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New exact solution of Gilson–Pickering equation in plasma

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Abstract

In this paper, we use Paul-Painlev´e approach method, extended rational sine-cosine method and extended rational sinh-cosh method to construct the exact solution of the nonlinear Gilson-Pickering (GP) equation in plasma. The exact solution of GP equation obtained by the above three methods is new, and we use mathematical software to draw the two-dimensional and three-dimensional graphs of the new exact solutions. Through the study of nonlinear equations in plasma, this study will enrich the research and connotation of nonlinear development equations in plasma.

Keywords: Paul-Painlev´e approach method; Extended rational sine-cosine method; E xtended rational sinh-cosh method; Gilson–Pickering equation; Nonlinear equation; Exact solution; Plasma

MSC 2020 No.: 35Q99, 35G20

1. Introduction

The evolution of human understanding of nature from linear to nonlinear phenomena is a sign of the development of nonlinear science. Nonlinear research in plasma is a potential research topic, which covers a lot of domains, including natural science, humanities and social science, also has greater scientific value and profound philosophical methodological significance.

A team of materials scientists claimed to have achieved magnetic confinement-free stability in an impermeable plasma in 2013. Obtaining spectroscopic data on plasma properties is challenging under high pressure, but the plasma's passive influence on nanostructure creation indicates efficient confinement. Maintaining impermeability for a few tens of seconds resulted in a significant secondary mode of heating, distinct reaction kinetics, and complex nanomaterials.

The GP model given in the next formula is what we are studying in this context. The form is as follows:

$$-r_1\frac{\partial^3\varphi}{\partial x^2\partial t}-r_4\frac{\partial\varphi}{\partial x}\frac{\partial^2\varphi}{\partial x^2}-\varphi\frac{\partial^3\varphi}{\partial x^3}-r_3\varphi\frac{\partial\varphi}{\partial x}+2r_2\frac{\partial\varphi}{\partial x}+\frac{\partial\varphi}{\partial t}=0, \tag{1}$$

while describes wave propagation in plasma physics and crystal lattice theory, r_i, i = 1, 2, 3, 4 are arbitrary parameters.

Ayesh et al. (2022) thought of this article is to achieve new soliton solutions of the Gilson–Pickering equation (GPE) with the assistance of Sardar's subequation method (SSM) and Jacobi elliptic function method (JEFM). Rehman et al. (2022) extended simple equation method (ESEM) and the generalized Riccati equation mapping (GREM) method are applied to the nonlinear third-order Gilson–Pickering (GP) model to obtain a variety of new exact wave solutions. Liu et al. (2023) derived some sets of nonlinear ordinary differential equation, along with some analytical solutions, based on the auxiliary transformations. A lot of scholars have conducted extensive research on this.

2. Analytical methods

Three different analysis methods are briefly introduced in the section that follows.

2.1. Paul-Painlev´e approach method

Suppose the form of the nonlinear development equation is as follows:

$$T\left(\phi,\phi_x,\phi_t,\phi_{xx},\phi_{tt,\cdots}\right)=0. \tag{2}$$

By admitting the transformation, the form is as follows:

$$\varphi(x,t)=\varphi(\zeta), \tag{3}$$

$$\zeta=\lambda x+kt. \tag{4}$$

Plugging the Equations (3) – (4) into the Equation (2), we get the equation as follows:

$$S\left(\phi',\phi'',\phi''',...\right)=0. \tag{5}$$

The solution to the nonlinear development equation is presented as follows:

$$\varphi(\zeta)=A_0+W(X)\mathrm{e}^{-N\zeta}, \tag{6}$$

or like so:

$$\varphi(\zeta)=A_0+A_1W(X)\mathrm{e}^{-N\zeta}+A_2W^2(X)\mathrm{e}^{-2N\zeta}. \tag{7}$$

Equations (6) – (7) satisfy the following conditions:

$$X=T(\zeta)=E-\frac{\mathrm{e}^{-N\zeta}}{N}. \tag{8}$$

By using Riccati's equation, the form is as follows:

$$W_X+FW^2=0. \tag{9}$$

One set of exact solutions is as follows:

$$W(X)=\frac{1}{FX+X_0}, \tag{10}$$

where A_0, A_1, A_2, N, E, F and X_0 are constants.

2.2. Expanded rational sine-cosine method

In some nonlinear development equation, like so:

$$U\left(\phi,\phi_x,\phi_t,\phi_{xx},\phi_{tt},\cdots\right)=0. \tag{11}$$

By admitting the transformation, the form is as follows:

$$\varphi(x,t)=\varphi(\zeta), \tag{12}$$

$$\zeta = \lambda x + kt. \tag{13}$$

Plugging the Equations (12) – (13) into the Equation (11), we get the ordinary differential equation, the form is as follows:

$$Q(\varphi, \varphi_{\zeta}, \varphi_{\zeta\zeta}, \varphi_{\zeta\zeta\zeta}, \ldots) = 0. \tag{14}$$

The solution of the nonlinear development equation be this:

$$\varphi(\zeta) = \frac{A\sin(\mu\zeta)}{C + B\cos(\mu\zeta)}, \tag{15}$$

$$\cos(\mu\zeta) \neq -\frac{C}{B}. \tag{16}$$

Or like the following:

$$\varphi(\zeta) = \frac{A\cos(\mu\zeta)}{C + B\sin(\mu\zeta)}, \tag{17}$$

$$\sin(\mu\zeta) \neq -\frac{C}{B}, \tag{18}$$

where μ, A, B and C are constants.

2.3. Expanded rational sinh-cosh method

In some nonlinear development equation, like so:

$$U(\phi, \phi_x, \phi_t, \phi_{xx}, \phi_{tt}, \cdots) = 0. \tag{19}$$

By admitting the transformation, the form is as follows:

$$\varphi(x,t) = \varphi(\zeta), \tag{20}$$

$$\zeta = \lambda x + kt. \tag{21}$$

Plugging the Equations (20) – (21) into the Equation (19), we get the ordinary differential equation, the form is as follows:

$$Q(\varphi, \varphi_{\zeta}, \varphi_{\zeta\zeta}, \varphi_{\zeta\zeta\zeta}, \ldots) = 0. \tag{22}$$

The solution of the nonlinear development equation be this:

$$\varphi(\zeta) = \frac{A \sinh(\mu\zeta)}{C + B \cosh(\mu\zeta)}, \tag{23}$$

$$\cosh(\mu\zeta) \neq -\frac{C}{B}. \tag{24}$$

Or like this:

$$\varphi(\zeta) = \frac{A \cosh(\mu\zeta)}{C + B \sinh(\mu\zeta)}, \tag{25}$$

$$\sinh(\mu\zeta) \neq -\frac{C}{B}, \tag{26}$$

where μ, A, B and C are constants.

3. Application

In this part, we will apply the Paul-Painlev´ e approach method, expanded rational sine-cosine method and expanded rational sinh-cosh method as the new ideas to solve the GP equation. A new traveling wave solution can be obtained when these variables have the specific value.

By acknowledging the transformation, the form is as follows:

$$\zeta = \lambda x + kt, \tag{27}$$

where λ and k are constants.

Plugging the Equation (27) into the Equation (1), we get the following form:

$$-r_1\varphi''''\lambda^2 k - r_4\varphi'\varphi''\lambda^3 - \varphi\varphi'''\lambda^3 - r_3\varphi\varphi'\lambda + 2r_2\varphi'\lambda + \varphi' k = 0. \tag{28}$$

Integrating once, we get the following form:

$$-r_1\lambda^2 k\varphi'' - \lambda^3\varphi\varphi'' - \frac{1}{2}(r_4 - 1)\lambda^3(\varphi')^2 - \frac{1}{2}r_3\lambda\varphi^2 + (2r_2\lambda + k)\varphi = 0. \tag{29}$$

3.1. Application of Paul-Painlev´e approach method

Calculation Equation (6) to get φ in other forms as follows:

$$\varphi = A_0 + e^{-N\zeta}W, \tag{30}$$

$$\varphi'' = N^2 e^{-N\zeta}W + 3FN e^{-2N\zeta}W^2 + 2F^2 e^{-3N\zeta}W^3, \tag{31}$$

$$\varphi\varphi'' = \left(A_0 + e^{-N\zeta}W\right)\left(N^2 e^{-N\zeta}W + 3FN e^{-2N\zeta}W^2 + 2F^2 e^{-3N\zeta}W^3\right), \tag{32}$$

$$\left(\varphi'\right)^2 = N^2 e^{-2N\zeta}W^2 + F^2 e^{-4N\zeta}W^4 + 2FN e^{-3N\zeta}W^3, \tag{33}$$

$$\varphi^2 = A_0^{\ 2} + 2A_0 e^{-N\zeta}W + e^{-2N\zeta}W^2. \tag{34}$$

By plugging Equations (30) – (34) into Equation (29), we get the equation as follows:

$$\begin{aligned} &-\frac{1}{2}r_3\lambda A_0^{\ 2} + \left(2r_2\lambda + k\right)A_0 + e^{-N\zeta}W\left(-r_1\lambda^2 kN^2 - \lambda^3 A_0 N^2 - r_3\lambda A_0 + 2r_2\lambda + k\right) \\ &+ e^{-2N\zeta}W^2\left[-3r_1\lambda^2 kFN - \lambda^3\left(3A_0FN + N^2\right) - \frac{1}{2}\left(r_4 - 1\right)\lambda^3 N^2 - \frac{1}{2}r_3\lambda\right] \\ &+ e^{-3N\zeta}W^3\left[-2r_1\lambda^2 kF^2 - \lambda^3\left(2A_0F^2 + 3FN\right) - \left(r_4 - 1\right)\lambda^3 FN\right] \\ &+ e^{-4N\zeta}W^4\left[-2\lambda^3F^2 - \frac{1}{2}\left(r_4 - 1\right)\lambda^3F^2\right] = 0. \end{aligned} \tag{35}$$

Let constant, $e^{-N\zeta}W$, $e^{-2N\zeta}W^2$, $e^{-3N\zeta}W^3$, $e^{-4N\zeta}W^4$ term to be zero, we get the equations as follows:

$$-\frac{1}{2}r_3\lambda A_0^{\ 2} + \left(2r_2\lambda + k\right)A_0 = 0, \tag{36}$$

$$-r_1\lambda^2 kN^2 - \lambda^3 A_0 N^2 - r_3\lambda A_0 + 2r_2\lambda + k = 0, \tag{37}$$

$$-3r_1\lambda^2 kFN - \lambda^3\left(3A_0FN + N^2\right) - \frac{1}{2}\left(r_4 - 1\right)\lambda^3 N^2 - \frac{1}{2}r_3\lambda = 0, \tag{38}$$

$$-2r_1\lambda^2 kF^2 - \lambda^3\left(2A_0F^2 + 3FN\right) - \left(r_4 - 1\right)\lambda^3 FN = 0, \tag{39}$$

$$-2\lambda^3F^2 - \frac{1}{2}\left(r_4 - 1\right)\lambda^3F^2 = 0. \tag{40}$$

By using Wolfram Mathematica software, we get the equations as follows:

$$A_0 = 0, N \neq 0, r_2 \neq 0, r_4 \neq -1,$$
$$k = \pm \frac{2ir_2\sqrt{r_3}\sqrt{1+r_4}}{\sqrt{N^2 + 2r_1r_3N^2 + r_1{}^2r_3{}^2N^2 + 2r_4N^2 + 2r_1r_3r_4N^2 + r_4{}^2N^2}}, \tag{41}$$
$$\lambda = \frac{-k - r_1r_3k - r_4k}{2r_2(1+r_4)}, \text{and } F = 0,$$

$$r_3 \neq 0, A_0 = \frac{4r_1r_2}{1+r_1r_3-r_4}, (1+r_4)N \neq 0,$$
$$k = \pm \frac{i\sqrt{r_3}(4r_2 - r_3A_0)}{\sqrt{4N^2 + 4r_4N^2}}, \lambda = \frac{(1+r_1r_3-r_4)k}{2r_2(-1+r_4)}, \text{and } F = 0, \tag{42}$$

$$r_4 = -3, 1 + r_1r_3 \neq 0, N = -\frac{4r_1r_2F}{1+r_1r_3}, A_0 = 0, r_1F \neq 0,$$
$$k = \pm \frac{i\sqrt{r_3}}{2r_1F}, r_2 \neq 0, \lambda = \frac{-k - r_1r_3k}{2r_2}, \text{and } r_3 \neq 0, \tag{43}$$

$$r_4 = -3, 1 + r_1r_3 \neq 0, N = \frac{4r_1r_2F}{1+r_1r_3}, A_0 = \frac{4r_1r_2}{1+r_1r_3}, r_1F \neq 0,$$
$$k = \pm \frac{i\sqrt{r_3}}{2r_1F}, r_2 \neq 0, \lambda = \frac{-k - r_1r_3k}{2r_2}, \text{and } r_3 \neq 0. \tag{44}$$

The form of the solution is as follows:

$$\phi(x,t) = A_0 + \frac{\mathrm{e}^{-N(\lambda x + kt)}}{F\left(E - \frac{\mathrm{e}^{-N(\lambda x + kt)}}{N}\right) + X_0}. \tag{45}$$

By plugging Equations (41) – (44) into Equation (45), we get the equations as follows:

$$\varphi(x,t) = \frac{\mathrm{e}^{-N\left(\left(\frac{(-1-r_1r_3-r_4)}{2r_2(1+r_4)}x+t\right)\frac{2ir_2\sqrt{r_3}\sqrt{1+r_4}}{\sqrt{N^2+2r_1r_3N^2+r_1{}^2r_3{}^2N^2+2r_4N^2+2r_1r_3r_4N^2+r_4{}^2N^2}}\right)}}{X_0}, \tag{46}$$

$$\varphi(x,t) = \frac{\mathrm{e}^{-N\left(\left(\frac{(-1-r_1r_3-r_4)}{2r_2(1+r_4)}x+t\right)\frac{-2ir_2\sqrt{r_3}\sqrt{1+r_4}}{\sqrt{N^2+2r_1r_3N^2+r_1{}^2r_3{}^2N^2+2r_4N^2+2r_1r_3r_4N^2+r_4{}^2N^2}}\right)}}{X_0}, \tag{47}$$

$$\varphi(x,t)=\frac{4r_1r_2}{1+r_1r_3-r_4}+\frac{\mathrm{e}^{-N\left(\frac{(1+r_1r_3-r_4)}{2r_2(-1+r_4)}\frac{i\sqrt{r_3}(4r_2-r_3A_0)}{\sqrt{4N^2+4r_4N^2}}x+\frac{i\sqrt{r_3}(4r_2-r_3A_0)}{\sqrt{4N^2+4r_4N^2}}t\right)}}{X_0}, \tag{48}$$

$$\varphi(x,t)=\frac{4r_1r_2}{1+r_1r_3-r_4}+\frac{\mathrm{e}^{-N\left(\frac{(1+r_1r_3-r_4)}{2r_2(-1+r_4)}\frac{-i\sqrt{r_3}(4r_2-r_3A_0)}{\sqrt{4N^2+4r_4N^2}}x+\frac{-i\sqrt{r_3}(4r_2-r_3A_0)}{\sqrt{4N^2+4r_4N^2}}t\right)}}{X_0}, \tag{49}$$

$$\varphi(x,t)=\frac{\mathrm{e}^{--\frac{4r_1r_2F}{1+r_1r_3}\left(\frac{-1-r_1r_3}{2r_2}\frac{i\sqrt{r_3}}{2r_1F}+\frac{i\sqrt{r_3}}{2r_1F}t\right)}}{F\left(E-\frac{\mathrm{e}^{--\frac{4r_1r_2F}{1+r_1r_3}\left(\frac{-1-r_1r_3}{2r_2}\frac{i\sqrt{r_3}}{2r_1F}+\frac{i\sqrt{r_3}}{2r_1F}t\right)}}{-\frac{4r_1r_2F}{1+r_1r_3}}\right)+X_0}, \tag{50}$$

$$\varphi(x,t)=\frac{\mathrm{e}^{--\frac{4r_1r_2F}{1+r_1r_3}\left(\frac{-1-r_1r_3}{2r_2}\frac{-i\sqrt{r_3}}{2r_1F}+\frac{-i\sqrt{r_3}}{2r_1F}t\right)}}{F\left(E-\frac{\mathrm{e}^{--\frac{4r_1r_2F}{1+r_1r_3}\left(\frac{-1-r_1r_3}{2r_2}\frac{-i\sqrt{r_3}}{2r_1F}+\frac{-i\sqrt{r_3}}{2r_1F}t\right)}}{-\frac{4r_1r_2F}{1+r_1r_3}}\right)+X_0}, \tag{51}$$

$$\varphi(x,t)=\frac{4r_1r_2}{1+r_1r_3}+\frac{\mathrm{e}^{-\frac{4r_1r_2F}{1+r_1r_3}\left(\frac{-1-r_1r_3}{2r_2}\frac{i\sqrt{r_3}}{2r_1F}x+\frac{i\sqrt{r_3}}{2r_1F}t\right)}}{F\left(E-\frac{\mathrm{e}^{-\frac{4r_1r_2F}{1+r_1r_3}\left(\frac{-1-r_1r_3}{2r_2}\frac{i\sqrt{r_3}}{2r_1F}x+\frac{i\sqrt{r_3}}{2r_1F}t\right)}}{\frac{4r_1r_2F}{1+r_1r_3}}\right)+X_0}, \tag{52}$$

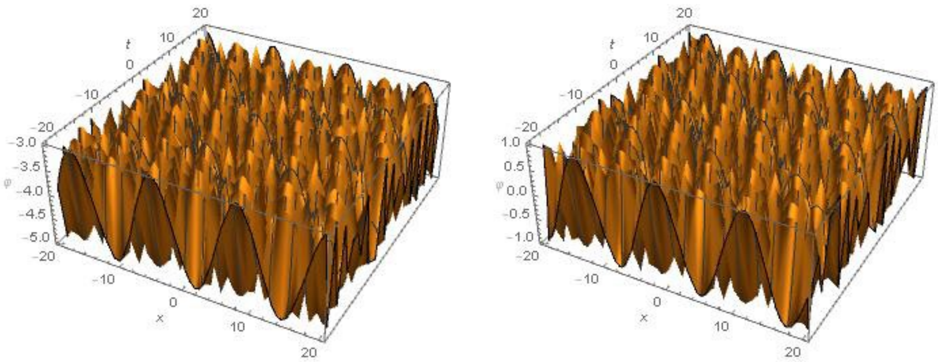
$$\varphi(x,t)=\frac{4r_1r_2}{1+r_1r_3}+\frac{\mathrm{e}^{-\frac{4r_1r_2F}{1+r_1r_3}\left(\frac{-1-r_1r_3}{2r_2}\frac{-i\sqrt{r_3}}{2r_1F}x+\frac{-i\sqrt{r_3}}{2r_1F}t\right)}}{F\left(E-\frac{\mathrm{e}^{-\frac{4r_1r_2F}{1+r_1r_3}\left(\frac{-1-r_1r_3}{2r_2}\frac{-i\sqrt{r_3}}{2r_1F}x+\frac{-i\sqrt{r_3}}{2r_1F}t\right)}}{\frac{4r_1r_2F}{1+r_1r_3}}\right)+X_0}. \tag{53}$$

Through the induction and summary of the Equations (46) – (53), the types of new exact solutions obtained by Paul-Painlev´e approach method can be divided into two categories. We choose two exact solutions to draw the image.

Taking Equation (48) for example, we use Wolfram Mathematica software to draw the two-dimensional and three-dimensional graphs.

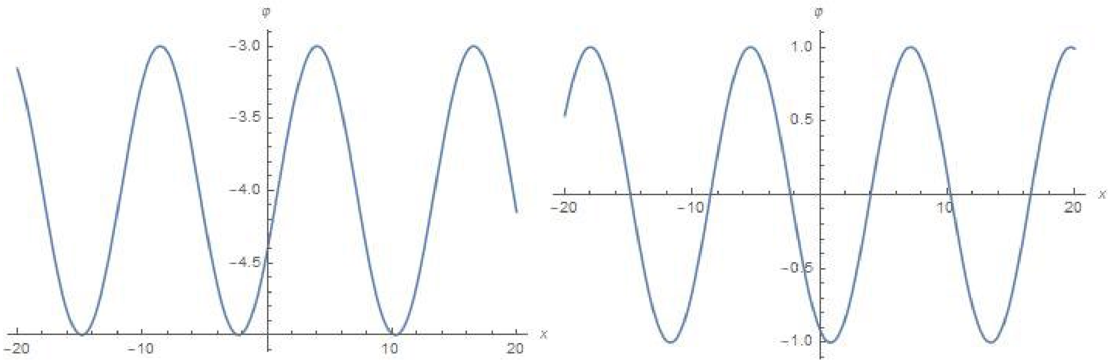
By plugging $r_2 = 1$, $r_3 = 1$, $r_1 = 1$, $r_4 = 3$, $A_0 = -4$, $F = 0$, $E = 1$, $X_0 = 1$, and $N = 1$ into Equation (48), we get the equation as follows:

$$\varphi(x,t) = -4 + \mathrm{e}^{-\left(-\frac{i}{2}x+2it\right)}. \tag{54}$$



(a) (b)

Figure 1. Picture of Equation (54). (a) and (b) are three-dimensional graphs with real and imaginary parts



(a) (b)

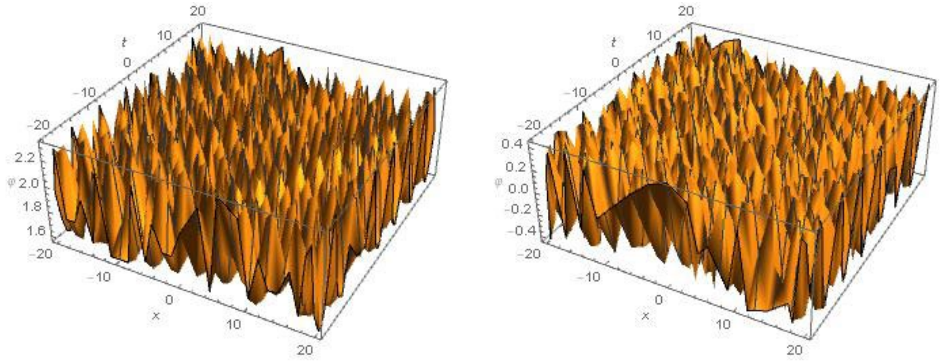
Figure 2. Picture of Equation (54). (a)and(b) are two-dimensional graphs with real and imaginary parts, where t is equal to 1

In Figures 1 and 2, Equation (55) is a soliton solution of Equation (1), and other soliton type solutions can be obtained for different values of r_1, r_2, r_3, r_4, A_0, E, F, X_0, t, and N.

Taking Equation (52) for example, we use Wolfram Mathematica software to draw the two-dimensional and three-dimensional graphs.

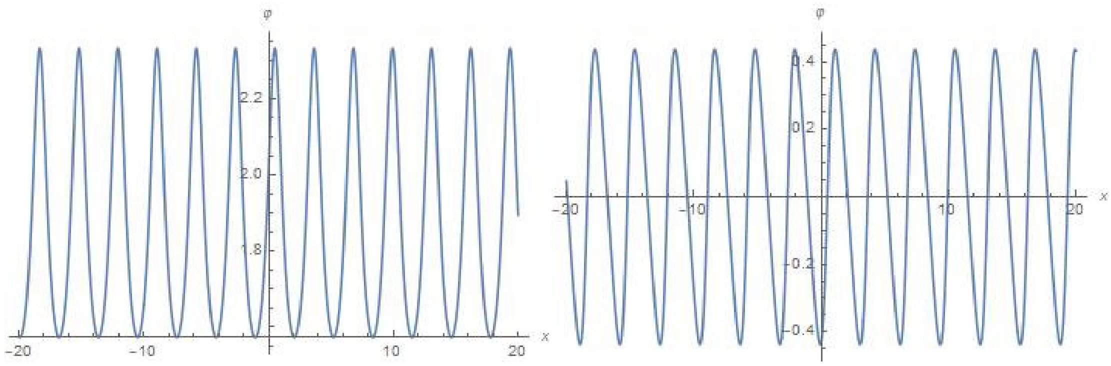
By plugging $r_2 = 1$, $r_3 = 1$, $r_1 = 1$, $r_4 = -3$, $A_0 = 2$, $F = 1$, $E = 1$, $X_0 = 1$, and $N = 2$ into Equation (52), we get the equation as follows:

$$\varphi(x,t) = 2 + \frac{\mathrm{e}^{-2i\left(-x+\frac{1}{2}t\right)}}{2 - \mathrm{e}^{-2i\left(-x+\frac{1}{2}t\right)}}. \tag{55}$$



(a) (b)

Figure 3. Picture of Equation (55). (a) and (b) are three-dimensional graphs with real and imaginary parts



(a) (b)

Figure 4. Picture of Equation (55). (a) and (b) are two-dimensional graphs with real and imaginary parts, where t is equal to 1

In Figures 3 and 4, Equation (55) is a periodic solution of Equation (1), and the periodic solution reflects the periodicity in time of the new exact solution.

3.2. Application of expanded rational sine-cosine method

Calculation Equation (15) to get φ in other forms as follows:

$$\varphi(\zeta)=\frac{A\sin(\mu\zeta)}{C+B\cos(\mu\zeta)}, \tag{56}$$

$$\varphi''(\zeta)=\frac{A\mu^2\left(2B^2-C^2+BC\cos(\mu\zeta)\right)\sin(\mu\zeta)}{\left(C+B\cos(\mu\zeta)\right)^3}, \tag{57}$$

$$\varphi(\zeta)\varphi''(\zeta)=\frac{A^2\mu^2\sin^2(\mu\zeta)\left(2B^2-C^2+BC\cos(\mu\zeta)\right)}{\left(C+B\cos(\mu\zeta)\right)^4}, \tag{58}$$

$$\left(\varphi'(\zeta)\right)^2=\frac{A^2\mu^2\left(B^2+2BC\cos(\mu\zeta)+C^2\cos^2(\mu\zeta)\right)}{\left(C+B\cos(\mu\zeta)\right)^4}, \tag{59}$$

$$\varphi^2(\zeta)=\frac{A^2\sin^2(\mu\zeta)}{(C+B\cos(\mu\zeta))^2}. \tag{60}$$

By plugging Equations (56) – (60) into Equation (29), we get the equation as follows:

$$\begin{aligned}
&-\frac{1}{2}(r_4-1)\lambda^3A^2\mu^2B^2+\sin(\mu\varsigma)\left[-r_1\lambda^2kA\mu^2\left(2B^2-C^2\right)C+(2r_2\lambda+k)AC^3\right]\\
&+\sin^2(\mu\zeta)\left[-\lambda^3A^2\mu^2\left(2B^2-C^2\right)-\frac{1}{2}r_3\lambda A^2C^2\right]\\
&+\sin(\mu\zeta)\cos(\mu\zeta)\left[-r_1\lambda^2kA\mu^2\left(\left(2B^2-C^2\right)B+BC^2\right)+3(2r_2\lambda+k)ABC^2\right]\\
&+\sin(\mu\zeta)\cos^2(\mu\zeta)\left[-r_1\lambda^2kA\mu^2B^2C+3(2r_2\lambda+k)AB^2C\right]\\
&+\sin(\mu\zeta)\cos^3(\mu\zeta)\left[(2r_2\lambda+k)AB^3\right]+\sin^2(\mu\zeta)\cos(\mu\zeta)\left[-\lambda^3A^2\mu^2BC-r_3\lambda A^2BC\right]\\
&+\sin^2(\mu\zeta)\cos^2(\mu\zeta)\left[-\frac{1}{2}r_3\lambda A^2B^2\right]+\cos(\mu\zeta)\left[-(r_4-1)\lambda^3A^2\mu^2BC\right]\\
&+\cos^2(\mu\zeta)\left[-\frac{1}{2}(r_4-1)\lambda^3A^2\mu^2C^2\right]=0.
\end{aligned} \tag{61}$$

Let the constant $\sin^m(\mu\zeta)$, $\cos^n(\mu\zeta)$, $\sin^m(\mu\zeta)\cos^n(\mu\zeta)$ be zeros, then we get the equations as follows:

$$-\frac{1}{2}(r_4-1)\lambda^3A^2\mu^2B^2=0, \tag{62}$$

$$-r_1\lambda^2kA\mu^2\left(2B^2-C^2\right)C+(2r_2\lambda+k)AC^3=0, \tag{63}$$

$$-\lambda^3A^2\mu^2\left(2B^2-C^2\right)-\frac{1}{2}r_3\lambda A^2C^2=0, \tag{64}$$

$$-r_1\lambda^2kA\mu^2\left(\left(2B^2-C^2\right)B+BC^2\right)+3(2r_2\lambda+k)ABC^2=0, \tag{65}$$

$$-r_1\lambda^2kA\mu^2B^2C+3(2r_2\lambda+k)AB^2C=0, \tag{66}$$

$$(2r_2\lambda+k)AB^3=0, \tag{67}$$

$$-\lambda^3A^2\mu^2BC-r_3\lambda A^2BC=0, \tag{68}$$

$$-\frac{1}{2}r_3\lambda A^2B^2=0, \tag{69}$$

$$-(r_4-1)\lambda^3A^2\mu^2BC=0, \tag{70}$$

$$-\frac{1}{2}(r_4-1)\lambda^3A^2\mu^2C^2=0. \tag{71}$$

By using mathematical calculation software, we get the equation as follows:

$$A \neq 0, B = 0, C \neq 0, \mu = \pm\sqrt{\sqrt{\frac{-(2r_2\lambda + k)}{k\lambda^2 r_1}}\sqrt{\frac{r_3}{2\lambda^2}}}, r_1 = \frac{-(4r_2\lambda + 2k)}{kr_3}, \text{and } r_4 = 1. \quad (72)$$

The form of the solution is as follows:

$$\varphi(x,t) = \frac{A\sin(\mu(\lambda x + kt))}{C + B\cos(\mu(\lambda x + kt))}. \quad (73)$$

By plugging Equation (72) into Equation (73), we get the equations as follows:

$$\varphi(x,t) = \frac{A\sin\left(\sqrt{\sqrt{\frac{-(2r_2\lambda + k)}{k\lambda^2 r_1}}\sqrt{\frac{r_3}{2\lambda^2}}}(\lambda x + kt)\right)}{C}, \quad (74)$$

$$\varphi(x,t) = \frac{A\sin\left(-\sqrt{\sqrt{\frac{-(2r_2\lambda + k)}{k\lambda^2 r_1}}\sqrt{\frac{r_3}{2\lambda^2}}}(\lambda x + kt)\right)}{C}. \quad (75)$$

Through the induction and summary of the Equations (74) – (75), the types of new exact solutions obtained by expanded rational sine-cosine method can be divided into one category. We choose a representative exact solution to draw the image.

Taking Equation (74) for example, we use Wolfram Mathematica software to draw the two-dimensional and three-dimensional graphs.

By plugging $r_1 = 2$, $r_2 = -1$, $r_3 = 1$, $r_4 = 1$, $k = 1$, $\lambda = 1$, $A = 2$, $B = 0$, and $C = 1$ into Equation (74), we get the equation as follows:

$$\varphi(x,t) = 2\sin\left(\frac{1}{2}(x + t)\right). \quad (76)$$

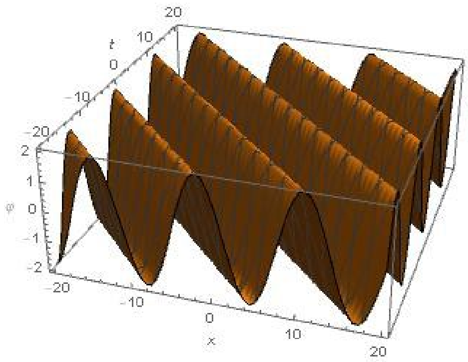


Figure 5. The three-dimensional graph of Equation (76)

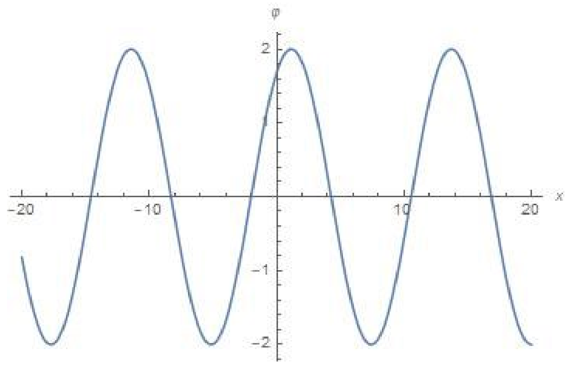


Figure 6. The two-dimensional graph of Equation (76), where t is equal to 1

In Figures 5 and 6, Equation (76) is a periodic solution of Equation (1), and the periodic solution reflects the periodicity in time of the new exact solution.

Calculation Equation (17) to get φ in other forms as follows:

$$\varphi(\zeta)=\frac{A\cos(\mu\zeta)}{C+B\sin(\mu\zeta)}, \tag{77}$$

$$\varphi^2(\zeta)=\frac{A^2\cos^2(\mu\zeta)}{(C+B\sin(\mu\zeta))^2}, \tag{78}$$

$$\varphi'(\zeta)=\frac{-A\mu(B+C\sin(\mu\zeta))}{(C+B\sin(\mu\zeta))^2}, \tag{79}$$

$$(\varphi'(\zeta))^2=\frac{A^2\mu^2(B^2+2BC\sin(\mu\zeta)+C^2\sin^2(\mu\zeta))}{(C+B\sin(\mu\zeta))^4}, \tag{80}$$

$$\varphi''(\zeta)=\frac{A\mu^2\cos(\mu\zeta)(2B^2-C^2+BC\sin(\mu\zeta))}{(C+B\sin(\mu\zeta))^3}, \tag{81}$$

$$\varphi(\zeta)\varphi''(\zeta)=\frac{A^2\mu^2\cos^2(\mu\zeta)\left(2B^2-C^2+BC\sin(\mu\zeta)\right)}{\left(C+B\sin(\mu\zeta)\right)^4}. \tag{82}$$

By plugging Equations (77) – (82) into Equation (29), we get the equation as follows:

$$\begin{aligned}
&-\frac{1}{2}(r_4-1)\lambda^3A^2\mu^2B^2+\cos(\mu\varsigma)\left[-r_1\lambda^2kA\mu^2\left(2B^2-C^2\right)C+(2r_2\lambda+k)AC^3\right]\\
&+\cos^2(\mu\zeta)\left[-\lambda^3A^2\mu^2\left(2B^2-C^2\right)-\frac{1}{2}r_3\lambda A^2C^2\right]\\
&+\cos(\mu\zeta)\sin(\mu\zeta)\left[-r_1\lambda^2kA\mu^2\left(\left(2B^2-C^2\right)B+BC^2\right)+3(2r_2\lambda+k)ABC^2\right]\\
&+\cos(\mu\zeta)\sin^2(\mu\zeta)\left[-r_1\lambda^2kA\mu^2B^2C+3(2r_2\lambda+k)AB^2C\right]\\
&+\cos(\mu\zeta)\sin^3(\mu\zeta)\left[(2r_2\lambda+k)AB^3\right]+\cos^2(\mu\zeta)\sin(\mu\zeta)\left[-\lambda^3A^2\mu^2BC-r_3\lambda A^2BC\right]\\
&+\cos^2(\mu\zeta)\sin^2(\mu\zeta)\left[-\frac{1}{2}r_3\lambda A^2B^2\right]+\sin(\mu\zeta)\left[-(r_4-1)\lambda^3A^2\mu^2BC\right]\\
&+\sin^2(\mu\zeta)\left[-\frac{1}{2}(r_4-1)\lambda^3A^2\mu^2C^2\right]=0.
\end{aligned} \tag{83}$$

Let constants $\sin^m(\mu\zeta)$, $\cos^n(\mu\zeta)$, $\sin^m(\mu\zeta)\cos^n(\mu\zeta)$ be zeros, then we get the equations as follows:

$$-\frac{1}{2}(r_4-1)\lambda^3A^2\mu^2B^2=0, \tag{84}$$

$$-r_1\lambda^2kA\mu^2\left(2B^2-C^2\right)C+(2r_2\lambda+k)AC^3=0, \tag{85}$$

$$-\lambda^3A^2\mu^2\left(2B^2-C^2\right)-\frac{1}{2}r_3\lambda A^2C^2=0, \tag{86}$$

$$-r_1\lambda^2kA\mu^2\left(\left(2B^2-C^2\right)B+BC^2\right)+3(2r_2\lambda+k)ABC^2=0, \tag{87}$$

$$-r_1\lambda^2kA\mu^2B^2C+3(2r_2\lambda+k)AB^2C=0, \tag{88}$$

$$(2r_2\lambda+k)AB^3=0, \tag{89}$$

$$-\lambda^3A^2\mu^2BC-r_3\lambda A^2BC=0, \tag{90}$$

$$-\frac{1}{2}r_3\lambda A^2B^2=0, \tag{91}$$

$$-(r_4-1)\lambda^3A^2\mu^2BC=0, \tag{92}$$

$$-\frac{1}{2}(r_4-1)\lambda^3A^2\mu^2C^2=0. \tag{93}$$

By using mathematical calculation software, we get the equation as follows:

$$A \neq 0, B = 0, C \neq 0, \mu = \pm\sqrt{\sqrt{\frac{-(2r_2\lambda + k)}{k\lambda^2 r_1}}\sqrt{\frac{r_3}{2\lambda^2}}}, r_1 = \frac{-(4r_2\lambda + 2k)}{kr_3}, \text{and } r_4 = 1. \quad (94)$$

The form of the solution is as follows:

$$\varphi(x,t) = \frac{A\cos(\mu(\lambda x + kt))}{C + B\sin(\mu(\lambda x + kt))}. \quad (95)$$

By plugging Equation (94) into Equation (95), we get the equations as follows:

$$\varphi(x,t) = \frac{A\cos\left(\sqrt{\sqrt{\frac{-(2r_2\lambda + k)}{k\lambda^2 r_1}}\sqrt{\frac{r_3}{2\lambda^2}}}(\lambda x + kt)\right)}{C}, \quad (96)$$

$$\varphi(x,t) = \frac{A\cos\left(-\sqrt{\sqrt{\frac{-(2r_2\lambda + k)}{k\lambda^2 r_1}}\sqrt{\frac{r_3}{2\lambda^2}}}(\lambda x + kt)\right)}{C}. \quad (97)$$

Through the induction and summary of the Equations (96) – (97), the types of new exact solutions obtained by expanded rational sine-cosine method can be divided into one category. We choose a representative exact solution to draw the image.

Taking Equation (96) for example, we use Wolfram Mathematica software to draw the two-dimensional and three-dimensional graphs.

By plugging $r_1 = 2$, $r_2 = -1$, $r_3 = 1$, $r_4 = 1$, $k = 1$, $\lambda = 1$, $A = 2$, $B = 0$, and $C = 1$ into Equation (96), we get the equation as follows:

$$\varphi(x,t) = 2\cos\left(\frac{1}{2}(x + t)\right). \quad (98)$$

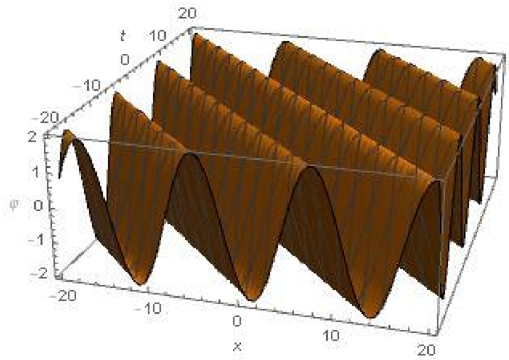


Figure 7. The three-dimensional graph of Equation (98)

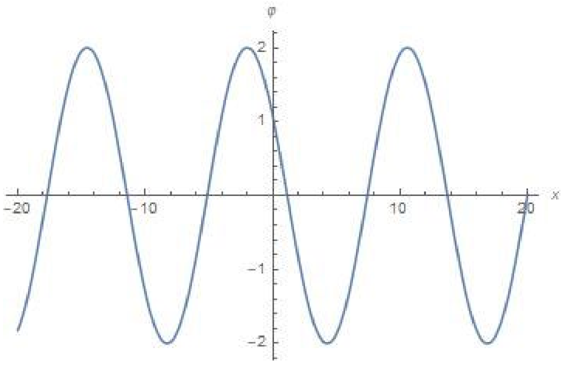


Figure 8. The two-dimensional graph of Equation (98), where t is equal to 1

In Figures 7 and 8, Equation (98) is a periodic solution of Equation (1), and other soliton type solutions can be obtained for different values of r_1, r_2, r_3, r_4, A, B, C, k, λ, t, and μ.

3.3. Application of expanded rational sinh-cosh method

Calculation Equation (23) to get φ in other forms as follows:

$$\varphi(\zeta)=\frac{A\sinh(\mu\zeta)}{C+B\cosh(\mu\zeta)}, \tag{99}$$

$$\varphi^2(\zeta)=\frac{A^2\sinh^2(\mu\zeta)}{(C+B\cosh(\mu\zeta))^2}, \tag{100}$$

$$\varphi'(\zeta)=\frac{A\mu(B+C\cosh(\mu\zeta))}{(C+B\cosh(\mu\zeta))^2}, \tag{101}$$

$$(\varphi'(\zeta))^2=\frac{A^2\mu^2\left(B^2+2BC\cosh(\mu\zeta)+C^2\cosh^2(\mu\zeta)\right)}{(C+B\cosh(\mu\zeta))^4}, \tag{102}$$

$$\varphi''(\zeta)=\frac{-A\mu^2\left(2B^2-C^2+BC\cosh(\mu\zeta)\right)\sinh(\mu\zeta)}{(C+B\cosh(\mu\zeta))^3}, \tag{103}$$

$$\varphi(\zeta)\varphi''(\zeta)=\frac{-A^2\mu^2\sinh^2(\mu\zeta)\left(2B^2-C^2+BC\cosh(\mu\zeta)\right)}{\left(C+B\cosh(\mu\zeta)\right)^4}. \tag{104}$$

By plugging Equations (99) – (104) into Equation (29), we get the equation as follows:

$$\begin{aligned}
&-\frac{1}{2}(r_4-1)\lambda^3A^2\mu^2B^2+\sinh(\mu\varsigma)\left[r_1\lambda^2kA\mu^2\left(2B^2-C^2\right)C+(2r_2\lambda+k)AC^3\right]\\
&+\sinh^2(\mu\zeta)\left[\lambda^3A^2\mu^2\left(2B^2-C^2\right)-\frac{1}{2}r_3\lambda A^2C^2\right]\\
&+\sinh(\mu\zeta)\cosh(\mu\zeta)\left[r_1\lambda^2kA\mu^2\left(\left(2B^2-C^2\right)B+BC^2\right)+3(2r_2\lambda+k)ABC^2\right]\\
&+\sinh(\mu\zeta)\cosh^2(\mu\zeta)\left[r_1\lambda^2kA\mu^2B^2C+3(2r_2\lambda+k)AB^2C\right]\\
&+\sinh(\mu\zeta)\cosh^3(\mu\zeta)\left[(2r_2\lambda+k)AB^3\right]+\sinh^2(\mu\zeta)\cosh(\mu\zeta)\left[\lambda^3A^2\mu^2BC-r_3\lambda A^2BC\right]\\
&+\sinh^2(\mu\zeta)\cosh^2(\mu\zeta)\left[-\frac{1}{2}r_3\lambda A^2B^2\right]+\cosh(\mu\zeta)\left[-(r_4-1)\lambda^3A^2\mu^2BC\right]\\
&+\cosh^2(\mu\zeta)\left[-\frac{1}{2}(r_4-1)\lambda^3A^2\mu^2C^2\right]=0.
\end{aligned} \tag{105}$$

Let constants $\sinh^m(\mu\zeta)$, $\cosh^n(\mu\zeta)$, $\sinh^m(\mu\zeta)\cosh^n(\mu\zeta)$ be zeros, then we get the equations as follows:

$$-\frac{1}{2}(r_4-1)\lambda^3A^2\mu^2B^2=0, \tag{106}$$

$$r_1\lambda^2kA\mu^2\left(2B^2-C^2\right)C+(2r_2\lambda+k)AC^3=0, \tag{107}$$

$$\lambda^3A^2\mu^2\left(2B^2-C^2\right)-\frac{1}{2}r_3\lambda A^2C^2=0, \tag{108}$$

$$r_1\lambda^2kA\mu^2\left(\left(2B^2-C^2\right)B+BC^2\right)+3(2r_2\lambda+k)ABC^2=0, \tag{109}$$

$$r_1\lambda^2kA\mu^2B^2C+3(2r_2\lambda+k)AB^2C=0, \tag{110}$$

$$(2r_2\lambda+k)AB^3=0, \tag{111}$$

$$\lambda^3A^2\mu^2BC-r_3\lambda A^2BC=0, \tag{112}$$

$$-\frac{1}{2}r_3\lambda A^2B^2=0, \tag{113}$$

$$\cosh(\mu\zeta):-(r_4-1)\lambda^3A^2\mu^2BC=0, \tag{114}$$

$$-\frac{1}{2}(r_4-1)\lambda^3A^2\mu^2C^2=0. \tag{115}$$

By using mathematical calculation software, we get the equation as follows:

$$A \neq 0, B = 0, C \neq 0, \mu = \pm \sqrt{\sqrt{\frac{2r_2\lambda + k}{k\lambda^2 r_1}} \sqrt{\frac{-r_3}{2\lambda^2}}}, r_1 = \frac{-(4r_2\lambda + 2k)}{kr_3}, \text{and } r_4 = 1. \quad (116)$$

The form of the solution is as follows:

$$\varphi(x,t) = \frac{A\sinh(\mu(\lambda x + kt))}{C + B\cosh(\mu(\lambda x + kt))}. \quad (117)$$

By plugging Equation (116) into Equation (117), we get the equations as follows:

$$\varphi(x,t) = \frac{A\sinh\left(\sqrt{\sqrt{\frac{2r_2\lambda + k}{k\lambda^2 r_1}} \sqrt{\frac{-r_3}{2\lambda^2}}}(\lambda x + kt)\right)}{C}, \quad (118)$$

$$\varphi(x,t) = \frac{A\sinh\left(-\sqrt{\sqrt{\frac{2r_2\lambda + k}{k\lambda^2 r_1}} \sqrt{\frac{-r_3}{2\lambda^2}}}(\lambda x + kt)\right)}{C}. \quad (119)$$

Through the induction and summary of the Equations (118) – (119), the types of new exact solutions obtained by expanded rational sine-cosine method can be divided into one category. We choose a representative exact solution to draw the image.

Taking Equation (118) for example, we use Wolfram Mathematica software to draw the two-dimensional and three-dimensional graphs.

By plugging $r_1 = 3$, $r_2 = 1$, $r_3 = -2$, $r_4 = 1$, $k = 1$, $\lambda = 1$, $A = 2$, $B = 0$, and $C = 1$ into Equation (118), we get the equation as follows:

$$\varphi(x,t) = 2\sinh(x + t). \quad (120)$$

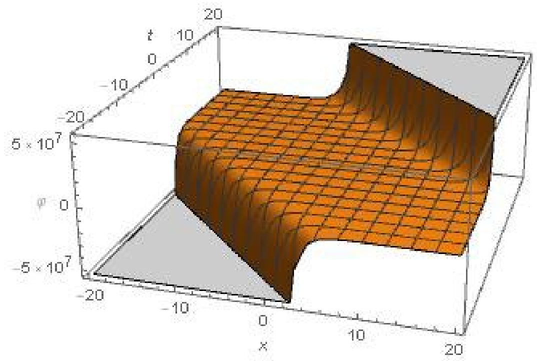


Figure 9. The three-dimensional graph of Equation (120)

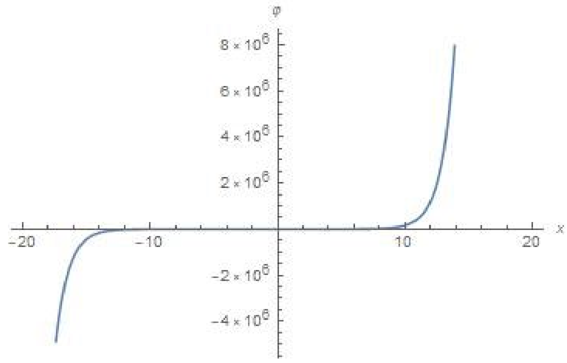


Figure 10. The two-dimensional graph of Equation (120) ,where *t* is equal to 1

In Figures 9-10, Equation (120) is a periodic solution of Equation (1), and the periodic solution reflects the periodicity in time of the new exact solution.

Calculation Equation (25) to get φ in other forms as follows:

$$\varphi(\zeta)=\frac{A\cosh(\mu\zeta)}{C+B\sinh(\mu\zeta)}, \tag{121}$$

$$\varphi^2(\zeta)=\frac{A^2\cosh^2(\mu\zeta)}{(C+B\sinh(\mu\zeta))^2}, \tag{122}$$

$$\varphi'(\zeta)=\frac{A\mu(-C+B\sinh(\mu\zeta))}{(C+B\sinh(\mu\zeta))^2}, \tag{123}$$

$$(\varphi'(\zeta))^2=\frac{A^2\mu^2(B^2-2BC\sinh(\mu\zeta)+C^2\sinh^2(\mu\zeta))}{(C+B\sinh(\mu\zeta))^4}, \tag{124}$$

$$\varphi''(\zeta)=\frac{A\mu^2(2B^2+C^2-BC\sinh(\mu\zeta))\cosh(\mu\zeta)}{(C+B\sinh(\mu\zeta))^3}, \tag{125}$$

$$\varphi(\zeta)\varphi''(\zeta)=\frac{A^2\mu^2\cosh^2(\mu\zeta)\left(2B^2+C^2-BC\sinh(\mu\zeta)\right)}{\left(C+B\sinh(\mu\zeta)\right)^4}. \tag{126}$$

By plugging Equations (121) – (126) into Equation (29), we get the equation as follows:

$$\begin{aligned}
&-\frac{1}{2}(r_4-1)\lambda^3A^2\mu^2B^2+\cosh(\mu\varsigma)\left[-r_1\lambda^2kA\mu^2\left(2B^2-C^2\right)C+(2r_2\lambda+k)AC^3\right]\\
&+\cosh^2(\mu\zeta)\left[-\lambda^3A^2\mu^2\left(2B^2-C^2\right)-\frac{1}{2}r_3\lambda A^2C^2\right]\\
&+\cosh(\mu\zeta)\sinh(\mu\zeta)\left[-r_1\lambda^2kA\mu^2\left(\left(2B^2-C^2\right)B+BC^2\right)+3(2r_2\lambda+k)ABC^2\right]\\
&+\cosh(\mu\zeta)\sinh^2(\mu\zeta)\left[r_1\lambda^2kA\mu^2B^2C+3(2r_2\lambda+k)AB^2C\right]\\
&+\cosh(\mu\zeta)\sinh^3(\mu\zeta)\left[(2r_2\lambda+k)AB^3\right]+\cosh^2(\mu\zeta)\sinh(\mu\zeta)\left[\lambda^3A^2\mu^2BC-r_3\lambda A^2BC\right]\\
&+\cosh^2(\mu\zeta)\sinh^2(\mu\zeta)\left[-\frac{1}{2}r_3\lambda A^2B^2\right]+\sinh(\mu\zeta)\left[(r_4-1)\lambda^3A^2\mu^2BC\right]\\
&+\sinh^2(\mu\zeta)\left[-\frac{1}{2}(r_4-1)\lambda^3A^2\mu^2C^2\right]=0.
\end{aligned} \tag{127}$$

Let constant sinh$^m(\mu\zeta)$, cosh$^n(\mu\zeta)$, sinh$^m(\mu\zeta)$cosh$^n(\mu\zeta)$ terms to be zeros, thenwe get the equations as follows:

$$-\frac{1}{2}(r_4-1)\lambda^3A^2\mu^2B^2=0, \tag{128}$$

$$-r_1\lambda^2kA\mu^2\left(2B^2+C^2\right)C+(2r_2\lambda+k)AC^3=0, \tag{129}$$

$$-\lambda^3A^2\mu^2\left(2B^2+C^2\right)-\frac{1}{2}r_3\lambda A^2C^2=0, \tag{130}$$

$$-r_1\lambda^2kA\mu^2\left(\left(2B^2+C^2\right)B-BC^2\right)+3(2r_2\lambda+k)ABC^2=0, \tag{131}$$

$$r_1\lambda^2kA\mu^2B^2C+3(2r_2\lambda+k)AB^2C=0, \tag{132}$$

$$(2r_2\lambda+k)AB^3=0, \tag{133}$$

$$\lambda^3A^2\mu^2BC-r_3\lambda A^2BC=0, \tag{134}$$

$$-\frac{1}{2}r_3\lambda A^2B^2=0, \tag{135}$$

$$(r_4-1)\lambda^3A^2\mu^2BC=0, \tag{136}$$

$$-\frac{1}{2}(r_4-1)\lambda^3A^2\mu^2C^2=0. \tag{137}$$

By using mathematical calculation software, we get the equation as follows:

$$A \neq 0, B = 0, C \neq 0, \mu = \pm\sqrt{\sqrt{\frac{2r_2\lambda + k}{k\lambda^2 r_1}}\sqrt{\frac{-r_3}{2\lambda^2}}}, r_1 = \frac{-(4r_2\lambda + 2k)}{kr_3}, \text{and } r_4 = 1. \quad (138)$$

The form of the solution is as follows:

$$\varphi(x,t) = \frac{A\cosh(\mu(\lambda x + kt))}{C + B\sinh(\mu(\lambda x + kt))}. \quad (139)$$

By plugging Equation (138) into Equation (139), we get the equations as follows:

$$\varphi(x,t) = \frac{A\cosh\left(\sqrt{\sqrt{\frac{2r_2\lambda + k}{k\lambda^2 r_1}}\sqrt{\frac{-r_3}{2\lambda^2}}}(\lambda x + kt)\right)}{C}, \quad (140)$$

$$\varphi(x,t) = \frac{A\cosh\left(\sqrt{\sqrt{\frac{2r_2\lambda + k}{k\lambda^2 r_1}}\sqrt{\frac{-r_3}{2\lambda^2}}}(\lambda x + kt)\right)}{C}. \quad (141)$$

Through the induction and summary of the Equations (140) – (141), the types of new exact solutions obtained by expanded rational sine-cosine method can be divided into one category. We choose a representative exact solution to draw the image.

Taking Equation (140) for example, we use Wolfram Mathematica software to draw the two-dimensional and three-dimensional graphs.

By plugging $r_1 = 3$, $r_2 = 1$, $r_3 = -2$, $r_4 = 1$, $k = 1$, $\lambda = 1$, $A = 2$, $B = 0$, and $C = 1$ into Equation (140), we get the equation as follows:

$$\varphi(x,t) = 2\cosh(x + t). \quad (142)$$

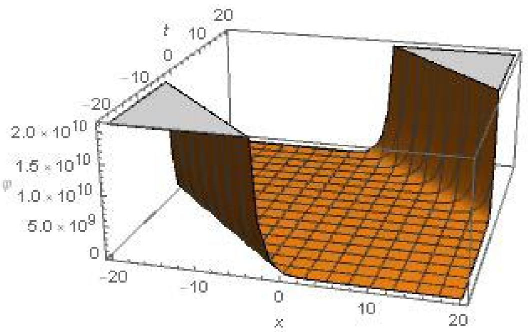


Figure 11. The three-dimensional graph of Equation (142)

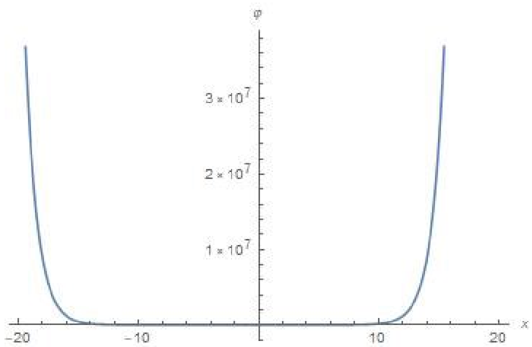


Figure 12. The two-dimensional graph of Equation (142), where t is equal to 1

In Figures 11-12, Equation (142) is a periodic solution of Equation (1), and other soliton type solutions can be obtained for different values of r_1, r_2, r_3, r_4, A, B, C, k, λ, t, and μ.

4. Conclusion

In this work, the exact solution of the GP equation was obtained using the Paul-Painlev´e approach method, extended rational sine-cosine and extended rational sinh-cosh method. The exact solution of GP equation obtained by the above three methods is new, and we use mathematical software to draw the two-dimensional and three-dimensional graphs of the new exact solutions. This paper not only extends the exact solution of GP equation, but also provides a new way to study more exact solutions of nonlinear equation in the future. But different methods will get different exact solutions, the method we use only gets part of the exact solution of the equation, so finding more exact solutions will be a problem to be further studied.

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