 2:** $m \neq n$. We have $\deg(u_i) = \deg(v_j) = 1, \deg(u) = m+2, \deg(v) = n+2$ and $\deg(x) = \deg(y) = 3$, for all $1 \leq i \leq m$ and $1 \leq j \leq n$. So, if $\phi$ is an automorphism of $J(m,n)$, then we have the following possible cases for $\phi$:

(i) The vertices $u_1, u_2, \ldots, u_n$ can be mapped to each other by $\phi$, also the vertices $v_1, v_2, \ldots, v_n$ can be mapped to each other by $\phi$ to each other by $\phi$, $\phi(x) = y$ & $\phi(y) = x$, and the vertices $u, v$ are fixed by $\phi$.

(ii) The vertices $u_1, u_2, \ldots, u_n$ can be mapped to each other by $\phi$, also the vertices $v_1, v_2, \ldots, v_n$ can mapped to each other by $\phi$, and vertices $u, v, x, y$ are fixed by $\phi$.

Thus, $|Aut\,(J(m,n))| = 2m!n!$ and $Aut\,(J(m,n)) \cong \mathbb{Z}_2 \times S_m \times S_n$. Since $u_1, u_2, \ldots, u_m$ can be mapped to each other by any non-trivial automorphism $\phi$ of $J(m,n)$, we have to assign the vertices $u_1, u_2, \ldots, u_m$ with different labels. We assign the vertices $u_1, u_2, \ldots, u_m$ with labels $1, 2, \ldots, m$, respectively. Similarly, $v_1, v_2, \ldots, v_n$ can be mapped to each other by any non-trivial automorphism $\phi$ of $J(m,n)$, so we have to assign the vertices $v_1, v_2, \ldots, v_n$ with different labels. We assign the vertices $v_1, v_2, \ldots, v_n$ with labels $1, 2, \ldots, n$, respectively.

Further, assign the vertices $u, x$ with label $1$, and the vertices $v, y$ with label $2$. Thus, the assignment of the above-mentioned labeling is distinguishing because all the labeled vertices are not fixed by any non-trivial automorphism $\phi$ of $J(m,n)$. Thus, the minimum number of different labels which are only preserved by the trivial automorphism of $J(m,n)$ is larger value from $m$ and $n$. Hence, $D\,[J(m,n)] = \max\{m,n\}$. ■

**Example 2.4.**

Figures 5 and 6 demonstrate the distinguishing number of the jellyfish graph $F\,(1,1)$ and $F\,(5,5)$, respectively, when $m = n$. Further, Figure 7 demonstrates the distinguishing number of jellyfish graph $F\,(4,5)$ when $m \neq n$. In all three figures, the left figure shows the vertex labeling while the right figure shows their corresponding distinguishing labeling for jellyfish graph.

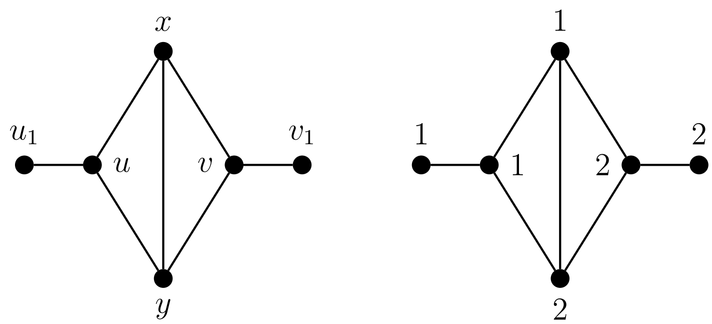


**Figure 5.** The vertex labeling and distinguishing labeling for jellyfish graph $F\,(1,1)$ when $m = n$, and $n = 1$

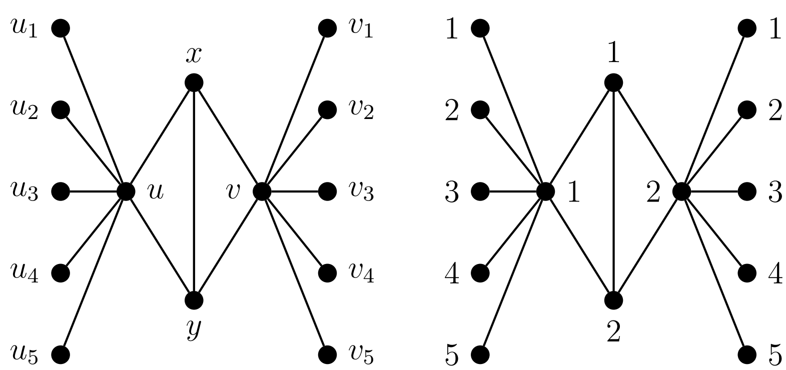


**Figure 6.** The vertex labeling and distinguishing labeling for jellyfish graph $F\,(5,5)$ when $m = n$, and $n > 1$

**Theorem 2.5.**

For a banana tree graph $BT(m,n)$,

(1) $D\,[BT(m,n)] = n-1$ when $m \leq n-1$.
(2) $D\,[BT(m,n)] = m$ when $m > n-1$.

***Proof:***

Let $BT(m,n)$ be a banana tree graph constructed by connecting one leaf of each of $m$-copies of

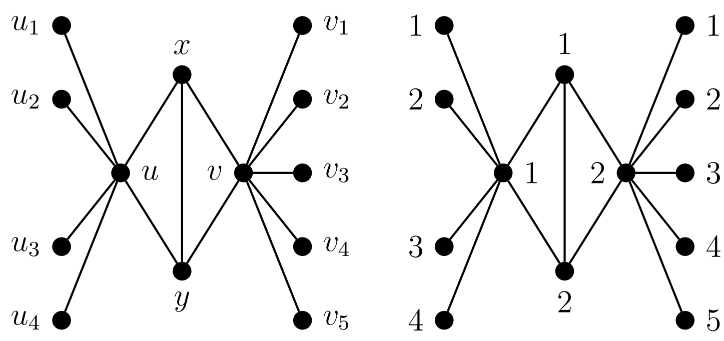


**Figure 7.** The vertex labeling and distinguishing labeling for jellyfish graph $F\,(4, 5)$ when $m \neq n$

a star graph $K_{1,n}$ with a new single root vertex $v$. Let $u_1, u_2, \ldots, u_m$ be the central vertices of $m$-copies of star graphs $K_{1,n}$ in $BT(m, n)$. For all $1 \leq i \leq m$, let $u_i^1, u_i^2, \ldots, u_i^{n-1}$ be the pendant vertices of $i^{th}$ copy of star graph $K_{1,n}$ in $BT(m, n)$, and $u_i^n$ be the vertex of $i^{th}$ copy of star graph $K_{1,n}$ which is connected to $v$ in $BT(m, n)$. We have $\deg(v) = m$, $\deg(u_i) = n$, $\deg\left(u_i^j\right) = 1$ and $\deg(u_i^n) = 2$, for all $1 \leq j \leq n-1$ and $1 \leq i \leq m$. So, if $\phi$ is an automorphism of $BT(m, n)$, then we have the following possible cases for $\phi$:

(i) The vertices $u_i^1, u_i^2, \ldots, u_i^{n-1}$ of star graph $K_{1,n}$ can be mapped to each other by $\phi$, and the vertices $u_i, u_i^n, v$ are fixed by $\phi$, for all $1 \leq i \leq m$.

(ii) The vertex $u_i$ can be mapped to $u_j$ by $\phi$. If $u_i$ mapped to $u_j$ by $\phi$, then $\phi$ have to map the vertices of star graph $K_{1,n}$ corresponding to $u_i$ to the vertices of star graph $K_{1,n}$ corresponding to $u_j$, for all $1 \leq i, j \leq m$.

Thus, $|Aut\,(BT(m, n))| = m!\,((n-1)!)^m$ and $Aut\,(BT(m, n)) \cong S_m \times S_{n-1}^m$. Since the vertices $u_i^1, u_i^2, \ldots, u_i^{n-1}$ of star graph $K_{1,n}$ can be mapped to each other by any non-trivial automorphism $\phi$ of $BT(m, n)$, we have to assign these vertices with different labels, for all $1 \leq i \leq m$. We assign the vertices $u_i^1, u_i^2, \ldots, u_i^{n-1}$ with labels $1, 2, \ldots, n-1$, respectively.

Moreover, the vertices $u_1, u_2, \ldots, u_m$ can mapped to each other by $\phi$, along with the vertices of star graph corresponding to $u_i$ will also be mapped to the vertices of star graph corresponding to $u_j$, for all $1 \leq i, j \leq m$. So, we have to assign the vertices $u_1, u_2, \ldots, u_m$ with different labels. We assign the vertices $u_1, u_2, \ldots, u_m$ with labels $1, 2, \ldots, m,$ respectively. So, the minimum number of different labels which are only preserved by the trivial automorphism of $BT(m, n)$ is $n-1$ when $m \leq n-1$, and $m$ when $m > n-1$. Hence, $D\,[BT(m, n)] = n-1$ when $m \leq n-1$, and $D\,[BT(m, n)] = m$ when $m > n-1$. ■

**Example 2.5.**

Figure 8 demonstrates the distinguishing number of banana tree graph $BT(3, 5)$ when $m \leq n-1$. Further, Figure 9 demonstrates the distinguishing number of banana tree graph $BT(3, 3)$ when $m > n-1$. In both figures, the left figure shows the vertex labeling while the right figure shows their corresponding distinguishing labeling for banana tree graph.

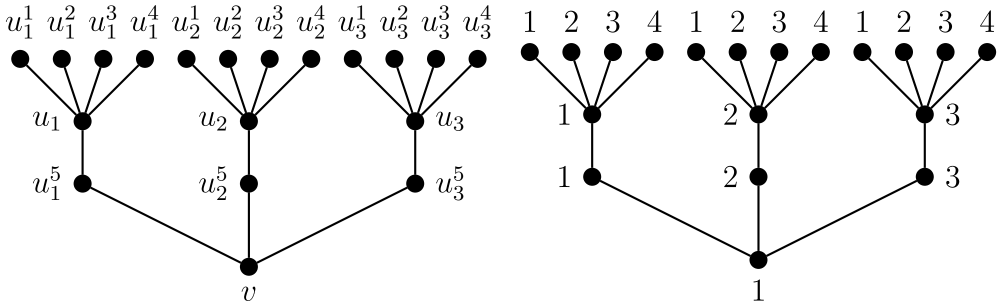


**Figure 8.** The vertex labeling and distinguishing labeling for banana tree graph $B(3,5)$ when $m \leq n-1$

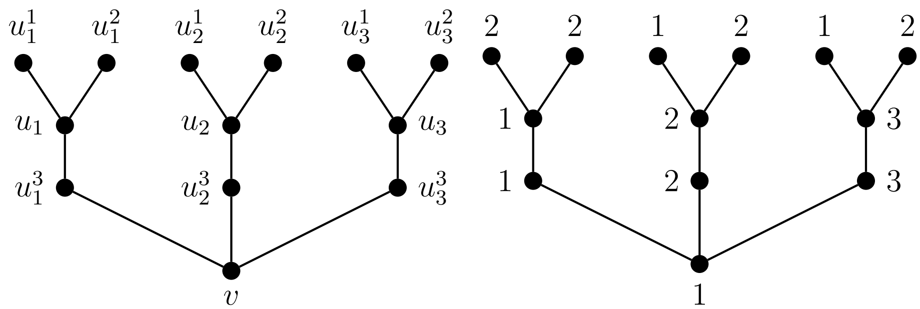


**Figure 9.** The vertex labeling and distinguishing labeling for banana tree graph $B(3,3)$ when $m > n-1$

## 3. Conclusion

In this paper, the number of automorphisms and automorphism group of some graphs like the coconut tree graph, firecracker graph, jellyfish graph, triangular book graph, and banana tree graph have been studied. Further, the distinguishing number of coconut tree graph, fire cracker graph, jelly fish graph, triangular book graph, and banana tree graph have been determined by the behaviour of automorphisms of that graphs. Since the distinguishing index of the above-mentioned graphs are not determined in this paper, then other researchers can determine the distinguishing index of the above-mentioned graphs.

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