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A Comparative Study of Two Novel Analytical Methods for Solving Time-Fractional Coupled Boussinesq-Burger Equation

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Abstract

In this paper, a comparative study between two different methods for solving nonlinear time-fractional coupled Boussinesq-Burger equation is conducted. The techniques are denoted as the Natural Transform Decomposition Method (NTDM) and the Variational Iteration Transform Method (VITM). To showcase the efficacy and precision of the proposed approaches, a pair of different numerical examples are presented. The outcomes garnered indicate that both methods exhibit robustness and efficiency, yielding approximations of heightened accuracy and the solutions in a closed form. Nevertheless, the VITM boasts a distinct advantage over the NTDM by addressing nonlinear predicaments without recourse to the application of Adomian polynomials. Furthermore, the VITM's capacity to surmount challenges arising from the identification of the overarching Lagrange multiplier stands as an augmenting facet, amplifying its advantage over the NTDM technique.

Keywords: Adomain decomposition method; Nonlinear Caputo time-fractional Boussinesq-Burger equation; Variational iteration method; Series solution

MSC 2010 No.: 35R11, 35F55, 35C05

1. Introduction

Fractional differential equations are widely used in the interpretation and modeling of many realism matters that appeared in applied mathematics and physics including fluid mechanics, plasma physics, optical fibers, dispersive wave, acoustic, thermoelastic materials, medical sciences, biology, electricity channels, rusting physical chemistry, shock wave, mathematics, fluid mechanics and polymers, have performed studies in fractional behavior (Abu Arqub (2019); Momani et al. (2020a); Momani et al. (2020b); Maayah et al. (2022); Arqub and Maayah (2023); Abu Arqub et al. (2022)). Several authors have employed various techniques for solving fractional order of differential equations in linear or nonlinear forms. Some techniques include the homotopy analysis method (Shah and Singh (2019)), Shehu decomposition method (Chu et al. (2021)), homotopy perturbation method (Jani and Singh (2022); Choksi and Singh (2019)), variational iteration transform method (Chu et al. (2021); Chandru and Radhakrishnan (2022)), adomian decomposition Method (Yadav and Singh (2023c); Yadav and Singh (2023d); Khalouta and Kadem (2019)), new laplace variational iteration method (Anjum and He (2019)), modified variational iteration method (Yadav and Singh (2023b)), natural transform decomposition method and iterative shehu transform method (Kanth et al. (2021)).

The ADM was developed by G. Adomian, an American mathematician. It aims to find results in the form of a series and decompose the nonlinear operator into a sequence, with the terms being examined recursively using Adomian polynomials (Wazwaz and El-Sayed (2001)). By incorporating the Natural transformation, this technique has been modified to become the Natural transform decomposition method (NTDM). It has been successfully applied to various fractional higher-order partial differential equations (Kanth et al. (2021)).

He (1998), a Chinese mathematician, introduced the variational iteration method (VIM) as a modification of a general Lagrange multiplier technique (Inokuti et al. (1978)). This approach has proven to be effective in solving various problems. Similarly, the Shehu transform method has been used to modify this method, resulting in the variational iteration transform method (VITM). VITM has been successfully applied to solve different types of differential and partial differential equations. For example, it has been used to solve non-linear fractional dispersive partial differential equations in Wu and Baleanu (2013) and Chu et al. (2021) and nonlinear oscillator equations in Anjum and He (2019). The Elzaki transform with VIM is examined for the time-fractional derivative of a nonlinear PDE (Ziane et al. (2017)). Compared to Adomian's decomposition process, the computation of Adomian's polynomials is not necessary for VITM, and it provides a closed solution to the problem instead of an approximation at mesh points like the (Dehghan (2006)) mesh point methods. Additionally, VITM can provide an accurate approximation of the exact solution.

Fractional non-linear differential equations are important as they can accurately describe phenomena that cannot be explained by integer-order differential equations. The Boussinesq-Burgers equation describes the behavior of certain types of nonlinear waves in fluid dynamics and other fields like nonlinear acoustics and traffic flow. It combines the nonlinear and dispersive effects of the Boussinesq equation with the dissipative effects (due to viscosity) of the Burgers equation. Many numerical and semi-analytical methods, for example, Sarhan et al. used Laplace Residue Power series (Sarhan et al. (2022)), Gupta et al. used homotopy perturbation method and optimal homotopy asymptotic method (Gupta and Ray (2014)), Kumar et al. used homotopy analysis transform method (Kumar et al. (2016)), Khater et al. used generalized Kudryashov method (Khater and Kumar (2017)), and so on, are suggested in the open literature to obtain an approximated solution of the Boussinesq-Burgers Equation to obtain.

This work introduces a natural transform with the Adomian decomposition method, called this conjunction NTDM, and the Shehu transform with a variational iteration method, called this conjunction VITM. The proposed approaches are used to solve non-linear PDE without perturbation, discretizing them, with less computation which leads to a more realistic representation. The main motivation of this paper is to compare the NTDM and VITM solutions for solving the time-fractional coupled Boussinesq-Burgers (TF-CBB) equation in Caputo fractional derivative with a singular kernel. One of the great advantages of the Caputo fractional derivative is that it allows traditional initial and boundary conditions to be included in the formulation of the problem. In addition, its derivative for a constant is zero (Podlubny (1998)).

Different numerical examples are considered to validate the efficiency of the proposed methods.

1.1. Basic Definitions

In this section, necessary definitions and preliminary results about fractional calculus, Natural transform, and Shehu transform, which are used further in this paper, are referenced. For more details, see Yadav and Singh (2023d), Yadav and Singh (2023c), Kilbas et al. (2006), Maitama and Zhao (2019), Khalouta and Kadem (2019), and Belgacem and Silambarasan (2012).

2. Solution of nonlinear time-fractional coupled differential equations by the VITM

Theorem 2.1.

Consider the nonlinear Caputo time-fractional coupled partial differential equations (1) and (2) with the initial conditions. Then, by the VITM the exact solution of Equations (1) and (2) is given as a limit of the successive approximations $\theta_m(\varsigma,\tau)$ and $\eta_m(\varsigma,\tau)$, $m=0,1,2,\cdots$; in other words,

$$\theta(\varsigma,\tau)=\lim_{m\to\infty}\theta_m(\varsigma,\tau),$$

$$\eta(\varsigma,\tau)=\lim_{m\to\infty}\eta_m(\varsigma,\tau).$$

Proof:

The general form of the nonlinear non-homogeneous fractional partial differential equation (FPDE) is considered with an order of $\mu\in(m-1,m]$ to demonstrate the basic operation of the proposed methods, NTDM and VITM, which aim to solve the TF-CBB equation

$$D_\tau^\mu\theta(\varsigma,\tau)+\mathbf{N}_1\left[\theta(\varsigma,\tau)\right]+\mathbf{L}_1\left[\theta(\varsigma,\tau)\right]=h(\varsigma,\tau), \tag{1}$$

$$D_\tau^\mu \eta(\varsigma,\tau) + \mathbf{N}_2\left[\eta(\varsigma,\tau)\right] + \mathbf{L}_2\left[\eta(\varsigma,\tau)\right] = g(\varsigma,\tau), \tag{2}$$

with initial condition

$$\theta(\varsigma,0) = \psi(\varsigma), \quad \theta_\tau(\varsigma,0) = \phi(\varsigma), \;\; \eta(\varsigma,0) = \vartheta(\varsigma), \quad \eta_\tau(\varsigma,0) = \varphi(\varsigma), \tag{3}$$

where $D_\tau^\mu \theta(\varsigma,\tau)$ and $D_\tau^\mu \eta(\varsigma,\tau)$ is the Caputo fractional derivative of the function $\theta(\varsigma,\tau)$ and $\eta(\varsigma,\tau)$, respectively, $\mathbf{L}_1$ and $\mathbf{L}_2$ are the linear differential operators, $h(\varsigma,\tau)$ and $g(\varsigma,\tau)$ are the non-homogeneous terms and $\mathbf{N}_1$ and $\mathbf{N}_2$ represent the non-linear differential operators.

Equations (1) and (2), when subjected to the Shehu transform $(\mathbb{S})$ procedure for fractional derivative, lead to

$$\mathbb{S}\left[D_\tau^\mu \theta(\varsigma,\tau)\right] = \mathbb{S}[h(\varsigma,\tau)] + \mathbb{S}\left[\mathbf{L}_1\left[\theta(\varsigma,\tau)\right] + \mathbf{N}_1\left[\theta(\varsigma,\tau)\right]\right], \tag{4}$$

$$\mathbb{S}\left[D_\tau^\mu \eta(\varsigma,\tau)\right] = \mathbb{S}[g(\varsigma,\tau)] + \mathbb{S}\left[\mathbf{L}_2\left[\eta(\varsigma,\tau)\right] + \mathbf{N}_2\left[\eta(\varsigma,\tau)\right]\right]. \tag{5}$$

Applying the differentiation property of the Shehu transform, we obtain

$$\left(\frac{p}{q}\right)^\mu \mathbb{S}\left[\theta(\varsigma,\tau)\right] - \left(\frac{p}{q}\right)^{\mu-1} \psi(\varsigma) - \left(\frac{p}{q}\right)^{\mu-2} \phi(\varsigma) = \mathbb{S}[h(\varsigma,\tau)] + \mathbb{S}[\mathfrak{W}_1(\varsigma,\tau)], \tag{6}$$

$$\left(\frac{p}{q}\right)^\mu \mathbb{S}\left[\eta(\varsigma,\tau)\right] - \left(\frac{p}{q}\right)^{\mu-1} \vartheta(\varsigma) - \left(\frac{p}{q}\right)^{\mu-2} \varphi(\varsigma) = \mathbb{S}[g(\varsigma,\tau)] + \mathbb{S}[\mathfrak{W}_2(\varsigma,\tau)], \tag{7}$$

where

$$\mathfrak{W}_2(\varsigma,\tau) = \mathbf{N}_1(\theta(\varsigma,\tau)) + \mathbf{L}_1(\theta(\varsigma,\tau))) \text{ and } \mathfrak{W}(\varsigma,\tau) = \mathbf{N}_2(\eta(\varsigma,\tau)) + \mathbf{L}_2(\eta(\varsigma,\tau))).$$

After this, apply the inverse Shehu transform $(\mathbb{S})$ in the above equation. We have

$$\theta(\varsigma,\tau) = Q(\varsigma,\tau) + \mathbb{S}^{-1}\Big(\frac{q^\mu}{p^\mu}\mathbb{S}[\mathfrak{W}_1(\varsigma,\tau) + h(\varsigma,\tau)]\Big), \tag{8}$$

$$\eta(\varsigma,\tau) = R(\varsigma,\tau) + \mathbb{S}^{-1}\Big(\frac{q^\mu}{p^\mu}\mathbb{S}[\mathfrak{W}_2(\varsigma,\tau) + g(\varsigma,\tau)]\Big), \tag{9}$$

where $Q(\varsigma,\tau)$ and $R(\varsigma,\tau)$ are a term arising from the source term and the prescribed initial conditions. Take the first partial derivative with respect to τ in the above equation to obtain

$$\frac{\partial}{\partial\tau}\theta(\varsigma,\tau) - \frac{\partial}{\partial\tau}Q(\varsigma,\tau) - \frac{\partial}{\partial\tau}\mathbb{S}^{-1}\Big(\frac{q^\mu}{p^\mu}\mathbb{S}[\mathfrak{W}_1(\varsigma,\tau)]\Big) = 0, \tag{10}$$

$$\frac{\partial}{\partial\tau}\eta(\varsigma,\tau) - \frac{\partial}{\partial\tau}R(\varsigma,\tau) - \frac{\partial}{\partial\tau}\mathbb{S}^{-1}\Big(\frac{q^\mu}{p^\mu}\mathbb{S}[\mathfrak{W}_2(\varsigma,\tau)]\Big) = 0. \tag{11}$$

According to the variational iteration method (Biazar et al. (2010)), we can construct a correct functional as follows

$$\theta_{m+1}(\varsigma,\tau) = \theta_m(\varsigma,\tau) - \int_0^\tau \left[\frac{\partial\theta_m}{\partial\xi} - \frac{\partial Q}{\partial\tau} - \frac{\partial}{\partial\tau}\mathbb{S}^{-1}\left(\frac{q^\mu}{p^\mu}\mathbb{S}[\mathbf{N}_1(\theta_m) + \mathbf{L}_1(\theta_m))]\right)\right] d\xi, \tag{12}$$

$$\eta_{m+1}(\varsigma,\tau) = \eta_m(\varsigma,\tau) - \int_0^\tau \left[\frac{\partial\eta_m}{\partial\xi} - \frac{\partial R}{\partial\tau} - \frac{\partial}{\partial\tau}\mathbb{S}^{-1}\left(\frac{q^\mu}{p^\mu}\mathbb{S}[\mathbf{N}_1(\eta_m) + \mathbf{L}_1(\eta_m))]\right)\right] d\xi, \tag{13}$$

or

$$\theta_{m+1}(\varsigma,\tau) = Q(\varsigma,\tau) + \mathbb{S}^{-1}\left(\frac{q^\mu}{p^\mu}\mathbb{S}[\mathbf{N}_1(\theta_m) + \mathbf{L}_1(\theta_m))]\right), \tag{14}$$

$$\eta_{m+1}(\varsigma,\tau) = R(\varsigma,\tau) + \mathbb{S}^{-1}\left(\frac{q^\mu}{p^\mu}\mathbb{S}[\mathbf{N}_1(\eta_m) + \mathbf{L}_1(\eta_m))]\right). \tag{15}$$

Finally, the exact solution of Equations (30) and (31) is given as a limit of the successive approximations $\theta_m(\varsigma,\tau)$ and $\eta_m(\varsigma,\tau)$, $m = 0, 1, 2, ...$, in other words,

$$\theta(\varsigma,\tau) = \lim_{m\to\infty}\theta_m(\varsigma,\tau), \tag{16}$$

$$\eta(\varsigma,\tau) = \lim_{m\to\infty}\eta_m(\varsigma,\tau). \tag{17}$$

■

3. Solution of nonlinear time-fractional coupled differential equations by the NTDM

Theorem 3.1.

Consider the nonlinear Caputo time-fractional coupled partial differential equations (1) and (2) with the initial conditions. Then, by the VITM the exact solution of Equations (1) and (2) is given as a limit of the successive approximations $\theta_m(\varsigma,\tau)$ and $\eta_m(\varsigma,\tau)$, $m = 0, 1, 2, \cdots$; in other words,

$$\theta(\varsigma,\tau) = \sum_{m=0}^{\infty}\theta_m(\varsigma,\tau),$$

$$\eta(\varsigma,\tau) = \sum_{m=0}^{\infty}\eta_m(\varsigma,\tau).$$

Proof:

In this section, we see the methodology of NTDM. Applying natural transform $(\mathcal{NT})$ in Equation (1) and (2), we have

$$\left(\frac{p}{q}\right)^\mu\left(\mathcal{NT}\left[\theta(\varsigma,\tau)\right] - \frac{\psi(\varsigma)}{p} - \frac{q\phi(\varsigma)}{p^2}\right) = \mathcal{NT}\left[\mathbf{L}_1(\theta(\varsigma,\tau)) + \mathbf{N}_1(\theta(\varsigma,\tau))\right], \tag{18}$$

$$\left(\frac{p}{q}\right)^\mu\left(\mathcal{NT}\left[\eta(\varsigma,\tau)\right] - \frac{\vartheta(\varsigma)}{p} - \frac{q\varphi(\varsigma)}{p^2}\right) = \mathcal{NT}\left[\mathbf{L}_2(\eta(\varsigma,\tau)) + \mathbf{N}_2(\eta(\varsigma,\tau))\right], \tag{19}$$

where

$$\mathfrak{W}_1(\varsigma,\tau) = \mathbf{N}_1(\theta(\varsigma,\tau)) + \mathbf{L}_1(\theta(\varsigma,\tau)) \text{ and } \mathfrak{W}_2(\varsigma,\tau) = \mathbf{N}_2(\eta(\varsigma,\tau)) + \mathbf{L}_2(\eta(\varsigma,\tau)).$$

Express the nonlinear operator $\mathbf{N}_1$ and $\mathbf{N}_2$ as a decomposition into (Wazwaz and El-Sayed (2001))

$$\mathbf{N}_1(\theta(\varsigma,\tau)) = \sum_{m=0}^{\infty}\mathtt{A}_m \text{ and } \mathbf{N}_2(\eta(\varsigma,\tau)) = \sum_{m=0}^{\infty}\mathtt{B}_m. \tag{20}$$

The definition of the Adomian polynomials' nonlinearity is as follows

$$\mathtt{A}_m = \frac{1}{m!}\left[\frac{\partial^m}{\partial\lambda^m}\left(\mathbf{N_1}\left(\sum_{i=0}^{\infty}\lambda^i\theta_i\right)\right)\right]_{\lambda=0} \text{ and } \mathtt{B}_m = \frac{1}{m!}\left[\frac{\partial^m}{\partial\lambda^m}\left(\mathbf{N_2}\left(\sum_{i=0}^{\infty}\lambda^i\theta_i\right)\right)\right]_{\lambda=0}, \quad (21)$$

where $m = 0, 1, 2, 3, \ldots$. The analytical expansion for Equations (1) and (2) is assumed to be represented by the Adomian polynomials, which are denoted by $\mathtt{A}_m$ and $\mathtt{B}_m$,

$$\theta(\varsigma,\tau) = \sum_{m=0}^{\infty}\theta_m(\varsigma,\tau) \text{ and } \eta(\varsigma,\tau) = \sum_{m=0}^{\infty}\eta_m(\varsigma,\tau). \quad (22)$$

By inserting Equations (20) through (22) into (18) and (19), we obtain:

$$\begin{aligned}\theta(\varsigma,\tau) = \sum_{m=0}^{\infty}\theta_m(\varsigma,\tau) &= \mathcal{NT}^{-1}\left[\frac{q\phi(\varsigma)}{p^2} + \frac{\psi(\varsigma)}{p}\right] \\ &+ \mathcal{NT}^{-1}\left[\left(\frac{q}{p}\right)^{\mu}\mathcal{NT}\left[\sum_{m=0}^{\infty}\mathbf{L}_1(\theta_m(\varsigma,\tau)) + \mathtt{A}_m\right]\right], \end{aligned} \quad (23)$$

$$\begin{aligned}\eta(\varsigma,\tau) = \sum_{m=0}^{\infty}\eta_m(\varsigma,\tau) &= \mathcal{NT}^{-1}\left[\frac{q\varphi(\varsigma)}{p^2} + \frac{\vartheta(\varsigma)}{p}\right] \\ &+ \mathcal{NT}^{-1}\left[\left(\frac{q}{p}\right)^{\mu}\mathcal{NT}\left[\sum_{m=0}^{\infty}\mathbf{L}_2(\eta_m(\varsigma,\tau)) + \mathtt{B}_m\right]\right]. \end{aligned} \quad (24)$$

From Equations (23) and (24) , we get

$$\theta_0(\varsigma,\tau) = \mathcal{NT}^{-1}\left[\frac{\psi(\varsigma)}{p} + \frac{q\,\phi(\varsigma)}{p^2}\right],$$

$$\eta_0(\varsigma,\tau) = \mathcal{NT}^{-1}\left[\frac{\vartheta(\varsigma)}{p} + \frac{q\,\varphi(\varsigma)}{p^2}\right],$$

$$\theta_1(\varsigma,\tau) = \mathcal{NT}^{-1}\left[\left(\frac{q}{p}\right)^{\mu}\mathcal{NT}[\mathbf{L}_1(\theta_0(\varsigma,\tau)) + \mathtt{A}_0]\right],$$

$$\eta_1(\varsigma,\tau) = \mathcal{NT}^{-1}\left[\left(\frac{q}{p}\right)^{\mu}\mathcal{NT}[\mathbf{L}_2(\eta_0(\varsigma,\tau)) + \mathtt{B}_0]\right],$$

$$\vdots$$

$$\theta_{m+1}(\varsigma,\tau) = \mathcal{NT}^{-1}\left[\left(\frac{q^{\mu}}{p^{\mu}}\right)\mathcal{NT}[\mathbf{L}_1(\theta_m(\varsigma,\tau)) + \mathtt{A}_m]\right], \quad m \geq 1, \quad (25)$$

$$\eta_{m+1}(\varsigma,\tau) = \mathcal{NT}^{-1}\left[\left(\frac{q^{\mu}}{p^{\mu}}\right)\mathcal{NT}[\mathbf{L}_2(\eta_m(\varsigma,\tau)) + \mathtt{B}_m]\right], \quad m \geq 1. \quad (26)$$

To obtain the NTDM solution of Equations (1) through (3), we insert (25) and (26) into (22). The solution in series form can be obtained by Equations (25) and (26) as

$$\theta(\varsigma,\tau) = \theta_0(\varsigma,\tau) + \theta_1(\varsigma,\tau) + \theta_2(\varsigma,\tau) + \ldots = \sum_{m=0}^{\infty}\theta_m(\varsigma,\tau), \quad (27)$$

$$\eta(\varsigma,\tau)=\eta_0(\varsigma,\tau)+\eta_1(\varsigma,\tau)+\eta_2(\varsigma,\tau)+...=\sum_{m=0}^{\infty}\eta_m(\varsigma,\tau). \tag{28}$$

■

4. Convergence analysis

For the convergence of the NTDM and VITM, solutions have been used this definition.

Definition 4.1.

$\forall\, k \in N \cup 0; \upsilon_k$ can be obtain as (Yadav and Singh (2023d))

$$\upsilon_k=\begin{cases}\frac{\|\theta_k+1\|}{\|\theta_k\|} & \text{if } \|\theta_k\|\neq 0,\\ 0 & \text{if } \|\theta_k\|=0.\end{cases} \tag{29}$$

5. Numerical Application of time-fractional coupled Boussinesq-Burgers equation

In this work, we look at the time-fractional coupled Boussinesq-Burgers equation

$$D_\tau^\mu\theta-0.5\eta_\varsigma+2\theta\theta_\varsigma=0, \tag{30}$$

$$D_\tau^\mu\eta-0.5\theta_{\varsigma\varsigma\varsigma}+2(\theta\eta)_\varsigma=0, \tag{31}$$

$\tau>0$, $0<\mu\leq 1$, with initial condition

$$\theta(\varsigma,0)=\frac{ck}{2}+\frac{ck}{2}\tanh\left(\frac{-k\varsigma-\ln b}{2}\right), \tag{32}$$

$$\eta(\varsigma,0)=\frac{-k^2}{8}\text{sech}^2\left(\frac{k\varsigma+\ln b}{2}\right), \tag{33}$$

where $\theta(\varsigma,\tau)$ is the horizontal velocity field and $\eta(\varsigma,\tau)$ is the height of the water surface above a horizontal level at the bottom. At $\mu=1$, the exact solution of Equations (30) and (31) is given by

$$\theta(\varsigma,\tau)=\frac{ck}{2}+\frac{ck}{2}\tanh\left(\frac{ck^2\tau-k\varsigma-\ln b}{2}\right), \tag{34}$$

$$\eta(\varsigma,\tau)=\frac{-k^2}{8}\text{sech}^2\left(\frac{k\varsigma-ck^2\tau+\ln b}{2}\right). \tag{35}$$

Here Equations (30) and (31) are the general form of time-fractional coupled Boussinesq-Burgers equation as (Kumar et al. (2016)), (Sarhan et al. (2022)) with initial conditions (32) and (33). Equations (34) and (35) are the exact solution of the TF-CBB equation.

Now, we apply the VITM and NTDM to solve two examples of nonlinear Caputo time-fractional Boussinesq-Burger equations with different parameters and then compare our approximate solutions with the exact solutions.

Example 5.1.

Consider the nonlinear Caputo time-fractional coupled Boussinesq-Burgers equations (30) and (31) with initial conditions (32) and (33) at $c = 0.5$, $k = -1$ and $b = 2$.

5.1. Application of the VITM

Applying the steps involved in the VITM as presented in Section (2) to Equations (30) and (31), we obtain the iteration formula as follows

$$\theta_{m+1}(\varsigma,\tau) = -\frac{1}{4} - \frac{1}{4}\tanh\left(\frac{\varsigma - \ln 2}{2}\right) + \mathbb{S}^{-1}\left(\frac{q^{\mu}}{p^{\mu}}\mathbb{S}\left[0.5\frac{\partial}{\partial\varsigma}(\eta_m) - 2\theta_m\frac{\partial}{\partial\varsigma}(\theta_m)\right]\right), \tag{36}$$

$$\eta_{m+1}(\varsigma,\tau) = \frac{-1}{8}\text{sech}\left(\frac{\varsigma - \ln 2}{2}\right) + \mathbb{S}^{-1}\left(\frac{q^{\mu}}{p^{\mu}}\mathbb{S}\left[0.5\frac{\partial^3}{\partial\varsigma^3}(\eta_m) - 2\frac{\partial}{\partial\varsigma}(\theta_m\eta_m)\right]\right), \tag{37}$$

and

$$\theta_0(\varsigma,\tau) = -\frac{1}{4} - \frac{1}{4}\tanh\left(\frac{\varsigma - \ln 2}{2}\right),$$
$$\eta_0(\varsigma,\tau) = \frac{-1}{8}\,\text{sech}^2\left(\frac{\varsigma - \ln 2}{2}\right),$$

$$\theta_1(\varsigma,\tau) = -\frac{1}{4} - \frac{1}{4}\tanh\left(\frac{\varsigma - \ln 2}{2}\right) + \frac{0.0625\,\tau^{\mu}}{\cosh^3\left(\frac{\varsigma-\ln(2)}{2}\right)\Gamma(\mu+1)},$$

$$\eta_1(\varsigma,\tau) = \frac{-1}{8}\,\text{sech}^2\left(\frac{\varsigma - \ln 2}{2}\right) + \frac{0.0625\,\tau^{\mu}\,\sinh\left(\frac{\varsigma-\ln(2)}{2}\right)}{\cosh^3\left(\frac{\varsigma-\ln(2)}{2}\right)\Gamma(\mu+1)},$$

$$\theta_2(\varsigma,\tau) = -\frac{1}{4} - \frac{1}{4}\tanh\left(\frac{\varsigma - \ln 2}{2}\right) + \frac{0.0625\,\tau^{\mu}}{\cosh^3\left(\frac{\varsigma-\ln(2)}{2}\right)\Gamma(\mu+1)} + \frac{0.0312s\,\tau^{2\mu}\,\sinh\left(\frac{\varsigma-\ln(2)}{2}\right)}{\cosh^3\left(\frac{\varsigma-\ln(2)}{2}\right)\Gamma(2\mu+1)},$$

$$\eta_2(\varsigma,\tau) = \frac{-1}{8}\,\text{sech}^2\left(\frac{\varsigma - \ln 2}{2}\right) + \frac{0.0625\,\tau^{\mu}\,\sinh\left(\frac{\varsigma-\ln(2)}{2}\right)}{\cosh^3\left(\frac{\varsigma-\ln(2)}{2}\right)\Gamma(\mu+1)}$$
$$- \frac{0.015625\,\tau^{2\mu}\,\left(2\cosh\left(\frac{\varsigma-\ln(2)}{2}\right)-3\right)}{\cosh^4\left(\frac{\varsigma-\ln(2)}{2}\right)\Gamma(2\mu+1)},$$
$$\vdots$$

The VITM solution is obtained by incorporating $\theta_m(\varsigma,\tau)$ and $\eta_m(\varsigma,\tau)$ in Equations (16) and (17),

$$\theta(\varsigma,\tau) = \lim_{m\to\infty}\theta_m(\varsigma,\tau) \approx -\frac{1}{4} - \frac{1}{4}\tanh\left(\frac{x-\ln 2}{2}\right) + \frac{0.0625\,\tau^{\mu}}{\cosh^3\left(\frac{\varsigma-\ln(2)}{2}\right)\Gamma(\mu+1)} + \frac{0.03125\,\tau^{2\mu}\,\sinh\left(\frac{\varsigma-\ln(2)}{2}\right)}{\cosh^3\left(\frac{\varsigma-\ln(2)}{2}\right)\Gamma(2\mu+1)} + \cdots, \tag{38}$$

$$\eta(\varsigma,\tau) = \lim_{m\to\infty}\eta_m(\varsigma,\tau) \approx \frac{-1}{8}\,\text{sech}^2\left(\frac{\varsigma-\ln 2}{2}\right) + \frac{0.0625\,\tau^{\mu}\,\sinh\left(\frac{\varsigma-\ln(2)}{2}\right)}{\cosh^3\left(\frac{\varsigma-\ln(2)}{2}\right)\Gamma(\mu+1)} - \frac{0.015625\,\tau^{2\mu}\,\left(2\cosh\left(\frac{\varsigma-\ln(2)}{2}\right)-3\right)}{\cosh^4\left(\frac{\varsigma-\ln(2)}{2}\right)\Gamma(2\mu+1)} + \cdots. \tag{39}$$

Finally, as $m\to\infty$, the exact solution of Equations (30) and (31) is

$$\theta(\varsigma,\tau) = -\frac{1}{4} - \frac{1}{4}\tanh\left(\frac{0.5\tau+\varsigma-\ln 2}{2}\right), \tag{40}$$

$$\eta(\varsigma,\tau) = -\frac{1}{8}\text{sech}^2\left(\frac{-\varsigma-0.5\tau+\ln 2}{2}\right). \tag{41}$$

5.2. Application of the NTDM

Apply the steps involved in the NTDM as presented in Section (3) to Equations (30) and (31). The Natural transform is applied in governing equation with initial and boundary condition

$$\left(\frac{p}{q}\right)^{\alpha}\left[\mathcal{NT}[\theta(\varsigma,\tau)] - \left(\frac{1}{p}\right)\theta(\varsigma,0)\right] = \mathcal{NT}\left[0.5\eta_{\varsigma} - 2\theta\theta_{\varsigma}\right],$$
$$\left(\frac{p}{q}\right)^{\alpha}\left[\mathcal{NT}[\eta(\varsigma,\tau)] - \left(\frac{1}{p}\right)\eta(\varsigma,0)\right] = \mathcal{NT}\left[0.5\theta_{\varsigma\varsigma\varsigma} - 2(\theta\eta)_{\varsigma}\right].$$

Applying the inverse natural transform

$$\theta(\varsigma,\tau) = \mathcal{NT}^{-1}\left[\frac{\theta(\varsigma,0)}{p} + \frac{q^{\alpha}}{p^{\alpha}}\mathcal{NT}\left\{0.5\eta_{\varsigma} - 2\theta\theta_{\varsigma}\right\}\right],$$

$$\eta(\varsigma,\tau) = \mathcal{NT}^{-1}\left[\frac{\eta(\varsigma,0)}{p} + \frac{q^{\alpha}}{p^{\alpha}}\mathcal{NT}\left\{0.5\theta_{\varsigma\varsigma\varsigma} - 2(\theta\eta)_{\varsigma}\right\}\right].$$

Applying ADM approach

$$\sum_{m=0}^{\infty}\theta(\varsigma,\tau)=-\frac{1}{4}-\frac{1}{4}\tanh\left(\frac{\varsigma-\ln 2}{2}\right)+\mathcal{NT}^{-1}\left(\frac{q^{\mu}}{p^{\mu}}\mathcal{NT}\left[0.5\frac{\partial}{\partial\varsigma}(\eta_m)-2\sum_{m=0}^{\infty}A_m\right]\right), \tag{42}$$

$$\sum_{m=0}^{\infty}\theta_{m+1}(\varsigma,\tau)=\frac{-1}{8}\text{sech}\left(\frac{-\varsigma+\ln 2}{2}\right)+\mathcal{NT}^{-1}\left(\frac{q^{\mu}}{p^{\mu}}\mathcal{NT}\left[0.5\frac{\partial^3}{\partial\varsigma^3}(\eta_m)-2\frac{\partial}{\partial\varsigma}(\sum_{m=0}^{\infty}B_m)\right]\right), \tag{43}$$

where $\sum_{m=0}^{\infty}A_m = \theta\theta_{\varsigma}, \sum_{m=0}^{\infty}B_m = (\theta\eta)_{\varsigma}$, are the Adomian polynomials that represent the nonlinear terms, and

$$\theta_0(\varsigma,\tau)=-\frac{1}{4}-\frac{1}{4}\tanh\left(\frac{\varsigma-\ln 2}{2}\right),$$
$$\eta_0(\varsigma,\tau)=\frac{-1}{8}\,\text{sech}^2\left(\frac{\varsigma-\ln 2}{2}\right),$$

$$\theta_1(\varsigma,\tau)=\frac{0.0625\,\tau^{\mu}}{\cosh^3\left(\frac{\varsigma-\ln(2)}{2}\right)\Gamma(\mu+1)},$$

$$\eta_1(\varsigma,\tau)=\frac{0.0625\,\tau^{\mu}\,\sinh\left(\frac{\varsigma-\ln(2)}{2}\right)}{\cosh^3\left(\frac{\varsigma-\ln(2)}{2}\right)\Gamma(\mu+1)},$$

$$\theta_2(\varsigma,\tau)=\frac{0.03125\,\tau^{2\mu}\,\sinh\left(\frac{\varsigma-\ln(2)}{2}\right)}{\cosh^3\left(\frac{\varsigma-\ln(2)}{2}\right)\Gamma(2\mu+1)},$$
$$\eta_2(\varsigma,\tau)=-\frac{0.015625\,\tau^{2\mu}\,\left(2\cosh\left(\frac{\varsigma-\ln(2)}{2}\right)-3\right)}{\cosh^4\left(\frac{\varsigma-\ln(2)}{2}\right)\Gamma(2\mu+1)},$$
$$\vdots$$

The NTDM solution is obtained by incorporating $\theta_0(\varsigma,\tau),\theta_1(\varsigma,\tau),...$ and $\eta_0(\varsigma,\tau),\eta_1(\varsigma,\tau)$ in

Equations (27) and (28),

$$\theta(\varsigma,\tau) = -\frac{1}{4} - \frac{1}{4}\tanh\left(\frac{\varsigma-\ln 2}{2}\right) + \frac{0.0625\,\tau^{\mu}}{\cosh^3\left(\frac{\varsigma-\ln(2)}{2}\right)\Gamma(\mu+1)} + \frac{0.03125\,\tau^{2\mu}\,\sinh\left(\frac{\varsigma-\ln(2)}{2}\right)}{\cosh^3\left(\frac{\varsigma-\ln(2)}{2}\right)\Gamma(2\mu+1)} + \cdots, \tag{44}$$

$$\eta(\varsigma,\tau) = \frac{-1}{8}\,\text{sech}^2\left(\frac{\varsigma-\ln 2}{2}\right) + \frac{0.0625\,\tau^{\mu}\,\sinh\left(\frac{\varsigma-\ln(2)}{2}\right)}{\cosh^3\left(\frac{\varsigma-\ln(2)}{2}\right)\Gamma(\mu+1)} - \frac{0.015625\,\tau^{2\mu}\left(2\cosh\left(\frac{\varsigma-\ln(2)}{2}\right)-3\right)}{\cosh^4\left(\frac{\varsigma-\ln(2)}{2}\right)\Gamma(2\mu+1)} + \cdots. \tag{45}$$

Finally, as $m\to\infty$, the exact solution of Equations (30) and (31) is

$$\theta(\varsigma,\tau) = -\frac{1}{4} - \frac{1}{4}\tanh\left(\frac{0.5\tau+\varsigma-\ln 2}{2}\right), \tag{46}$$

$$\eta(\varsigma,\tau) = -\frac{1}{8}\text{sech}^2\left(\frac{-\varsigma-0.5\tau+\ln 2}{2}\right). \tag{47}$$

Example 5.2.

Consider the nonlinear Caputo time-fractional Boussinesq-Burgers equation (30) and (31) with initial condition (32) and (33) at $c = 0.5$, $k = -1$ and $b = 9$.

5.3. Application of the VITM

Applying the steps involved in the VITM as presented in Section (2) to Equations (30) and (31), we obtain the iteration formula as follows,

$$\theta_{m+1}(\varsigma,\tau) = -\frac{1}{4} - \frac{1}{4}\tanh\left(\frac{\varsigma-\ln 9}{2}\right) + \mathbb{S}^{-1}\left(\frac{q^{\mu}}{p^{\mu}}\mathbb{S}\left[0.5\frac{\partial}{\partial\varsigma}(\eta_m) - 2\theta_m\frac{\partial}{\partial\varsigma}(\theta_m)\right]\right), \tag{48}$$

$$\eta_{m+1}(\varsigma,\tau) = \frac{-1}{8}\text{sech}\left(\frac{-\varsigma+\ln 9}{2}\right) + \mathbb{S}^{-1}\left(\frac{q^{\mu}}{p^{\mu}}\mathbb{S}\left[0.5\frac{\partial^3}{\partial\varsigma^3}(\eta_m) - 2\frac{\partial}{\partial\varsigma}(\theta_m\eta_m)\right]\right), \tag{49}$$

and

$$\theta_0(\varsigma,\tau) = -0.25 - 0.25\,\tanh\left(0.5\,\varsigma - 1.0986\right),$$

$$\eta_0(\varsigma,\tau) = -0.125\,\text{sech}(0.5\varsigma - 1.0986)^2,$$

$$\theta_1(\varsigma,\tau) = -0.25 - 0.25\,\tanh(0.5\,\varsigma - 1.0986) + \frac{-0.0625\,\tau^{\mu}}{\Gamma(\mu+1)\cosh^2(0.5\,\varsigma - 1.0986)},$$

$$\eta_1(\varsigma,\tau) = -0.125\,\text{sech}(0.5\varsigma - 1.0986)^2 + \frac{0.0625\,\tau^{\mu}\sinh(0.5\,\varsigma - 1.0986)}{\Gamma(\mu+1)\cosh^3(0.5\,\varsigma - 1.0986)},$$

$$\begin{aligned}\theta_2(\varsigma,\tau) = &-0.25 - 0.25\,\tanh(0.5\,\varsigma - 1.0986) + \frac{-0.0625\,\tau^{\mu}}{\Gamma(\mu+1)\cosh^2(0.5\,\varsigma - 1.0986)}\\ &+ \frac{0.03125\,\tau^{2\mu}\sinh(0.5\,\varsigma - 1.0986)}{\cosh^3(0.5\,\varsigma - 1.0986)\,\Gamma(2\mu+1)},\end{aligned}$$

$$\begin{aligned}\eta_2(\varsigma,\tau) = &-0.125\,\text{sech}(0.5\varsigma - 1.0986)^2 + \frac{0.0625\,\tau^{\mu}\sinh(0.5\,\varsigma - 1.0986)}{\Gamma(\mu+1)\cosh^3(0.5\,\varsigma - 1.0986)}\\ &+ \frac{-0.015625\,\tau^{2\mu}\left(2\,\cosh^2(0.5\,\varsigma - 1.0986) - 3\right)}{\Gamma(2\mu+1)\cosh^4(0.5\,\varsigma - 1.0986)},\end{aligned}$$

$$\vdots$$

The VITM solution is obtained by incorporating $\theta_m(\varsigma,\tau)$ and $\eta_m(\varsigma,\tau)$ in Equations (16) and (17),

$$\begin{aligned}\theta(\varsigma,\tau) = \lim_{m\to\infty}\theta_m(\varsigma,\tau) \approx &-0.25 - 0.25\,\tanh(0.5\,\varsigma - 1.0986) + \frac{-0.0625\,\tau^{\mu}}{\Gamma(\mu+1)\cosh^2(0.5\,\varsigma - 1.0986)}\\ &+ \frac{0.03125\,\tau^{2\mu}\sinh(0.5\,\varsigma - 1.0986)}{\cosh^3(0.5\,\varsigma - 1.0986)\,\Gamma(2\mu+1)} + \cdots,\end{aligned} \tag{50}$$

$$\begin{aligned}\eta(\varsigma,\tau) = \lim_{m\to\infty}\eta_m(\varsigma,\tau) \approx &-0.125\,\text{sech}(0.5\varsigma - 1.0986)^2 + \frac{0.0625\,\tau^{\mu}\sinh(0.5\,\varsigma - 1.0986)}{\Gamma(\mu+1)\cosh^3(0.5\,\varsigma - 1.0986)}\\ &+ \frac{-0.015625\,\tau^{2\mu}\left(2\,\cosh^2(0.5\,\varsigma - 1.0986) - 3\right)}{\Gamma(2\mu+1)\cosh^4(0.5\,\varsigma - 1.0986)} + \cdots.\end{aligned} \tag{51}$$

Finally, as $m \to \infty$ the exact solution of Equations (30) and (31) is

$$\theta(\varsigma,\tau) = -\frac{1}{4} - \frac{1}{4}\tanh\left(\frac{0.5\tau + \varsigma - \ln 9}{2}\right), \tag{52}$$

$$\eta(\varsigma,\tau) = -\frac{1}{8}\text{sech}^2\left(\frac{-\varsigma - 0.5\tau + \ln 9}{2}\right). \tag{53}$$

5.4. Application of the NTDM

Applying the steps involved in the NTDM as presented in Section (3) to Equations (30) and (31), we have

$$\sum_{m=0}^{\infty}\theta(\varsigma,\tau) = -\frac{1}{4}-\frac{1}{4}\tanh\left(\frac{\varsigma-\ln 9}{2}\right)+\mathcal{NT}^{-1}\left(\frac{q^{\mu}}{p^{\mu}}\mathcal{NT}\left[0.5\frac{\partial}{\partial\varsigma}(\eta_m)-2\sum_{m=0}^{\infty}A_m\right]\right), \tag{54}$$

$$\sum_{m=0}^{\infty}\theta_{m+1}(\varsigma,\tau) = \frac{-1}{8}\text{sech}\left(\frac{-\varsigma+\ln 9}{2}\right)+\mathcal{NT}^{-1}\left(\frac{q^{\mu}}{p^{\mu}}\mathcal{NT}\left[0.5\frac{\partial^3}{\partial\varsigma^3}(\eta_m)-2\frac{\partial}{\partial\varsigma}(\sum_{m=0}^{\infty}B_m)\right]\right), \tag{55}$$

where $\sum_{m=0}^{\infty}A_m = \theta\theta_\varsigma, \sum_{m=0}^{\infty}B_m = (\theta\eta)_\varsigma$, are the Adomian polynomials that represent the nonlinear terms, and

$$\begin{aligned}
\theta_0(\varsigma,\tau) &= -0.25-0.25\,\tanh\left(0.5\,\varsigma-1.0986\right),\\
\eta_0(\varsigma,\tau) &= -0.125\,\text{sech}(0.5\varsigma-1.0986)^2,\\
\theta_1(\varsigma,\tau) &= \frac{-0.0625\,\tau^{\mu}}{\Gamma\left(\mu+1\right)\cosh^2\left(0.5\,\varsigma-1.0986\right)},\\
\eta_1(\varsigma,\tau) &= \frac{0.0625\,\tau^{\mu}\sinh\left(0.5\,\varsigma-1.0986\right)}{\Gamma\left(\mu+1\right)\cosh^3\left(0.5\,\varsigma-1.0986\right)},\\
\theta_2(\varsigma,\tau) &= \frac{0.03125\,\tau^{2\,\mu}\sinh\left(0.5\,\varsigma-1.0986\right)}{\cosh^3\left(0.5\,\varsigma-1.0986\right)\,\Gamma\left(2\,\mu+1\right)},\\
\eta_2(\varsigma,\tau) &= \frac{-0.015625\,\tau^{2\,\mu}\left(2\,\cosh^2\left(0.5\,\varsigma-1.0986\right)-3\right)}{\Gamma\left(2\,\mu+1\right)\cosh^4\left(0.5\,\varsigma-1.0986\right)},\\
&\vdots
\end{aligned} \tag{56}$$

The NTDM solution is obtained by incorporating $\theta_0(\varsigma,\tau),\theta_1(\varsigma,\tau),...$ and $\eta_0(\varsigma,\tau),\eta_1(\varsigma,\tau)$ in Equations (27) and (28),

$$\begin{aligned}
\theta(\varsigma,\tau) = &-0.25-0.25\,\tanh\left(0.5\,\varsigma-1.0986\right)+\frac{-0.0625\,\tau^{\mu}}{\Gamma\left(\mu+1\right)\cosh^2\left(0.5\,\varsigma-1.0986\right)}\\
&+\frac{0.03125\,\tau^{2\,\mu}\sinh\left(0.5\,\varsigma-1.0986\right)}{\cosh^3\left(0.5\,\varsigma-1.0986\right)\,\Gamma\left(2\,\mu+1\right)}+\cdots,
\end{aligned} \tag{57}$$

$$\begin{aligned}
\eta(\varsigma,\tau) = &-0.125\,\text{sech}(0.5\varsigma-1.0986)^2+\frac{0.0625\,\tau^{\mu}\sinh\left(0.5\,\varsigma-1.0986\right)}{\Gamma\left(\mu+1\right)\cosh^3\left(0.5\,\varsigma-1.0986\right)}\\
&+\frac{-0.015625\,\tau^{2\,\mu}\left(2\,\cosh^2\left(0.5\,\varsigma-1.0986\right)-3\right)}{\Gamma\left(2\,\mu+1\right)\cosh^4\left(0.5\,\varsigma-1.0986\right)}+\cdots.
\end{aligned} \tag{58}$$

Finally, as $m \to \infty$ the exact solution of Equations (30) and (31) is

$$\theta(\varsigma,\tau) = -\frac{1}{4} - \frac{1}{4}\tanh\left(\frac{0.5\tau+\varsigma-\ln 9}{2}\right), \tag{59}$$

$$\eta(\varsigma,\tau) = -\frac{1}{8}\text{sech}^2\left(\frac{-\varsigma-0.5\tau+\ln 9}{2}\right). \tag{60}$$

6. Numerical results and discussion

In Example (5.1), we have used $b = 2$ to obtain the solution for the time-fractional coupled Boussinesq-Burgers equation (TF-CBB). Figure (1) shows a comparison between the VITM, NTDM, and Exact solution of $\theta(\varsigma,\tau)$ and $\eta(\varsigma,\tau)$. Figure (2) shows a comparison of a plot of maximum absolute errors between the VITM and NTDM solution of $\theta(\varsigma,\tau)$ and $\eta(\varsigma,\tau)$. Figure (3) displays a comparison between the three-dimensional NTDM, VITM, and exact solutions of the velocity function $\theta(\varsigma,\tau)$ for the nonlinear TF-CBB equation at $\mu = 1$, with $-10 < \varsigma < 10$ and $0 < \tau < 1$. Similarly, Figure (4) shows for the height $\eta(\varsigma,\tau)$. All the 3D and 2D plots of solutions obtained from our proposed methods exhibit behavior that is similar to the exact solution. Figure (5) compares the 3D plot of absolute errors between the NTDM and VITM solutions for $\theta(\varsigma,\tau)$ at $\mu = 1$. Similarly, Figure (6) compares the absolute errors between the NTDM and VITM solutions for $\eta(\varsigma,\tau)$ at $\mu = 1$. Figures (2), (5) and (6) indicate that the approximate solutions obtained by NTDM and VITM are the closest to the exact solution and have minimal errors. Furthermore, Figure (7) illustrates a comparison of the velocity $\theta(\varsigma,\tau)$ at different values of fractional orders $(\mu = 0.6, 0.7, 0.8, 0.9, 1)$ for $\varsigma = 5$. Similarly, Figure (8) shows a comparison of the heights $\eta(\varsigma,\tau)$ at the same fractional orders μ and $\varsigma = 5$. These figures collectively say that the proposed method yields highly accurate solutions, demonstrating excellent agreement with the exact solutions.

In Example (5.2), we have used $b = 9$ to obtain the solution for the time-fractional coupled Boussinesq-Burgers equation (TF-CBB). Figure (9) shows a comparison between the VITM, NTDM, and Exact solution of $\theta(\varsigma,\tau)$ and $\eta(\varsigma,\tau)$. Figure (10) shows a comparison of a plot of maximum absolute errors between the VITM and NTDM solution of $\theta(\varsigma,\tau)$ and $\eta(\varsigma,\tau)$. Figure (11) displays a comparison between the three-dimensional NTDM, VITM, and exact solutions of the velocity function $\theta(\varsigma,\tau)$ for the nonlinear TF-CBB equation at $\mu = 1$, $b = 2$, with $-10 < \varsigma < 10$ and $0 < \tau < 1$. Similarly, Figure (12) shows for the height $\eta(\varsigma,\tau)$. All the 3D and 2D plots of solutions obtained from our proposed methods exhibit behavior that is similar to the exact solution. Figure (13) compares the 3D plot of absolute errors between the NTDM and VITM solutions for $\theta(\varsigma,\tau)$ at $\mu = 1$. Similarly, Figure (14) compares the absolute errors between the NTDM and VITM solutions for $\eta(\varsigma,\tau)$ at $\mu = 1$. Figures (10), (13), and (14) indicate that the approximate solutions obtained by NTDM and VITM are the closest to the exact solution and have minimal errors. Furthermore, Figure (15) illustrates a comparison of the velocity $\theta(\varsigma,\tau)$ at different values of fractional orders $(\mu = 0.6, 0.7, 0.8, 0.9, 1)$ for $\varsigma = 5$. Similarly, Figure (16) compares the heights $\eta(\varsigma,\tau)$ at the same fractional orders μ and $\varsigma = 5$. These figures collectively say that the proposed method yields highly accurate solutions, demonstrating excellent agreement with the exact solutions.

Tables (1) and (2), we have computed the numerical comparison of absolute errors for differences between the exact solutions and the 3^{rd} order approximate solution by VITM and the 4^{rth} term approximate solution by NTDM at $\mu = 1$. The absolute errors obtained by VITM are the same results obtained by NTDM.

7. Conclusion

In this study, we have compared the solution obtained from the variational iteration Transform method (VITM) and the Natural transform decomposition method (NTDM) for solving the non-linear Caputo time-fractional coupled Boussinesq-Burgers equation. These two methods stand as reliable and effective approaches as both of them give solutions in approximations with higher precision. The correlation between the outcome of the third iteration using VITM and the fourth term within NTDM manifests a remarkable concordance. Nonetheless, a distinct advantage of VITM lies in its capacity to tackle nonlinear quandaries sans the utilization of Adomian polynomials. This attribute empowers VITM to surmount the challenges arising from the calculations of the general Lagrange multiplier, presenting an added edge over the decomposition method. It is concluded that these methods are very powerful mathematical tools for solving different kinds of nonlinear fractional differential equations. These methods are useful to solve real-life world problems.

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REFERENCES

Abu Arqub, O. (2019). Computational algorithm for solving singular Fredholm time-fractional partial integrodifferential equations with error estimates, Journal of Applied Mathematics and Computing, Vol. 59, No. 1-2, pp. 227-243.

Abu Arqub, O., Alsulami, H. and Alhodaly, M. (2022). Numerical Hilbert space solution of fractional Sobolev equation in $1 + 1$-dimensional space, Mathematical Sciences, pp. 1-12. https://doi.org/10.1007/s40096-022-00495-9

Adomian, G. (2013). Solving frontier problems of physics: The decomposition method, Springer Science and Business Media, Vol. 60.

Arqub, O. A. and Maayah, B. (2023). Adaptive the Dirichlet model of mobile/immobile advection/dispersion in a time-fractional sense with the reproducing kernel computational approach: Formulations and approximations, International Journal of Modern Physics B, Vol. 37, No. 18, 2350179.

Anjum, N. and He, J.H. (2019). Laplace transform: Making the variational iteration method easier, Applied Mathematics Letters, Vol. 92, pp. 134-138.

Belgacem, F.B.M. and Silambarasan, R. (2012). Theory of natural transform, Mathematical Engineering, Science and Aerospace, Vol 3, No. 2, pp. 99-124.

Biazar, J., Gholamin, P. and Hosseini, K. (2010). Variational iteration method for solving Fokker Planck equation, Journal of Franklin Institute, Vol. 347, No. 7, pp. 1137-1147.

Chandru, P. and Radhakrishnan, B. (2022). Comparison between the homotopy perturbation method and variational iteration method for fuzzy differential equations, Applications and Applied Mathematics: An International Journal (AAM), Vol. 17, No. 3, pp. 15.

Choksi, B.G. and Singh, T.R. (2019). A mathematical model of imbibition phenomenon in homogeneous porous media, Special Topics and Reviews in Porous Media: An International Journal, Vol. 10, No. 1, pp. 1-13.

Chu, Y.M., Bani Hani, E.H., El-Zahar, E.R., Ebaid, A. and Shah, N.A. (2021). Combination of Shehu decomposition and variational iteration transform methods for solving fractional third order dispersive partial differential equations, Numerical Methods for Partial Differential Equations, pp. 1-18.

Dehghan, M. (2006). Finite difference procedures for solving a problem arising in modeling and design of certain optoelectronic devices, Mathematics and Computers in Simulation, Vol. 71, No. 1, pp. 16-30.

Gupta, A.K. and Ray, S.S. (2014). Comparison between homotopy perturbation method and optimal homotopy asymptotic method for the soliton solutions of Boussinesq–Burger equations, Computers and Fluids, Vol. 103, pp. 34-41.

He, J.H. (1998). Approximate solution of nonlinear differential equations with convolution product nonlinearities, Computer methods in applied mechanics and engineering, Vol. 167, No. 1-2, pp. 69-73.

Inokuti, M., Sekine, H. and Mura, T. (1978). General use of the Lagrange multiplier in nonlinear mathematical physics, Variational Method in the Mechanics of Solids, Vol. 33, No. 5, pp. 156-162.

Jani, H.P. and Singh, T.R. (2022). A robust analytical method for regularized long wave equations, Iranian Journal of Science and Technology, Transactions A: Science, Vol. 46, No. 6, pp. 1667-1679.

Kanth, A.R., Aruna, K., Raghavendar, K., Rezazadeh, H. and İnç, M. (2021). Numerical solutions of nonlinear time fractional Klein-Gordon equation via natural transform decomposition method and iterative Shehu transform method, Journal of Ocean Engineering and Science, Vol. 12, No. 2, pp. 1-18.

Khalouta, A. and Kadem, A. (2019). A new method to solve fractional differential equations: Inverse fractional Shehu transform method, Appl. Appl. Math., Vol. 14, No. 2, pp. 926-941.

Khan, Z.H. and Khan, W.A. (2008). N-transform properties and applications, NUST Journal of Engineering Sciences, Vol. 1, No. 1, pp. 127-133.

Khater, M.M. and Kumar, D. (2017). New exact solutions for the time fractional coupled Boussinesq–Burger equation and approximate long water wave equation in shallow water, Journal of Ocean Engineering and Science, Vol. 2, No. 3, pp. 223-228.

Kilbas, A.A., Srivastava, H.M. and Trujillo, J.J. (2006). *Theory and Applications of Fractional*

Differential Equations (Vol. 204), Elsevier.

Kumar, S., Kumar, A. and Baleanu, D. (2016). Two analytical methods for time-fractional nonlinear coupled Boussinesq–Burger's equations arise in propagation of shallow water waves, Nonlinear Dynamics, Vol. 85, No. 2, pp. 699-715.

Maayah, B., Moussaoui, A., Bushnaq, S. and Arqub, O.A. (2022). The multistep Laplace optimized decomposition method for solving fractional-order coronavirus disease model (COVID-19) via the Caputo fractional approach, Demonstratio Mathematica, Vol. 55, No. 1, pp. 963-977.

Maitama, S. and Zhao, W. (2019), New integral transform: Shehu transform a generalization of Sumudu and Laplace transform for solving differential equations, International Journal of Analysis and Application, Vol. 17, No. 2, pp. 167-190.

Momani, S., Arqub, O. A. and Maayah, B. (2020a). Piecewise optimal fractional reproducing kernel solution and convergence analysis for the Atangana–Baleanu–Caputo model of the Lienard's equation, Fractals, Vol. 28, No. 8, pp. 2040007.

Momani, S., Maayah, B. and Arqub, O.A. (2020b). The reproducing kernel algorithm for numerical solution of Van der Pol damping model in view of the Atangana–Baleanu fractional approach, Fractals, Vol. 28, No. 8, pp. 2040010.

Podlubny, I. (1998). *Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications*, Elsevier.

Sarhan, A., Burqan, A., Saadeh, R. and Al-Zhour, Z. (2022). Analytical solutions of the nonlinear time-fractional coupled Boussinesq-Burger equations using Laplace residual power series technique, Fractal and Fractional, Vol. 6, No. 11, pp. 631.

Shah, K. and Singh, T. (2019). Modified approach to solve nonlinear equation arising in infiltration phenomenon, SeMA Journal, Vol. 76, No. 1, pp. 79-95.

Wazwaz, A. M. and El-Sayed, S. M. (2001). A new modification of the Adomian decomposition method for linear and nonlinear operators, Applied Mathematics and computation, Vol. 122, No. 3, pp. 393-405.

Wu, G.C. and Baleanu, D. (2013). Variational iteration method for fractional calculus-a universal approach by Laplace transform, Advances in Difference Equations, Vol. 2013, No. 1, pp. 1-9.

Yadav, J.U. and Singh, T.R. (2023a). An approximate analytical approach of water transport in an unsaturated porous medium by modified variational iteration method, New Frontiers in Communication and Intelligent Systems, SCRS, India, pp. 747-761. https://doi.org/10.52458/978-81-95502-00-4-74

Yadav, J.U. and Singh, T.R. (2023b). Solution of one-dimensional ground water recharge through porous media via natural transform decomposition method and variational iteration transform method, Mathematics in Engineering, Science and Aerospace, Vol. 14, No. 3, pp. 773-791.

Yadav, J.U. and Singh, T.R. (2023c). Solution of water infiltration phenomenon in unsaturated soils with fractional approach, Mathematics in Engineering, Science and Aerospace, Vol. 14, No. 4, pp. 1295-1317.

Yadav, J.U. and Singh, T.R. (2023d). Study of imbibition phenomenon in homogeneous porous medium model for oil recovery with temporal-fractional approach, Journal of Applied Physics, Vol. 134, No. 21, pp. 214901.

Ziane, D., Cherif, M.H. and Belghaba, K. (2017). Fractional higher dimensional initial boundary value problems via variational iteration method coupled with Elzaki transform, Nonlinear Studies, Vol. 24, No. 4, pp. 1-17.

Appendix

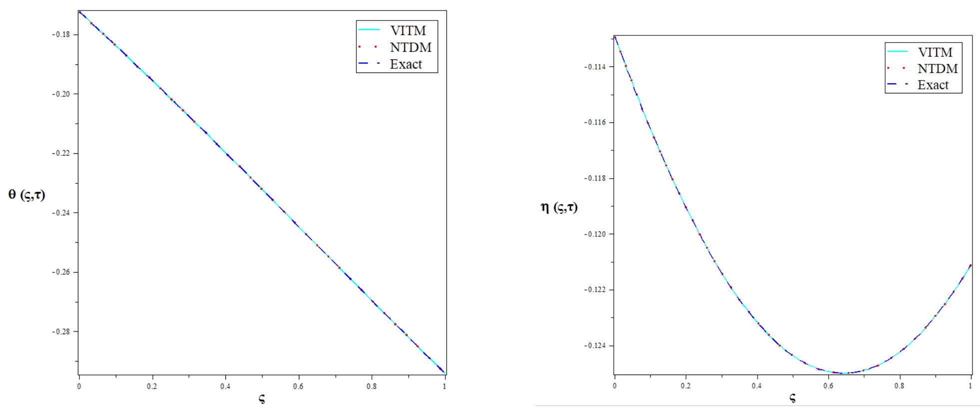

Figure 1. For example 1, comparison graph of VITM, NTDM and Exact solution at $\tau = 0.1$

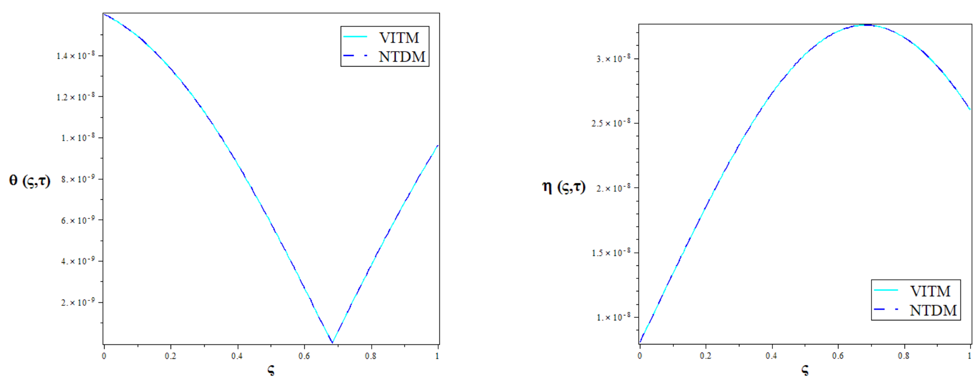

Figure 2. For example 1, plot of maximum absolute errors of VITM and NTDM solution at $\tau = 0.1$

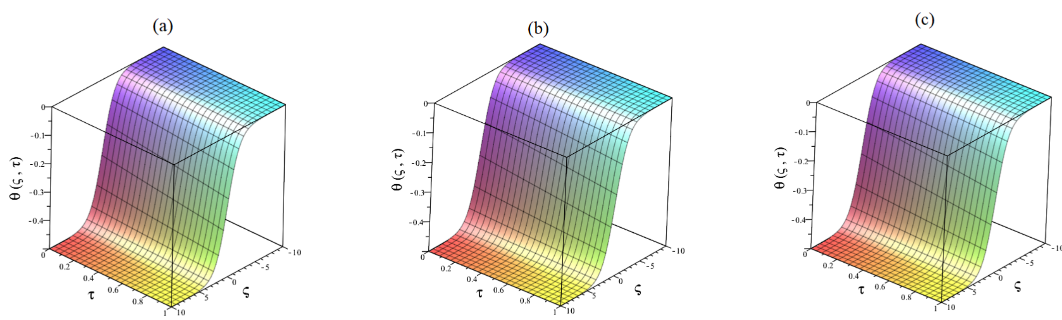

Figure 3. Surface plot of velocity $\theta(\varsigma, \tau)$ (a) Exact (b) NTDM and (c) VITM solution at $\mu = 1$

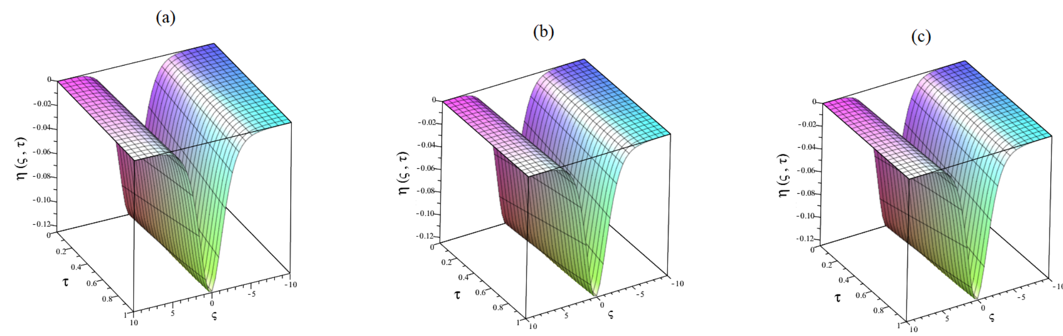

Figure 4. Surface plot of height $\eta(\varsigma, \tau)$ (a) Exact (b) NTDM and (c) VITM solution at $\mu = 1$

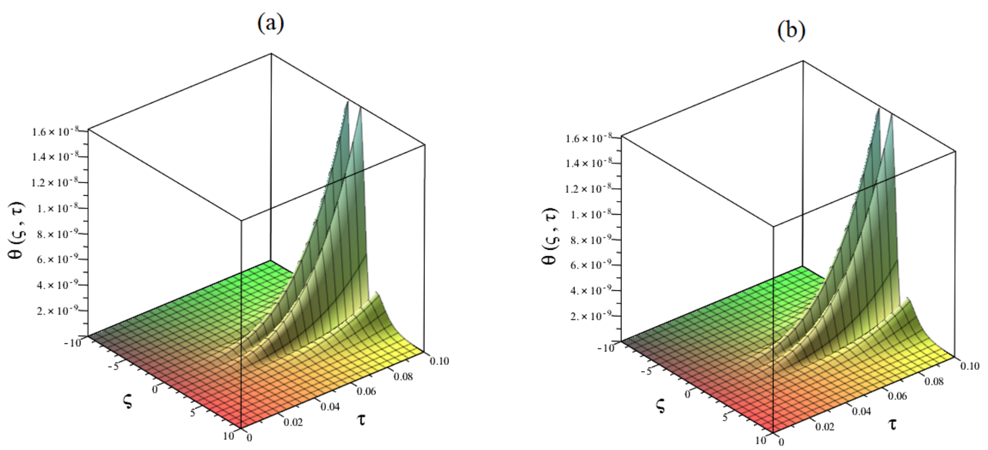

Figure 5. Comparison of absolute error of velocity $\theta(\varsigma, \tau)$ (a) NTDM and (b) VITM solution at $\mu = 1$

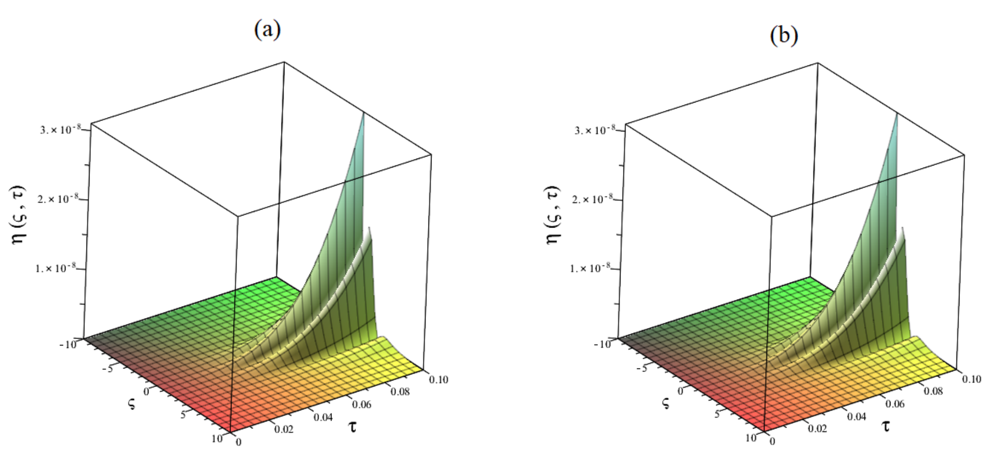

Figure 6. Comparison of absolute error of height $\eta(\varsigma, \tau)$ (a) NTDM and (b) VITM solution at $\mu = 1$

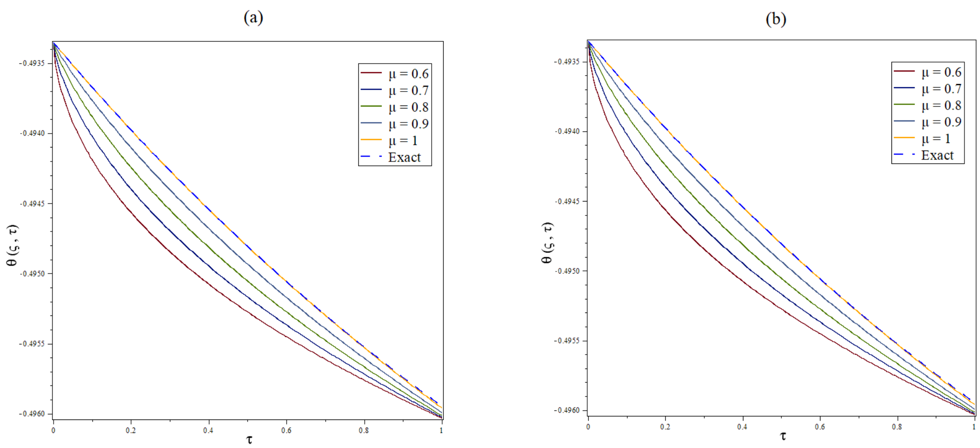

Figure 7. Comparison of velocity $\theta(\varsigma, \tau)$ (a) NTDM and (b) VITM solution at different fractional order μ

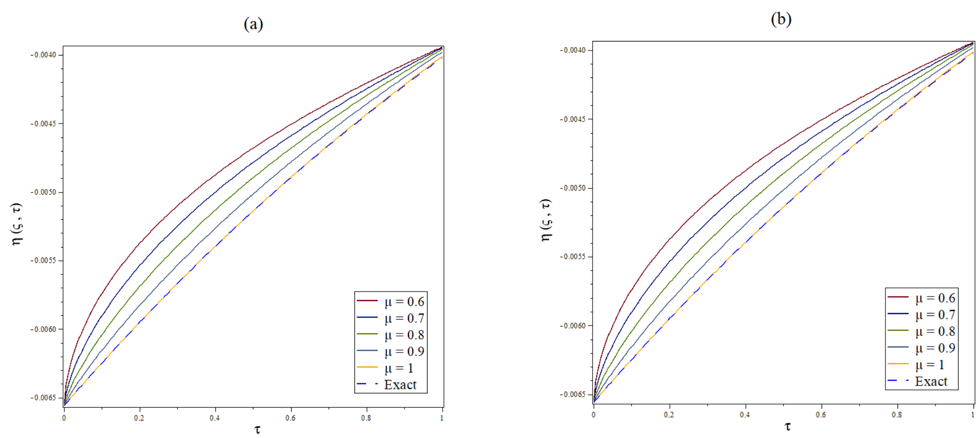

Figure 8. Comparison of height $\eta(\varsigma, \tau)$ (a) NTDM and (b) VITM solution at different fractional order μ

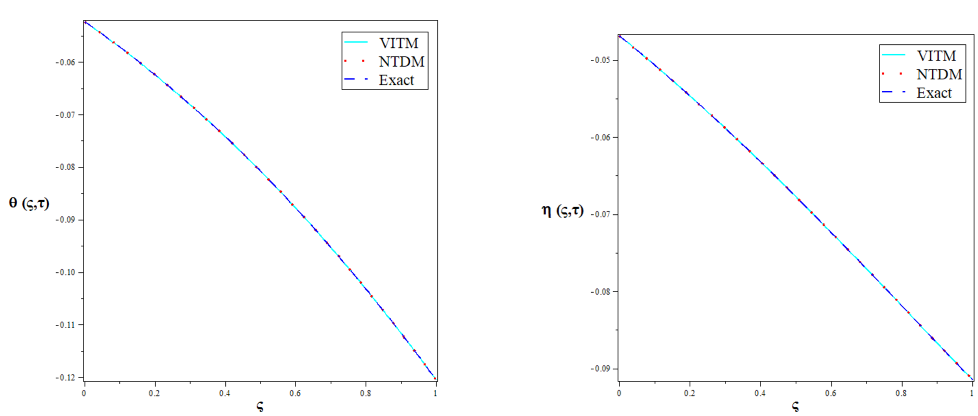

Figure 9. For example 2, comparison graph of VITM, NTDM and Exact solution at $\tau = 0.1$

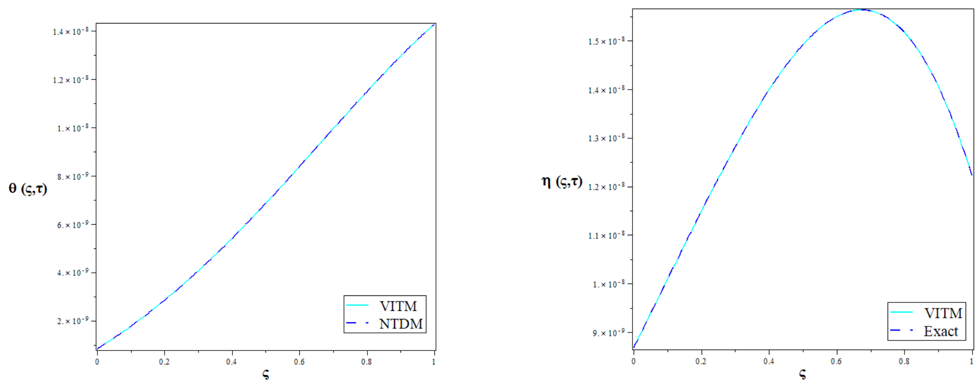

Figure 10. For example 2, plot of maximum absolute errors of VITM and NTDM solution at $\tau = 0.1$

Table 1. Comparision of the absolute errors for Example (5.1) at $\mu = 1$, $c = 0.5$, $k = -1$ and $b = 2$

(ς, τ)	$\lvert\theta_{Exact} - \theta_{NTDM}\rvert$	$\lvert\theta_{Exact} - \theta_{VITM}\rvert$	$\lvert\eta_{Exact} - \eta_{NTDM}\rvert$	$\lvert\eta_{Exact} - \eta_{VITM}\rvert$
(0.1,0.1)	1.5000×10^{-08}	1.5000×10^{-08}	1.3200×10^{-08}	1.3200×10^{-08}
(0.1,0.2)	2.3650×10^{-07}	2.3650×10^{-07}	2.2170×10^{-07}	2.2170×10^{-07}
(0.1,0.3)	1.1856×10^{-06}	1.1856×10^{-06}	1.1647×10^{-06}	1.1647×10^{-06}
(0.1,0.4)	3.7064×10^{-06}	3.7064×10^{-06}	3.8139×10^{-06}	3.8139×10^{-06}
(0.1,0.5)	8.9445×10^{-06}	8.9445×10^{-06}	9.6315×10^{-06}	9.6315×10^{-06}
(0.2,0.1)	1.3400×10^{-08}	1.3400×10^{-08}	1.8400×10^{-08}	1.8400×10^{-08}
(0.2,0.2)	2.1030×10^{-07}	2.1030×10^{-07}	3.0400×10^{-07}	3.0400×10^{-07}
(0.2,0.3)	1.0483×10^{-06}	1.0483×10^{-06}	1.5785×10^{-06}	1.5785×10^{-06}
(0.2,0.4)	3.2595×10^{-06}	3.2595×10^{-06}	5.1105×10^{-06}	5.1105×10^{-06}
(0.2,0.5)	7.8225×10^{-06}	7.8225×10^{-06}	1.2767×10^{-05}	1.2767×10^{-05}
(0.3,0.1)	1.1200×10^{-08}	1.1200×10^{-08}	2.3300×10^{-08}	2.3300×10^{-08}
(0.3,0.2)	1.7590×10^{-07}	1.7590×10^{-07}	3.7890×10^{-07}	3.7890×10^{-07}
(0.3,0.3)	8.7110×10^{-07}	8.7110×10^{-07}	1.9524×10^{-06}	1.9524×10^{-06}
(0.3,0.4)	2.6887×10^{-06}	2.6887×10^{-06}	6.2733×10^{-06}	6.2733×10^{-06}
(0.3,0.5)	6.4024×10^{-06}	6.4024×10^{-06}	1.5557×10^{-05}	1.5557×10^{-05}
(0.4,0.1)	8.8000×10^{-09}	8.8000×10^{-09}	2.7200×10^{-08}	2.7200×10^{-08}
(0.4,0.2)	1.3500×10^{-07}	1.3500×10^{-07}	4.4180×10^{-07}	4.4180×10^{-07}
(0.4,0.3)	6.6000×10^{-07}	6.6000×10^{-07}	2.2622×10^{-06}	2.2622×10^{-06}
(0.4,0.4)	2.0118×10^{-06}	2.0118×10^{-06}	7.2267×10^{-06}	7.2267×10^{-06}
(0.4,0.5)	4.7288×10^{-06}	4.7288×10^{-06}	1.7820×10^{-05}	1.7820×10^{-05}
(0.5,0.1)	5.8000×10^{-09}	5.8000×10^{-09}	3.0300×10^{-08}	3.0300×10^{-08}
(0.5,0.2)	8.8200×10^{-08}	8.8200×10^{-08}	4.8800×10^{-07}	4.8800×10^{-07}
(0.5,0.3)	4.2160×10^{-07}	4.2160×10^{-07}	2.4870×10^{-06}	2.4870×10^{-06}
(0.5,0.4)	1.2525×10^{-06}	1.2525×10^{-06}	5.9179×10^{-05}	5.9179×10^{-05}
(0.5,0.5)	2.8612×10^{-06}	2.8612×10^{-06}	1.9403×10^{-05}	1.9403×10^{-05}

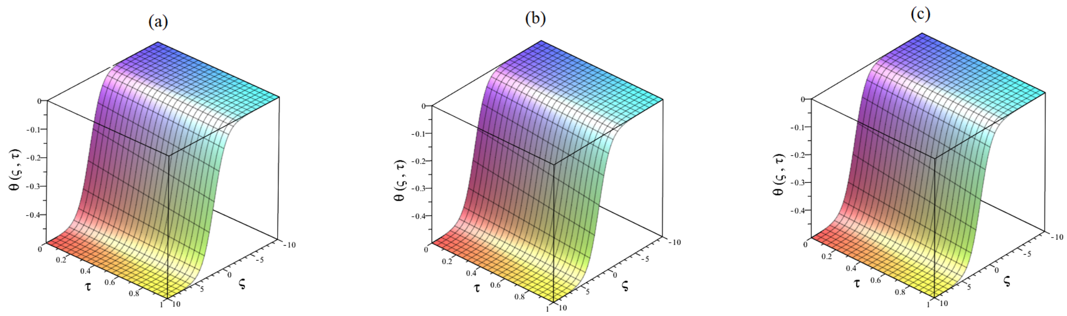

Figure 11. Surface plot of velocity $\theta(\varsigma, \tau)$ (a) Exact (b) NTDM and (c) VITM solution at $\mu = 1$

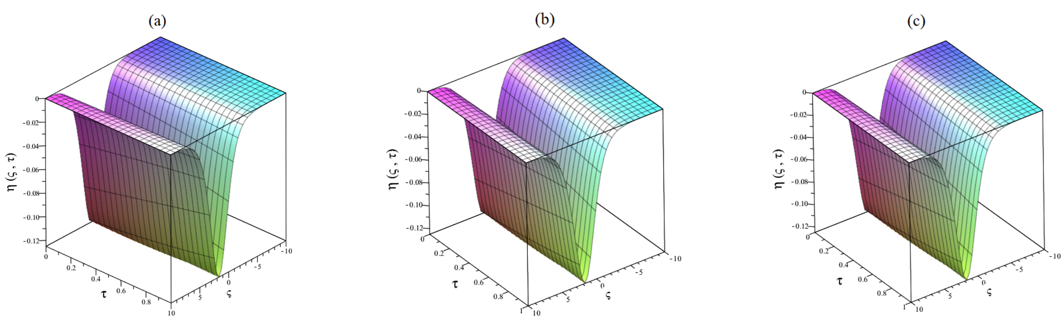

Figure 12. Surface plot of height $\eta(\varsigma, \tau)$ (a) Exact (b) NTDM and (c) VITM solution at $\mu = 1$

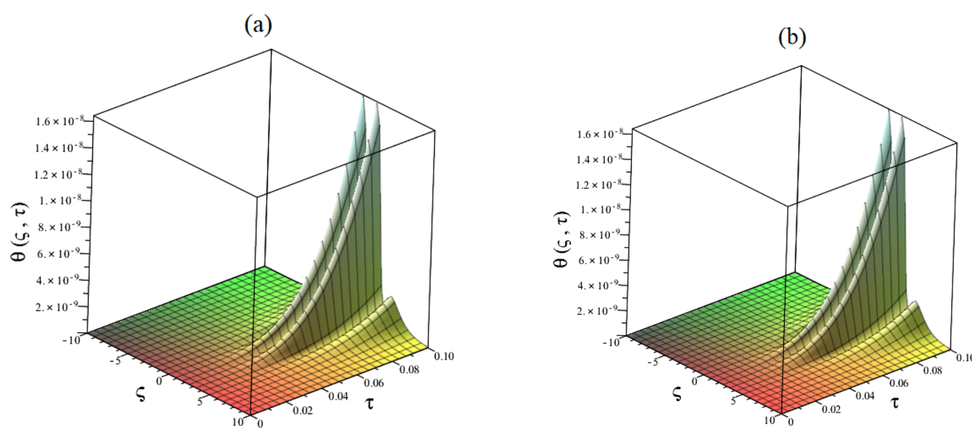

Figure 13. Comparison of absolute error of velocity $\theta(\varsigma, \tau)$ (a) NTDM and (b) VITM solution at $\mu = 1$

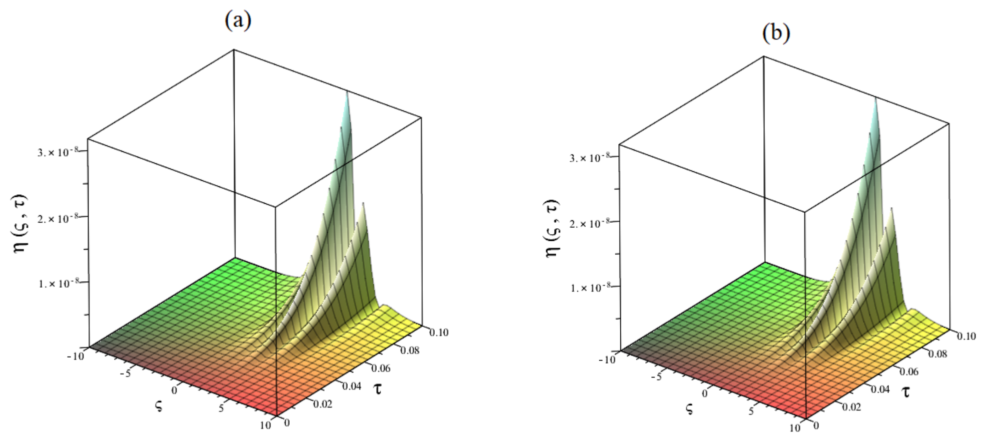

Figure 14. Comparison of absolute error of height $\eta(\varsigma,\tau)$ (a) NTDM and (b) VITM solution at $\mu = 1$

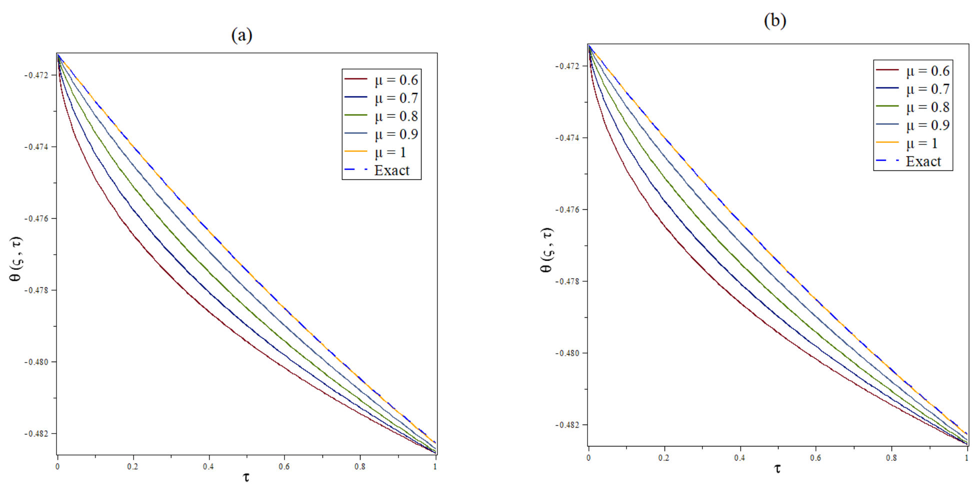

Figure 15. Comparison of velocity $\theta(\varsigma,\tau)$ (a) NTDM and (b) VITM solution at different fractional order μ

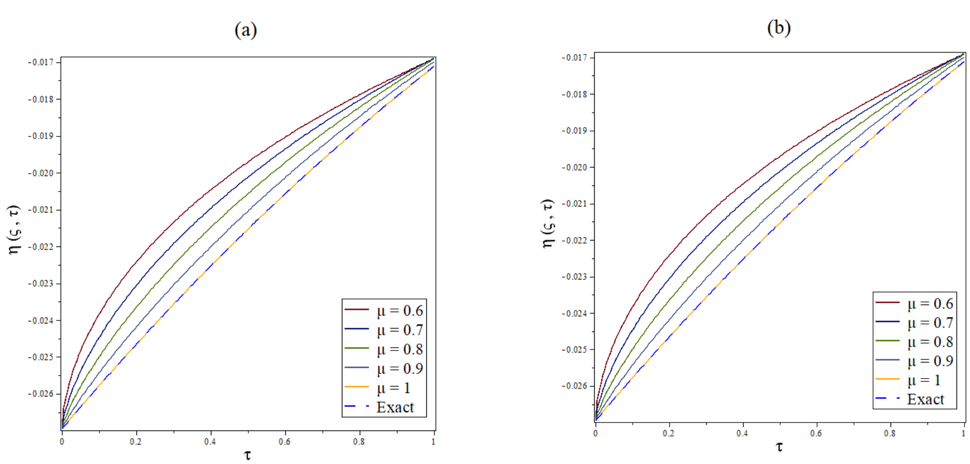

Figure 16. Comparison of height $\eta(\varsigma,\tau)$ (a) NTDM and (b) VITM solution at different fractional order μ

Table 2. Comparision of the absolute errors for Example (5.2) at $\mu = 1$, $c = 0.5$, $k = -1$ and $b = 9$

(ς, τ)	$\lvert\theta_{Exact} - \theta_{NTDM}\rvert$	$\lvert\theta_{Exact} - \theta_{VITM}\rvert$	$\lvert\eta_{Exact} - \eta_{NTDM}\rvert$	$\lvert\eta_{Exact} - \eta_{VITM}\rvert$
(0.1,0.1)	1.9300×10^{-09}	1.9300×10^{-09}	1.0080×10^{-08}	1.0080×10^{-08}
(0.1,0.2)	3.0100×10^{-08}	3.0100×10^{-08}	1.6353×10^{-07}	1.6353×10^{-07}
(0.1,0.3)	1.6064×10^{-07}	1.6064×10^{-07}	8.3935×10^{-07}	8.3935×10^{-07}
(0.1,0.4)	5.3494×10^{-07}	5.3494×10^{-07}	2.6887×10^{-06}	2.6887×10^{-06}
(0.1,0.5)	1.3747×10^{-06}	1.3747×10^{-06}	6.6512×10^{-06}	6.6512×10^{-06}
(0.2,0.1)	2.8300×10^{-09}	2.8300×10^{-09}	1.1500×10^{-08}	1.1500×10^{-08}
(0.2,0.2)	4.7350×10^{-08}	4.7350×10^{-08}	1.8592×10^{-07}	1.8592×10^{-07}
(0.2,0.3)	2.4992×10^{-07}	2.4992×10^{-07}	9.5204×10^{-07}	9.5204×10^{-07}
(0.2,0.4)	8.2166×10^{-07}	8.2166×10^{-07}	3.0429×10^{-06}	3.0429×10^{-06}
(0.2,0.5)	2.0831×10^{-06}	2.0831×10^{-06}	7.5107×10^{-06}	7.5107×10^{-06}
(0.3,0.1)	4.1900×10^{-09}	4.1900×10^{-09}	1.2790×10^{-08}	1.2790×10^{-08}
(0.3,0.2)	6.7320×10^{-08}	6.7320×10^{-08}	2.0692×10^{-07}	2.0692×10^{-07}
(0.3,0.3)	3.5079×10^{-07}	3.5079×10^{-07}	1.0575×10^{-06}	1.0575×10^{-06}
(0.3,0.4)	1.1427×10^{-06}	1.1427×10^{-06}	3.3725×10^{-06}	3.3725×10^{-06}
(0.3,0.5)	2.8748×10^{-06}	2.8748×10^{-06}	8.3049×10^{-06}	8.3049×10^{-06}
(0.4,0.1)	5.4900×10^{-09}	5.4900×10^{-09}	1.3980×10^{-08}	1.3980×10^{-08}
(0.4,0.2)	8.8870×10^{-08}	8.8870×10^{-08}	2.2543×10^{-07}	2.2543×10^{-07}
(0.4,0.3)	4.6126×10^{-07}	4.6126×10^{-07}	1.1492×10^{-06}	1.1492×10^{-06}
(0.4,0.4)	1.4947×10^{-06}	1.4947×10^{-06}	3.6560×10^{-06}	3.6560×10^{-06}
(0.4,0.5)	3.7403×10^{-06}	3.7403×10^{-06}	8.9809×10^{-06}	8.9809×10^{-06}
(0.5,0.1)	6.7800×10^{-09}	6.7800×10^{-09}	1.4910×10^{-08}	1.4910×10^{-08}
(0.5,0.2)	1.1206×10^{-07}	1.1206×10^{-07}	2.3980×10^{-07}	2.3980×10^{-07}
(0.5,0.3)	5.7976×10^{-07}	5.7976×10^{-07}	1.2194×10^{-06}	1.2194×10^{-06}
(0.5,0.4)	1.8715×10^{-06}	1.8715×10^{-06}	3.8688×10^{-06}	3.8688×10^{-06}
(0.5,0.5)	4.6648×10^{-06}	4.6648×10^{-06}	9.4775×10^{-06}	9.4775×10^{-06}