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Analysis of MMAP/PH(1), PH(2)/1 Preemptive Priority Queueing Model with Single Vacation, Repair and Impatient Customers

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Abstract

In this paper, we analyse a single server preemptive priority queue with phase-type vacation and repair, feedback, working breakdown, close-down and impatient customers. Customers arrive according to the Marked Markovian Arrival Process and their service time according to Phase-type distribution. If the High Priority customers need feedback, they lose their priority and join the Low Priority queue. At any instant, if the server is broken down, the server provide service with slow mode for that current customer and then the server will go into a repair process. When there are no customers present in both the queues, the server close-down the system and then goes on vacation. During the close-down and vacation period, high priority customers may balk. The Matrix Analytic Method is used to look into the number of consumers that are currently in the system. Analysis of the steady-state, the server active period, and the total cost are all discussed. Finally, some significant performance measures and numerical examples are given.

Keywords: Marked Markovian arrival process; Phase-type distribution; Single vacation; Working breakdown; Repair; Preemptive priority; Matrix analytic method

MSC 2010 No.: 60K25, 68M20, 90B22

1. Introduction

Neuts (1979) made the initial introduction to the flexible Markovian point process. Poisson, Markov-modulated Poissons, and PH-renewal are just a few of the well-known methods that belong to the broad family of point processes known as MAP. The underlying Markovian structure of the Markovian Arrival Process (MAP), which works effectively when stochastic models are analytically addressed using matrices, is one of the most important aspects of the MAP. Chakravarthy (2010) provided a detailed explanation of the phase-type distribution and the Markovian arrival process with various correlated and non-correlated arrival types. The techniques of matrix-analytic queueing theory were examined by Neuts (1984).

Nair et al. (2021) investigated the effects of feedback, MMAP arrivals, and PH-type service on priority loss for two single-server priority queueing systems. Both preemptive and non-preemptive priority models were investigated. Haghighi et al. (2008) considered Poisson arrival and delayed service with a single server queueing model. Haghighi and Mishev (2016) investigated the internal arrival tasks through direct feedback or by using a splitting procedure, both of which contain the concept of a delay. The reliability analysis of an embedded machining system with two different types of units and warm and cold standbys attracted the interest of Jain (2013). Ayyappan and Thilagavathy (2021b) discussed the MAP arrival and Phase type service queueing model with server vacation and immediate feedback. The investigation of non-preemptive queueing model, Krishnamoorthy et al. (2009) calculated the likelihood that priority-generated customers will receive n consecutive services as well as the typical wait time for a marked customer. Haghighi et al. (1986) proposed a multi-server design with state-dependent balking and reneging. Additionally, they created an average client loss over a predetermined period and produced an invariant probability vector. Consider Tian and Zhang (2006) and the survey work of Doshi (1986) when discussing vacation queueing models.

A model of non-preemptive priority retrial queueing with two categories of priority customers, negative arrival, breakdown, impatient customers, repair, single vacation, and backup server was discussed by Ayyappan and Thilagavathy (2021a). A capacity-limited queueing mechanism and reverse reneging were analysed by Kumar and Kumar Som (2015). A single-server M/M/1/N feedback queueing system was explored by Bouchentouf et al. (2019) in relation to vacation, balking, reneging, and maintenance of reneged customers.

Krishnamoorthy and Manjunath (2018) proposed queues with feedback-based prioritisation. The two-infinite lines, marked poisson arrival, and both preemptive and non-preemptive service discipline were all studied. Jain et al. (2015) retrial queue with finite capacity orbits was built for users who were both priority and non-priority. They also developed a cost model and looked at several performance indicators. Krishna Kumar et al. (2002) discuss a retrial queue with Bernoulli feedback and derive stability conditions and several system performance metrics. Kumar and Soodan (2019) have conducted a transient numerical investigation of a single server queuing model with associated reneging, baulking, and feedback. Melikov et al. (2020) computed the system state distribution with rapid feedback and fluctuating arrival rates in a Markov queueing model with a single server.

Ayyappan et al. (2020) looked at the two-way trial queueing system and the constant retrial policy for priority services. They also looked at how impatient customers affected the system. As part of their research on the queueing model, Kumar and Sharma (2019) discussed the retention of reneging clients. Senthil Kumar et al. (2013) combined two types of MAP arrivals using a preemptive priority retrial queueing mechanism, with type 1 being assigned a higher priority than type 2. Ayyappan and Archana (2023) discussed the two server queueing model with working breakdown. They consider arrival according to MAP and service according to Phase type distribution. Soodan and Kumar (2022) discussed a single server queuing system with feedback of served customers and correlated reneging. Kumar and Sharma (2021) examined the transient analysis of Markovian queueing system with multi-server and impatient customers.

2. Model Description

Consider a single server preemptive priority queueing model with two types of arrival. They are High Priority (HP) and Low Priority (LP) customers, and their arrival is followed by the Marked Markovian Arrival Process (MMAP) with representation (D_0, D_1, D_2) of order m. The square matrix D_0 governs no arrival in the system; the square matrix D_1 governs an HP customer's arrival in the system; and the square matrix D_2 governs an LP customer's arrival in the system. The HP customers are restricted to a finite capacity of size N, and the LP customers have infinite waiting space. The average arrival rate is $\lambda_i = \pi D_i e_m$, i=1,2 for HP and LP customers, respectively. The High Priority and Low Priority services offered by the server follow a PH-distribution with representation (α, T) and (β, S) of order n_1 and n_2 respectively, with $T^0 + Te = 0$ and $S^0 + Se = 0$, such that $T^0 = -Te$ and $S^0 = -Se$. The average service rate is $\mu_1 = [\alpha(-T)^{-1}e_{n_1}]^{-1}$ and $\mu_2 = [\beta(-S)^{-1}e_{n_2}]^{-1}$ for HP and LP customers, respectively. When the server provides service to HP or LP customers, at that moment, the server is struck with a breakdown, then the server provide a service slow mode for that current customer and the server will go into a repair process. The slow service for HP and LP customer follows PH-distribution accompanied by a depiction $(\alpha_1, \theta T)$ and $(\beta_1, \theta S)$ of order n_1 and n_2 respectively. The repair process follows the PH-distribution with representation (ψ, P) of order k_2 and the breakdown time follows an exponential distribution with parameter σ.

If the LP customer services are on-going at that time, HP customers arrive, they interrupt their service and the server provide service for HP customers. If no one presently logged into the HP queue, the server provide a service for LP customers. Feedback is only permitted for HP customers. If they need feedback with probability p, he joins the low priority queue; otherwise, if there is no feedback with probability q, he leaves the system. After completing the service process, if no one is presently logged into the system, the server will shut down the system and then go on vacation. Close-down time follows an exponential distribution with parameter δ and the vacation period follows a PH-distribution accompanied by a depiction (γ, V) of order k_1. After completion of vacation period, if there are no customers in the system, the service is idle; otherwise, if customers are present in both queues, the server starts the service to HP customers; if the HP queue is empty, the server starts the service to LP customers. During the close-down period and vacation period, the low priority customers may lose patience and balk the system with probability b, otherwise join

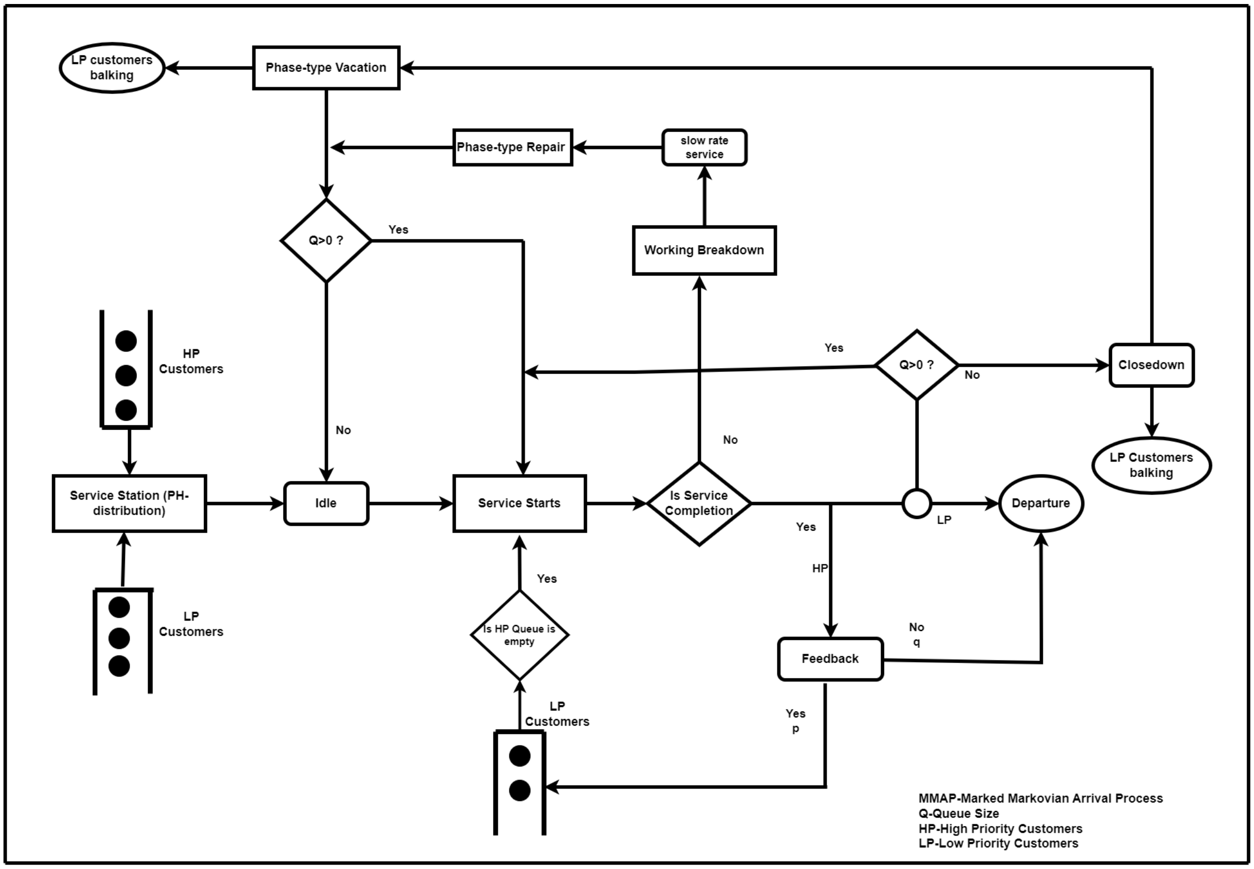

Figure 1. Pictorial model representation

the system with probability $(1 - b)$. The average vacation rate and average repair rate are given by η and ζ, respectively.

3. The Matrix Generation under QBD process

We will go over this section, which includes the notation that our model uses as its foundation for creating the QBD process.

Matrix Generation Notations

- $\otimes$ - A Kronecker product, which can be based on the works of Steeb and Hardy (2011), is the product of any two different order matrices.
- $\oplus$ - Any two of the various orders of matrices can be added up to form the Kronecker sum.
- I_k - It is a k-dimensional identity matrix.
- $e_i'(k)$ - It stands for a row vector of dimension k with 1 in the i^{th} position and 0 in the rest of the positions.
- $e_i(k)$ - It stands for a column vector of dimension k with 1 in the i^{th} position and 0 in the rest of the positions.
- e - The suitable dimension of the column vector, each element is 1.

- e_{n_1} - In a column vector of size n_1, each element is 1.
- e_{n_2} - In a column vector of size n_2, each element is 1.
- λ_i stands for the HP and LP customers arrival rate, which is defined by $\lambda_i = \pi D_i e_m$, for i=1,2, respectively.
- μ_1 stands for the High priorityv(HP) customer's service rate, which is defined by $\mu_1 = [\alpha(-T)^{-1}e_{n_1}]^{-1}$.
- μ_2 stands for the Low priorityv(LP) customer's service rate, which is defined by $\mu_2 = [\beta(-S)^{-1}e_{n_2}]^{-1}$.
- η stands for the vacation rate of the server, which is defined by $\eta = [\gamma(-V)^{-1}e_{k_1}]^{-1}$.
- ζ stands for the repair rate of the server, which is defined by $\zeta = [\psi(-P)^{-1}e_{k_2}]^{-1}$.
- Let $N_1(t)$ be the LP-customer count for the system at epoch t.
- Let $N_2(t)$ be the HP-customer count for the system at epoch t.
- Let V(t) represent the server's state at epoch t.

$$V(t) = \begin{cases} 0, \text{ if the server is on vacation,} \\ 1, \text{ if the server is in idle,} \\ 2, \text{ if the server busy with HP customers for normal mode,} \\ 3, \text{ if the server busy with LP customers for normal mode,} \\ 4, \text{ if the server busy with HP customers for slow mode,} \\ 5, \text{ if the server busy with LP customers for slow mode,} \\ 6, \text{ if the server is in repair process,} \\ 7, \text{ if the server is in close-down process.} \end{cases}$$

- $J_1(t)$ describes the vacation process as it is viewed by phases.
- $J_2(t)$ describes the repair process as it is viewed by phases.
- S(t) describes the service process as it is viewed by phases.
- M(t) describes the arrival process as it is viewed by phases.

Let { $N_1(t)$, $N_2(t)$, V(t), $J_1(t)$, $J_2(t)$, S(t), M(t) : $t \geq 0\}$ signify a Continuous Time Markov Chain (CTMC) with a state-level independent quasi-birth and death process, whose state space is as follows:

$$\Omega = l(0) \cup l(a_1),$$

where

$$\begin{aligned} l(0) =& \{(0, a_2, 0, j_1, k) : 0 \leq a_2 \leq N, 1 \leq j_1 \leq k_1, 1 \leq k \leq m\} \cup \{(0, 0, 1, k) : 1 \leq k \leq m\} \\ & \cup \{(0, a_2, 2, l_1, k) : 1 \leq a_2 \leq N, 1 \leq l_1 \leq n_1, 1 \leq k \leq m\} \\ & \cup \{(0, a_2, 4, l_1, k) : 1 \leq a_2 \leq N, 1 \leq l_1 \leq n_1, 1 \leq k \leq m\} \\ & \cup \{(0, a_2, 6, l_1, k) : 0 \leq a_2 \leq N, 1 \leq j_2 \leq k_2, 1 \leq k \leq m\} \\ & \cup \{(0, a_2, 7, k) : 0 \leq a_2 \leq N, 1 \leq k \leq m\}, \end{aligned}$$

for $a_1 \geq 1$,

$$\begin{aligned} l(a_1) =& \{(a_1, a_2, 0, j_1, k) : 0 \leq a_2 \leq N, 1 \leq j_1 \leq k_1, 1 \leq k \leq m\} \\ & \cup \{(a_1, a_2, 2, l_1, k) : 1 \leq a_2 \leq N, 1 \leq l_1 \leq n_1, 1 \leq k \leq m\} \\ & \cup \{(a_1, 0, 3, l_2, k) : 1 \leq l_2 \leq n_2, 1 \leq k \leq m\} \\ & \cup \{(a_1, a_2, 4, l_1, k) : 1 \leq a_2 \leq N, 1 \leq l_1 \leq n_1, 1 \leq k \leq m\} \\ & \cup \{(a_1, 0, 5, l_2, k) : 1 \leq l_2 \leq n_2, 1 \leq k \leq m\} \\ & \cup \{(a_1, a_2, 6, l_1, k) : 0 \leq a_2 \leq N, 1 \leq j_2 \leq k_2, 1 \leq k \leq m\} \\ & \cup \{(a_1, a_2, 7, k) : 0 \leq a_2 \leq N, 1 \leq k \leq m\}. \end{aligned}$$

The infinitesimal matrix generation of the QBD process is given by

$$Q = \begin{bmatrix} B_{00} & B_{01} & 0 & 0 & 0 & 0 & \dots \\ B_{10} & A_1 & A_0 & 0 & 0 & 0 & \dots \\ 0 & A_2 & A_1 & A_0 & 0 & 0 & \dots \\ 0 & 0 & A_2 & A_1 & A_0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \end{bmatrix}.$$

The elements in the block matrices of Q are defined as follows,

$$B_{00} = \begin{bmatrix} {B_{00}}^{11} & {B_{00}}^{12} & {B_{00}}^{13} & 0 & 0 & 0 \\ 0 & {B_{00}}^{22} & {B_{00}}^{23} & 0 & 0 & 0 \\ 0 & 0 & {B_{00}}^{33} & {B_{00}}^{34} & 0 & {B_{00}}^{36} \\ 0 & 0 & 0 & {B_{00}}^{44} & {B_{00}}^{45} & 0 \\ 0 & {B_{00}}^{52} & {B_{00}}^{53} & 0 & {B_{00}}^{55} & 0 \\ {B_{00}}^{61} & 0 & 0 & 0 & 0 & {B_{00}}^{66} \end{bmatrix},$$

$${B_{00}}^{11} = \begin{bmatrix} K_1 & K_2 & 0 & 0 & \dots & 0 \\ 0 & K_1 & K_2 & 0 & \dots & 0 \\ 0 & 0 & K_1 & K_2 & \dots & 0 \\ \vdots & \vdots & & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & 0 & K_1 & K_2 \\ 0 & \dots & \dots & 0 & 0 & K_1 + K_2 \end{bmatrix},$$

$${B_{00}}^{12} = e_1(N+1) \otimes V^0 \otimes I_m,$$

$$K_1 = V \oplus (D_0 + bD_2), \qquad K_2 = I_{k_1} \otimes D_1,$$

$${B_{00}}^{13} = \begin{bmatrix} 0 \\ I_N \otimes V^0 \alpha \otimes I_m \end{bmatrix}, \quad {B_{00}}^{22} = D_0, \quad {B_{00}}^{23} = {e_1}'(N) \otimes \alpha \otimes D_1,$$

$$B_{00}{}^{33} = \begin{bmatrix} K_3 & K_4 & 0 & 0 & \dots & 0 \\ K_5 & K_3 & K_4 & 0 & \dots & 0 \\ 0 & K_5 & K_3 & K_4 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & K_5 & K_3 & K_4 \\ 0 & \dots & 0 & 0 & K_5 & K_3 + K_4 \end{bmatrix},$$

$$B_{00}{}^{34} = I_N \otimes e_{n_1} \otimes \alpha_1 \otimes \sigma I_m,$$

$$K_3 = T \oplus D_0 - \sigma I_{n_1 m}, \quad K_4 = I_{n_1} \otimes D_1, \quad K_5 = qT^0\alpha \otimes I_m,$$

$$B_{00}{}^{36} = \left[e_1(N) \otimes q_2 T^0 \otimes I_m \; 0 \right],$$

$$B_{00}{}^{44} = \begin{bmatrix} K_6 & K_7 & 0 & 0 & \dots & 0 \\ 0 & K_6 & K_7 & 0 & \dots & 0 \\ 0 & 0 & K_6 & K_7 & \dots & 0 \\ \vdots & \vdots & & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & 0 & K_6 & K_7 \\ 0 & \dots & \dots & 0 & 0 & K_6 + K_7 \end{bmatrix},$$

$$K_6 = \theta T \oplus D_0, \quad K_7 = I_{n_1} \otimes D_1,$$

$$B_{00}{}^{45} = \left[I_N \otimes q\theta T^0 \psi \otimes I_m \; 0 \right], \quad B_{00}{}^{52} = e_1(N+1) \otimes P^0 \otimes I_m,$$

$$B_{00}{}^{53} = \begin{bmatrix} 0 \\ I_N \otimes P^0 \alpha \otimes I_m \end{bmatrix},$$

$$B_{00}{}^{55} = \begin{bmatrix} K_8 & K_9 & 0 & 0 & \dots & 0 \\ 0 & K_8 & K_9 & 0 & \dots & 0 \\ 0 & 0 & K_8 & K_9 & \dots & 0 \\ \vdots & \vdots & & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & 0 & K_8 & K_9 \\ 0 & \dots & \dots & 0 & 0 & K_8 + K_9 \end{bmatrix},$$

$$K_8 = P \oplus D_0, \quad K_9 = I_{k_2} \otimes D_1,$$

$$B_{00}{}^{61} = I_{N+1} \otimes \gamma \otimes \delta I_m,$$

$$
B_{00}{}^{66} = \begin{bmatrix} K_{10} & K_{11} & 0 & 0 & \dots & 0 \\ 0 & K_{10} & K_{11} & 0 & \dots & 0 \\ 0 & 0 & K_{10} & K_{11} & \dots & 0 \\ \vdots & \vdots & & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & 0 & K_{10} & K_{11} \\ 0 & \dots & \dots & 0 & 0 & K_{10} + K_{11} \end{bmatrix},
$$

$$
K_{10} = (D_0 + bD_2) - \delta I_m, \quad K_{11} = D_1,
$$

$$
B_{01} = \begin{bmatrix} B_{01}{}^{11} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & B_{01}{}^{23} & 0 & 0 & 0 & 0 \\ 0 & B_{01}{}^{32} & B_{01}{}^{33} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B_{01}{}^{44} & 0 & B_{01}{}^{46} & 0 \\ 0 & 0 & 0 & 0 & 0 & B_{01}{}^{56} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & B_{01}{}^{67} \end{bmatrix},
$$

$$
B_{01}{}^{11} = I_{N+1} \otimes I_{k_1} \otimes (1-b)D_2, \quad B_{01}{}^{23} = \beta \otimes D_2,
$$

$$
B_{01}{}^{32} = \begin{bmatrix} K_{12} & 0 & 0 & 0 & \dots & 0 \\ K_{13} & K_{12} & 0 & 0 & \dots & 0 \\ 0 & K_{13} & K_{12} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & K_{13} & K_{12} & 0 \\ 0 & \dots & 0 & 0 & K_{13} & K_{12} \end{bmatrix},
$$

$$
K_{12} = I_{n_1} \otimes D_2, \quad K_{13} = pT^0\alpha \otimes I_m,
$$

$$
B_{01}{}^{33} = e_1(N) \otimes pT^0\beta \otimes I_m, B_{01}{}^{44} = I_N \otimes I_{n_1} \otimes D_2, B_{01}{}^{46} = \left[I_N \otimes p\theta T^0 \psi \otimes I_m \; 0 \right],
$$

$$
B_{01}{}^{56} = I_{N+1} \otimes I_{k_2} \otimes D_2, \quad B_{01}{}^{67} = I_{N+1} \otimes (1-b)D_2,
$$

$$
B_{10} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & B_{10}{}^{36} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & B_{10}{}^{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},
$$

$$
B_{10}{}^{36} = e_1{}'(N+1) \otimes S^0 \otimes I_m, \quad B_{10}{}^{55} = e_1{}'(N+1) \otimes \theta S^0 \psi \otimes I_m,
$$

$$A_1 = \begin{bmatrix} {A_1}^{11} & {A_1}^{12} & {A_1}^{13} & 0 & 0 & 0 & 0 \\ 0 & {A_1}^{22} & {A_1}^{23} & {A_1}^{24} & 0 & 0 & 0 \\ 0 & 0 & {A_1}^{33} & 0 & {A_1}^{35} & 0 & 0 \\ 0 & 0 & 0 & {A_1}^{44} & 0 & {A_1}^{46} & 0 \\ 0 & 0 & 0 & 0 & {A_1}^{55} & 0 & 0 \\ 0 & {A_1}^{62} & {A_1}^{63} & 0 & 0 & {A_1}^{66} & 0 \\ {A_1}^{71} & 0 & 0 & 0 & 0 & 0 & {A_1}^{77} \end{bmatrix},$$

$${A_1}^{11} = {B_{00}}^{11}, \quad {A_1}^{12} = {B_{00}}^{13}, \quad {A_1}^{13} = e_1(N+1) \otimes V^0\beta \otimes I_m,$$

$${A_1}^{22} = {B_{00}}^{33}, \quad {A_1}^{23} = e_1(N) \otimes qT^0\beta \otimes I_m,$$

$${A_1}^{24} = {B_{00}}^{34}, \quad {A_1}^{33} = S \oplus (D_0 + D_1) - \sigma I_{n_2 m}, \quad {A_1}^{35} = e_{n_2} \otimes \beta_1 \otimes \sigma I_m,$$

$${A_1}^{44} = {B_{00}}^{44}, \quad {A_1}^{46} = {B_{00}}^{45}, \quad {A_1}^{55} = \theta S \oplus (D_0 + D_1), \quad {A_1}^{62} = {B_{00}}^{53},$$

$${A_1}^{63} = e_1(N+1) \otimes P^0\beta \otimes I_m, \quad {A_1}^{66} = {B_{00}}^{55}, \quad {A_1}^{71} = {B_{00}}^{61}, \quad {A_1}^{77} = {B_{00}}^{66},$$

$$A_0 = \begin{bmatrix} {A_0}^{11} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & {A_0}^{22} & {A_0}^{23} & 0 & 0 & 0 & 0 \\ 0 & 0 & {A_0}^{33} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & {A_0}^{44} & 0 & {A_0}^{46} & 0 \\ 0 & 0 & 0 & 0 & {A_0}^{55} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & {A_0}^{66} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & {A_0}^{77} \end{bmatrix},$$

$${A_0}^{11} = {B_{01}}^{11}, \quad {A_0}^{22} = {B_{01}}^{32}, \quad {A_0}^{23} = {B_{01}}^{33}, \quad {A_0}^{33} = I_{n_2} \otimes D_2,$$

$${A_0}^{44} = {B_{01}}^{44}, \quad {A_0}^{46} = {B_{01}}^{46}, \quad {A_0}^{55} = I_{n_2} \otimes D_2, \quad {A_0}^{66} = {B_{01}}^{56}, \quad {A_0}^{77} = {B_{01}}^{67},$$

$$A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & {A_2}^{33} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & {A_2}^{56} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$${A_2}^{33} = S^0\beta \otimes I_m, \quad {A_2}^{56} = {B_{10}}^{55}.$$

4. Invariant Analysis

We perform an analysis of our model under some stable system settings.

4.1. Condition for Stableness

Let's define the matrix A as $A = A_0 + A_1 + A_2$, which indicates that the square matrix A has dimensions of $((N+1)k_1m + 2Nn_1m + 2n_2m + (N+1)k_2m + (N+1)m)$ and is an irreducible infinitesimal generator matrix.

The vector φ denote the invariant probability vector of A fulfilling the condition $\varphi A = 0$ and $\varphi e = 1$. The vector φ is segmented by $\varphi = (\varphi_0, \varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5, \varphi_6) = (\varphi_{00}, \varphi_{01}, \ldots, \varphi_{0N}, \varphi_{11}, \varphi_{12}, \ldots, \varphi_{1N}, \varphi_2, \varphi_{31}, \varphi_{32}, \ldots, \varphi_{3N}, \varphi_4, \varphi_{50}, \varphi_{51}, \ldots, \varphi_{5N}, \varphi_{60}, \varphi_{61}, \ldots, \varphi_{6N})$, where φ_0 is of dimension $(N+1)k_1m$, φ_1 is of dimension Nn_1m, φ_2 is of dimension n_2m, φ_3 is of dimension Nn_1m, φ_4 is of dimension n_2m, φ_5 is of dimension $(N+1)k_2m$, φ_6 is of dimension $(N+1)m$. When the Markov process is studied under the quasi-birth-and-death structure, the stability of our model should meet the necessary and sufficient conditions $\varphi A_0 e < \varphi A_2 e$. The vector φ can be calculated by resolving the following equations:

$$
\begin{aligned}
&\varphi_{00}(V \oplus (D_0 + D_2)) + \varphi_{60}(\gamma \otimes \delta I_m) = 0,\\
&\varphi_{0j-1}(I_{k_1} \otimes D_1) + \varphi_{0j}(V \oplus (D_0 + D_2)) + \varphi_{6j}(\gamma \otimes \delta I_m) = 0 \;\; for \;\; 1 \leq j \leq N-1,\\
&\varphi_{0N-1}(I_{k_1} \otimes D_1) + \varphi_{0N}(V \oplus D) + \varphi_{6N}(\gamma \otimes \delta I_m) = 0,\\
&\varphi_{01}(V^0\alpha \otimes I_m) + \varphi_{11}(T \oplus (D_0 + D_2) - \sigma I_{n_1m}) + \varphi_{12}(T^0\alpha \otimes I_m)\\
&\qquad\qquad\qquad\qquad\qquad\qquad + \varphi_{51}(P^0\alpha \otimes I_m) = 0,\\
&\varphi_{0j}(V^0\alpha \otimes I_m) + \varphi_{1j-1}(I_{n_1} \otimes D_1) + \varphi_{1j}(T \oplus (D_0 + D_2) - \sigma I_{n_1m}) +\\
&\qquad\qquad \varphi_{1j+1}(T^0\alpha \otimes I_m) + \varphi_{5j}(P^0\alpha \otimes I_m) = 0 \;\; for \;\; 2 \leq j \leq N-1,\\
&\varphi_{0N}(V^0\alpha \otimes I_m) + \varphi_{1N-1}(I_{n_1} \otimes D_1) + \varphi_{1N}(T \oplus D - \sigma I_{n_1m}) + \varphi_{5N}(P^0\alpha \otimes I_m) = 0,\\
&\varphi_{00}(V^0\beta \otimes I_m) + \varphi_{11}(T^0\beta \otimes I_m) + \varphi_2((S + S^0\beta) \oplus D - \sigma I_{n_2m})\\
&\qquad\qquad\qquad\qquad\qquad\qquad + \varphi_{50}(P^0\beta \otimes I_m) = 0,\\
&\varphi_{11}(e_{n_1} \otimes \alpha_1 \otimes \sigma I_m) + \varphi_{31}(\theta T \oplus (D_0 + D_2)) = 0,\\
&\varphi_{1j}(e_{n_1} \otimes \alpha_1 \otimes \sigma I_m) + \varphi_{3j-1}(I_{n_1} \otimes D_1) + \varphi_{3j}(\theta T \oplus (D_0 + D_2)) = 0\\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{for } \; 2 \leq j \leq N-1,\\
&\varphi_{1N}(e_{n_1} \otimes \alpha_1 \otimes \sigma I_m) + \varphi_{3N-1}(I_{n_1} \otimes D_1) + \varphi_{3N}(\theta T \oplus D) = 0,\\
&\varphi_2(e_{n_2} \otimes \beta_1 \otimes \sigma I_m) + \varphi_4(\theta S \oplus D) = 0,\\
&\varphi_{31}(\theta T^0\psi \otimes I_m) + \varphi_4(\theta S^0\psi \otimes I_m) + \varphi_{50}(P \oplus (D_0 + D_2) = 0,\\
&\varphi_{3j+1}(\theta T^0\psi \otimes I_m) + \varphi_{5j-1}(I_{k_1} \otimes D_1) + \varphi_{5j}(P \oplus (D_0 + D_2)) = 0\\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{for } \; 1 \leq j \leq N-1,\\
&\varphi_{5N-1}(I_{k_1} \otimes D_1) + \varphi_{5N}(P \oplus D) = 0,\\
&\varphi_{60}((D_0 + D_2) - \delta I_m) = 0,\\
&\varphi_{6j-1}D_1 + \varphi_{6j}((D_0 + D_2) - \delta I_m) = 0 \;\; \text{for } \; 1 \leq j \leq N-1,\\
&\varphi_{6N-1}D_1 + \varphi_{6N}(D - \delta I_m) = 0,
\end{aligned}
$$

subject to normalizing condition

$$\sum_{j=0}^{N}\varphi_{0j}e_{k_1m}+\sum_{j=1}^{N}\varphi_{1j}e_{n_1m}+\varphi_2e_{n_2m}+\sum_{j=1}^{N}\varphi_{3j}e_{n_1m}+\varphi_4e_{n_2m}$$
$$+\sum_{j=0}^{N}\varphi_{5j}e_{k_2m}+\sum_{j=0}^{N}\varphi_{6j}e_m=1.$$

After a few algebraic simplifications, the stability condition $\varphi A_0e<\varphi A_2e$, which turns out to be

$$\sum_{j=0}^{N}\varphi_{0j}(e_{k_1}\otimes(1-b)D_2e_m)+\sum_{j=1}^{N}\varphi_{1j}(e_{n_1}\otimes D_2e_m+pT^0\otimes e_m)+\varphi_2(e_{n_2}\otimes D_2em)$$
$$+\sum_{j=1}^{N}\varphi_{3j}(e_{n_1}\otimes D_2e_m+p\theta T^0\otimes e_m)+\varphi_4(e_{n_2}\otimes D_2e_m)+\sum_{j=0}^{N}\varphi_{5j}(e_{k_2}\otimes D_2e_m)$$
$$+\sum_{j=0}^{N}\varphi_{6j}((1-b)D_2e_m)<\varphi_2(S^0\otimes e_m)+\varphi_4(\theta S^0\otimes e_m).$$

4.2. Invariant Probability Vector Analysis

Think about Q's invariant probability vector, which is represented by the symbol x and partitioned as $x=(x_0,x_1,x_2,\dots)$. x_0 has a dimension of $(N+1)k_1m+m+2Nn_1m+(N+1)k_2m+(N+1)m$ while $x_1,\ x_2,\dots$ have a dimension of $(N+1)k_1m+2Nn_1m+2n_2m+(N+1)k_2m+(N+1)m$. The requirement $xQ=0$ and $xe=1$ are satisfied by x.

In addition, the steady-state probability vector x could be derived using the following equation once the model's stability condition has been satisfied:

$$x_{a_1}=x_1R^{a_1-1},\ a_1\geq 2$$

R is the smallest non-negative solution of the quadratic matrix equation $R^2A_2+RA_1+A_0=0$, according to Neuts (1984). The rate matrix is derived from the quadratic matrix equation. It is derived that R of order $((N+1)k_1m+2Nn_1m+2n_2m+(N+1)k_2m+(N+1)m)$ fulfils the relation $RA_2e=A_0e$.

The sub vectors x_0 and x_1 can be found by solving the consequent equations.

$$x_0B_{00}+x_1B_{10}=0,$$

$$x_0B_{01}+x_1(A_1+RA_2)=0.$$

The normalizing condition is subject to

$$x_0e_{(N+1)k_1m+m+2Nn_1m+(N+1)k_2m+(N+1)m}$$
$$+x_1(I-R)^{-1}e_{(N+1)k_1m+2Nn_1m+2n_2m+(N+1)k_2m+(N+1)m}=1.$$

In light of Latouche and Ramaswami (1999), it is possible to generate the R matrix analytically by making use of crucial phases in the Logarithmic reduction process.

5. Analysis of Busy period

- An active period can be calculated as the period of time between customers entering the void system and the system becoming void once more after the first interval. The path from level one to level zero begins with this passage. Consequently, the busy cycle is defined as the first return to level 0, followed by at least one visit to a state at any subsequent level.
- Before going on to the active period, we provide a summary of the fundamental period. The first transition period, $a_1 \geq 2$, between level a_1 and level $a_1 - 1$, is considered by the QBD procedure.
- The boundary state scenarios corresponding to $a_1 = 0,\ 1$ must all be looked at separately. There are $((N+1)k_1m + 2Nn_1m + 2n_2m + (N+1)k_2m + (N+1)m)$ states that correspond for each level l with $a_1 \geq 2$ that should be taken into consideration. The state (a_1, l) at level a_1 denotes that the l^{th} state at level a_1 is mentioned when the states are arranged in lexicographic order.
- The QBD process conditional probability is starts in the state (a_1, l) at time $t = 0$ and reaches the level $a_1 - 1$ but not earlier time x, can make alterations a_1 transition to the left and reach the state (a_1, l') is represented by the variable $G_{ll'}(a_1, x)$. First, let's define the joint transform Neuts (1981),

$$\tilde{G}_{ll'}(z,s) = \sum_{a_1=1}^{\infty} z^{a_1} \int_0^{\infty} e^{-sx} dG_{ll'}(a_1, x); |z| \leq 1, Re(s) \geq 0,$$

and the matrix is shown as $\tilde{G}(z,s) = \tilde{G}_{ll'}(z,s)$, then the previously defined matrix $\tilde{G}(z,s)$ satisfied the equation

$$\tilde{G}(z,s) = z(sI - A_1)^{-1}A_2 + (sI - A_1)^{-1}A_0\tilde{G}^2(z,s).$$

- The first passage time would be calculated using the matrix $G = G_{ll'} = \tilde{G}(1,0)$, ignoring the boundary states. The matrix G may be found using the outcomes if we already have the matrix R,

$$G = -(A_1 + RA_2)^{-1}A_2.$$

Otherwise, the G matrix's values could be calculated using the concept of a logarithmic reduction procedure.

Notations

- $G_{ll'}^{(1,0)}(u,x)$ demonstrates that during the transition from level 1 to level 0, the conditional probability was stated for the first time at time $t = 0$.
- $G_{ll'}^{(0,0)}(u,x)$ demonstrates that the conditional probability for the return time to level 0 has been examined.
- $\mathfrak{P}_{1a_1}$ represents the process's average initial passage time between levels a_1 and $a_1 - 1$ if it is in the state (a_1, l) at time $t = 0$.
- $\vec{\mathfrak{P}}_1$ identifies the column vector containing the entries $\mathfrak{P}_{1a_1}$.
- $\mathfrak{P}_{2a_1}$ displays the typical number of clients anticipated to be served at the initial passage time from level a_1 to $a_1 - 1$, considering that the state's initial passage time has already started (a_1, l).
- $\vec{\mathfrak{P}}_2$ identifies the column vector containing the entries $\mathfrak{P}_{2a_1}$.
- $\vec{\mathfrak{P}}_1^{(1,0)}$ shows the average first passage time between level 1 and level 0.

- $\vec{\mathfrak{P}}_2^{(1,0)}$ shows the expected number of services finished during the first passage time from level 1 to level 0.
- $\vec{\mathfrak{P}}_1^{(0,0)}$ shows the initial return time to level 0.
- $\vec{\mathfrak{P}}_2^{(0,0)}$ shows the expected number of services finished between the first return time and level 0.

The subsequent equations, which are provided by $\tilde{G}^{(1,0)}(z,s)$ and $\tilde{G}^{(0,0)}(z,s)$, correspond to the boundary levels 1 and 0, respectively,

$$\tilde{G}^{(1,0)}(z,s) = z(sI - A_1)^{-1}B_{10} + (sI - A_1)^{-1}A_0\tilde{G}(z,s)\tilde{G}^{(1,0)}(z,s),$$
$$\tilde{G}^{(0,0)}(z,s) = (sI - B_{00})^{-1}B_{01}\tilde{G}^{(1,0)}(z,s).$$

Due to the stochastic character of G, $\tilde{G}^{(0,0)}(1,0)$ and $\tilde{G}^{(1,0)}(1,0)$, the matrices are utilised to calculate the subsequent cases. The instants can be calculated as follows:

$$\begin{aligned}
\vec{\mathfrak{P}}_1 &= -\frac{\partial}{\partial s}\tilde{G}(z,s)\Big|_{z=1,s=0} e = -[A_1 + A_0(I+G)]^{-1}e,\\
\vec{\mathfrak{P}}_2 &= \frac{\partial}{\partial z}\tilde{G}(z,s)\Big|_{z=1,s=0} e = -[A_1 + A_0(I+G)]^{-1}A_2 e,\\
\vec{\mathfrak{P}}_1^{(1,0)} &= -\frac{\partial}{\partial s}\tilde{G}^{(1,0)}(z,s)\Big|_{z=1,s=0} e = -[A_1 + A_0 G]^{-1}(A_0\vec{\mathfrak{P}}_1 + e),\\
\vec{\mathfrak{P}}_2^{(1,0)} &= \frac{\partial}{\partial z}\tilde{G}^{(1,0)}(z,s)\Big|_{z=1,s=0} e = -[A_1 + A_0 G]^{-1}(A_0\vec{\mathfrak{P}}_2 + B_{10}e),\\
\vec{\mathfrak{P}}_1^{(0,0)} &= -\frac{\partial}{\partial s}\tilde{G}^{(0,0)}(z,s)\Big|_{z=1,s=0} e = -B_{00}^{-1}[B_{01}\vec{\mathfrak{P}}_1^{(1,0)} + e],\\
\vec{\mathfrak{P}}_2^{(0,0)} &= \frac{\partial}{\partial z}\tilde{G}^{(0,0)}(z,s)\Big|_{z=1,s=0} e = -B_{00}^{-1}[B_{01}\vec{\mathfrak{P}}_2^{(1,0)}].
\end{aligned}$$

6. Performance Measures

- Mean number of LP customers in the system,

$$E_{LP} = \sum_{a_1=1}^{\infty} a_1 x_{a_1} e = x_1(I-R)^{-2}e,$$

  where $e = e_{(N+1)k_1m+2Nn_1m+2n_2m+(N+1)k_2m+(N+1)m}$.

- Mean number of HP customers in the system,

$$\begin{aligned}
E_{HP} = &\sum_{a_1=0}^{\infty}\sum_{a_2=1}^{N}\sum_{j_1=1}^{k_1}\sum_{k=1}^{m} a_2 x_{a_1a_20j_1k} + \sum_{a_1=0}^{\infty}\sum_{a_2=1}^{N}\sum_{l_1=1}^{n_1}\sum_{k=1}^{m} a_2 x_{a_1a_22l_1k}\\
&+ \sum_{a_1=0}^{\infty}\sum_{a_2=1}^{N}\sum_{l_1=1}^{n_1}\sum_{k=1}^{m} a_2 x_{a_1a_24l_1k} + \sum_{a_1=0}^{\infty}\sum_{a_2=1}^{N}\sum_{j_2=1}^{k_2}\sum_{k=1}^{m} a_2 x_{a_1a_26j_2k}\\
&+ \sum_{a_1=0}^{\infty}\sum_{a_2=1}^{N}\sum_{k=1}^{m} a_2 x_{a_1a_27k}.
\end{aligned}$$

- Probability that the server is on vacation,

$$P_{vac} = \sum_{a_1=0}^{\infty}\sum_{a_2=0}^{N}\sum_{j_1=1}^{k_1}\sum_{k=1}^{m} x_{a_1a_20j_1k} = x_0e_0(0) + x_1(I-R)^{-1}e(0),$$

where $e_0(0)$: a vector of dimension $(N+1)k_1m + m + 2Nn_1m + (N+1)k_2m + (N+1)m \times 1$ in which first $(N+1)k_1m$ elements are 1 and the rest of elements are 0,
$e(0)$: a vector of dimension $(N+1)k_1m + 2Nn_1m + 2n_2m + (N+1)k_2m + (N+1)m \times 1$ in which first $(N+1)k_1m$ elements are 1 and the rest of elements are 0.

- Probability that the server is idle,

$$P_{idle} = \sum_{k=1}^{m} x_{001k} = x_0e_0(1),$$

where $e_0(1)$: a vector of dimension $(N+1)k_1m + m + 2Nn_1m + (N+1)k_2m + (N+1)m \times 1$ in which $(N+1)k_1m + 1$ to $(N+1)k_1m + m$ elements are 1 and the rest of elements are 0.

- Probability that the server busy with normal mode service for HP customers,

$$P_{HPNB} = \sum_{a_1=0}^{\infty}\sum_{a_2=1}^{N}\sum_{l_1=1}^{n_1}\sum_{k=1}^{m} x_{a_1a_22l_1k} = x_0e_0(2) + x_1(I-R)^{-1}e(2),$$

where $e_0(2)$: a vector of dimension $(N+1)k_1m+m+2Nn_1m+(N+1)k_2m+(N+1)m \times 1$ in which $(N+1)k_1m+m+1$ to $(N+1)k_1m+m+Nn_1m$ elements are 1 and the rest of elements are 0,
$e(2)$: a vector of dimension $(N+1)k_1m + 2Nn_1m + 2n_2m + (N+1)k_2m + (N+1)m \times 1$ in which $(N+1)k_1m + 1$ to $(N+1)k_1m + Nn_1m$ elements are 1 and the rest of elements are 0.

- Probability that the server busy with normal mode service for LP customers,

$$P_{LPNB} = \sum_{a_1=1}^{\infty}\sum_{l_2=1}^{n_2}\sum_{k=1}^{m} x_{a_103l_2k} = x_1(I-R)^{-1}e(3),$$

where $e(3)$: a vector of dimension $(N+1)k_1m+2Nn_1m+2n_2m+(N+1)k_2m+(N+1)m \times 1$ in which $(N+1)k_1m + Nn_1m + 1$ to $(N+1)k_1m + Nn_1m + n_2m$ elements are 1 and the rest of elements are 0.

- Probability that the server busy with slow mode service for HP customers,

$$P_{HPSB} = \sum_{a_1=0}^{\infty}\sum_{a_2=1}^{N}\sum_{l_1=1}^{n_1}\sum_{k=1}^{m} x_{a_1a_24l_1k} = x_0e_0(4) + x_1(I-R)^{-1}e(4),$$

where $e_0(4)$: a vector of dimension $(N+1)k_1m + m + 2Nn_1m + (N+1)k_2m + (N+1)m \times 1$ in which $(N+1)k_1m + m + Nn_1m + 1$ to $(N+1)k_1m + m + Nn_1m + Nn_1m$ elements are 1 and the rest of elements are 0,
$e(4)$: a vector of dimension $(N+1)k_1m + 2Nn_1m + 2n_2m + (N+1)k_2m + (N+1)m \times 1$ in which $(N+1)k_1m + Nn_1m + n_2m + 1$ to $(N+1)k_1m + Nn_1m + n_2m + Nn_1m$ elements are

1 and the rest of elements are 0.

- Probability that the server busy with slow mode service for LP customers,

$$P_{LPSB} = \sum_{a_1=1}^{\infty}\sum_{l_2=1}^{n_2}\sum_{k=1}^{m} x_{a_1 0 5 l_2 k} = x_1(I-R)^{-1}e(5),$$

where $e(5)$: a vector of dimension $(N+1)k_1m + 2Nn_1m + 2n_2m + (N+1)k_2m + (N+1)m \times 1$ in which $(N+1)k_1m + 2Nn_1m + n_2m + 1$ to $(N+1)k_1m + 2Nn_1m + n_2m + n_2m$ elements are 1 and the rest of elements are 0.

- Probability that the server is repair process,

$$P_{rep} = \sum_{a_1=0}^{\infty}\sum_{a_2=0}^{N}\sum_{j_2=1}^{k_2}\sum_{k=1}^{m} x_{a_1 a_2 6 j_2 k} = x_0e_0(6) + x_1(I-R)^{-1}e(6),$$

where $e_0(6)$: a vector of dimension $(N+1)k_1m + m + 2Nn_1m + (N+1)k_2m + (N+1)m \times 1$ in which $(N+1)k_1m + m + 2Nn_1m + 1$ to $(N+1)k_1m + m + Nn_1m + 2Nn_1m + (N+1)k_2m$ elements are 1 and the rest of elements are 0,
$e(6)$: a vector of dimension $(N+1)k_1m + 2Nn_1m + 2n_2m + (N+1)k_2m + (N+1)m \times 1$ in which $(N+1)k_1m + 2Nn_1m + 2n_2m + 1$ to $(N+1)k_1m + 2Nn_1m + 2n_2m + (N+1)k_2m$ elements are 1 and the rest of elements are 0.

- Probability that the server is close-down process,

$$P_{cd} = \sum_{a_1=0}^{\infty}\sum_{a_2=0}^{N}\sum_{k=1}^{m} x_{a_1 a_2 7 k} = x_0e_0(7) + x_1(I-R)^{-1}e(7),$$

where $e_0(7)$: a vector of dimension $(N+1)k_1m + m + 2Nn_1m + (N+1)k_2m + (N+1)m \times 1$ in which $(N+1)k_1m + m + 2Nn_1m + (N+1)k_2m + 1$ to $(N+1)k_1m + m + Nn_1m + 2Nn_1m + (N+1)k_2m + (N+1)m$ elements are 1 and the rest of elements are 0,
$e(7)$: a vector of dimension $(N+1)k_1m + 2Nn_1m + 2n_2m + (N+1)k_2m + (N+1)m \times 1$ in which $(N+1)k_1m + 2Nn_1m + 2n_2m + (N+1)k_2m + 1$ to $(N+1)k_1m + 2Nn_1m + 2n_2m + (N+1)k_2m + (N+1)m$ elements are 1 and the rest of elements are 0.

- The rate at which LP customers balking during the server is on close-down and vacation period,

$$\mathfrak{B} = b\lambda_2\left(\sum_{a_1=0}^{\infty}\sum_{a_2=0}^{N}\sum_{j_1=1}^{k_1}\sum_{k=1}^{m} x_{a_1 a_2 0 j_1 k} + \sum_{a_1=0}^{\infty}\sum_{a_2=0}^{N}\sum_{k=1}^{m} x_{a_1 a_2 7 k}\right).$$

7. Cost Analysis

The cost function for our model has been created on the assumption that each cost element (per unit of time) corresponds to a different system measure,

$$\begin{aligned} TC = {} & C_{H_1}E_{HP} + C_{H_2}E_{LP} + C_{vac}P_{vac} + C_{idle}P_{idle} + C_{HPNB}P_{HPNB} + C_{LPNB}P_{LPNB} \\ & + C_{HPSB}P_{HPSB} + C_{LPSB}P_{LPSB} + C_{rep}P_{rep} + C_{cd}P_{cd} + \mu_1 C_1 + \mu_2 C_2 \\ & + \theta\mu_1 C_3 + \theta\mu_2 C_4 + \sigma C_5 + \zeta C_6 + \delta C_7 + \mathfrak{B} C_8, \end{aligned}$$

where

- TC - The total cost for each unit of time.
- C_{H_1} - Each HP customer's holding expense in the system.
- C_{H_2} - Each LP customer's holding expense in the system.
- C_{vac} - Cost obtained during server's vacation period.
- C_{idle} - Cost obtained due to server being idle.
- C_{HPNB} - Cost obtained by the system during server being normal mode busy with HP customers.
- C_{LPNB} - Cost obtained by the system during server being normal mode busy with LP customers.
- C_{HPSB} - Cost obtained by the system during server being slow mode busy with HP customers.
- C_{LPSB} - Cost obtained by the system during server being slow mode busy with LP customers.
- C_{rep} - Cost obtained by the server during repair process.
- C_{cd} - Cost that the server acquired during the shutdown process.
- C_1 - Cost obtained by the server for offering normal mode service to HP customers.
- C_2 - Cost obtained by the server for offering normal mode service to LP customers.
- C_3 - Cost obtained by the server for offering slow mode service to HP customers.
- C_4 - Cost obtained by the server for offering slow mode service to LP customers.
- C_5 - Cost received when a server broke.
- C_6 - Costs required during the repair process.
- C_7 - Costs required during the close-down procedure.
- C_8 - Cost obtained due to impatient behaviour of balking customers.

8. Numerical Results

In the following section, we are employing numerical and visual representations to analyze model behavior. The next five are different MAP representations whose mean value is identical, that is, 1 for all different arrival processes. These five sets of arrival values have been used as input data in Chakravarthy (2010).

Arrival in Erlang (A-ERL):

$$D_0 = \begin{bmatrix} -5 & 5 & 0 & 0 & 0 \\ 0 & -5 & 5 & 0 & 0 \\ 0 & 0 & -5 & 5 & 0 \\ 0 & 0 & 0 & -5 & 5 \\ 0 & 0 & 0 & 0 & -5 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Arrival in Exponential (A-EXP):

$$D_0 = \begin{bmatrix} -1 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0.6 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 0.4 \end{bmatrix}.$$

Arrival in Hyper-exponential (A-HEX):

$$D_0 = \begin{bmatrix} -1.90 & 0 \\ 0 & -0.19 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 1.026 & 0.114 \\ 0.1026 & 0.0114 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 0.684 & 0.076 \\ 0.0684 & 0.0076 \end{bmatrix}.$$

Arrival in MAP-Negative Correlation (A-MAPNC):

$$D_0 = \begin{bmatrix} -1.00243 & 1.00243 & 0 \\ 0 & -1.00243 & 0 \\ 0 & 0 & -225.797 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0.006012 & 0 & 0.595446 \\ 134.1234 & 0 & 1.3548 \end{bmatrix},$$
$$D_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0.004008 & 0 & 0.396964 \\ 89.4156 & 0 & 0.9032 \end{bmatrix}.$$

Arrival in MAP-Positive Correlation (A-MAPPC):

$$D_0 = \begin{bmatrix} -1.00243 & 1.00243 & 0 \\ 0 & -1.00243 & 0 \\ 0 & 0 & -225.797 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0.595446 & 0 & 0.006012 \\ 1.3548 & 0 & 134.1234 \end{bmatrix},$$
$$D_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0.396964 & 0 & 0.004008 \\ 0.9032 & 0 & 89.4156 \end{bmatrix}.$$

Let us consider three phase type distributions for the service, vacation and repair process. The following three different phase type distribution values have been taken from Chakravarthy (2010):

S-ERL (Erlang services):

$$\alpha = \beta = (1, 0), \quad T = S = \begin{bmatrix} -2 & 2 \\ 0 & -2 \end{bmatrix}.$$

R-ERL (Repair in Erlang):

$$\psi = (1, 0), \;\; P = \begin{bmatrix} -2 & 2 \\ 0 & -2 \end{bmatrix}.$$

V-ERL (Vacation in Erlang):

$$\gamma = (1, 0), \;\; V = \begin{bmatrix} -2 & 2 \\ 0 & -2 \end{bmatrix}.$$

S-EXP (Exponential services):

$$\alpha = \beta = [-1], \;\; T = S = [1].$$

R-EXP (Repair in Exponential):

$$\psi = [-1], \;\; P = [1].$$

V-EXP (Vacation in Exponential):

$$\gamma = [-1], \;\; V = [1].$$

S-HEX (Hyper-exponential services):

$$\alpha = \beta = (0.8, 0.2), \;\; T = S = \begin{bmatrix} -2.8 & 0 \\ 0 & -0.28 \end{bmatrix}.$$

R-HEX (Repair in Hyper-exponential):

$$\psi = (0.8, 0.2), \;\; P = \begin{bmatrix} -2.8 & 0 \\ 0 & -0.28 \end{bmatrix}.$$

V-HEX (Vacation in Hyper-exponential):

$$\gamma = (0.8, 0.2), \;\; V = \begin{bmatrix} -2.8 & 0 \\ 0 & -0.28 \end{bmatrix}.$$

8.1. Illustration 1

We looked into the impact of repair rate(ζ) versus the mean number of low priority customers in the system (E_{LP}). We fix $\lambda_1 = 1.2$, $\lambda_2 = 0.8$, $\mu_1 = 10$, $\mu_2 = 8$, $\delta = 2$, $\eta = 10$, $\sigma = 2$, $p = 0.4$, $q = 0.6$, $\theta = 0.8$, $b = 0.3$, $N = 5$ such that the system is stable.

- With the help of Tables 1 to 3, we analyze the repair rate versus expected number of LP customers in the system by combining the categories for arrival and service times.
- The repair rate (ζ) when it grows and the associated E_{LP} is decreases gradually in Hyper-exponential arrivals and MAP positive correlation arrivals and slowly in Erlang arrivals compared to all other arrivals with different service times.

8.2. Illustration 2

We looked into the impact of high priority service rate (μ_1) versus the total cost (TC). We fix $\lambda_1 = 1.2$, $\mu_1 = 10$, $\mu_2 = 8$, $\zeta = 8$, $\delta = 2$, $\eta = 10$, $\sigma = 2$, $p = 0.4$, $q = 0.6$, $\theta = 0.8$, $b = 0.3$, $N = 5$, $C_{H_1} = 10$, $C_{H_2} = 8$, $C_{idle} = 1$, $C_{vac} = 5$, $C_{HPNB} = 4$, $C_{LPNB} = 4$, $C_{HPSB} = 4$, $C_{LPSB} = 4$, $C_{rep} = 3$, $C_{cd} = 1$, $C_1 = 3$, $C_2 = 3$, $C_3 = 2$, $C_4 = 2$, $C_5 = 1$, $C_6 = 2$, $C_7 = 1$, $C_8 = 1$ such that the system is stable.

- With the help of Tables 4 to 6, we analyze the HP service rate versus total cost by combining the categories for arrival and service times.
- The HP service rate(μ_1) when it grows and the associated TC is increases rapidly in MAP positive correlation arrivals compared to all other arrivals with different service times.

8.3. Illustration 3

We looked into the impact of breakdown rate (σ) versus the mean number of low priority customers in the system (E_{LP}). We fix $\lambda_1 = 1.2$, $\lambda_2 = 0.8$, $\mu_1 = 10$, $\mu_2 = 8$, $\delta = 2$, $\eta = 10$, $\zeta = 8$, $p = 0.4$, $q = 0.6$, $\theta = 0.8$, $b = 0.3$, $N = 5$ such that the system is stable.

- With the help of Figures 2 to 6, we analyze the breakdown rate versus the mean number of LP customers in the system by combining the categories for arrival and service times.
- The breakdown rate (σ) when it grows and the associated E_{LP} is increases rapidly in Hyper-exponential arrivals and MAP positive correlation arrivals compared to all other arrivals with different service times.

8.4. Illustration 4

We looked into the impact of both repair rate (ζ) and high priority service rate (μ_1) versus the mean number of high priority customers in the system (E_{HP}). We fix $\lambda_1 = 1.2$, $\lambda_2 = 0.8$, $\mu_2 = 8$, $\delta = 2$, $\eta = 10$, $\sigma = 2$, $p = 0.4$, $q = 0.6$, $\theta = 0.8$, $b = 0.3$, $N = 5$ such that the system is stable.

- With the help of Figures 7 to 21, we analyze both repair rate and HP service rate versus mean number of HP customers in the system by combining the categories for arrival and service times.
- Both repair rate (ζ) and HP service rate (μ_1) when it grows, and the associated E_{HP} is decreases in different arrival and different service times.

Table 1. Repair rate(ζ) vs E_{LP} - Erlang service

ζ	A-ERL	A-EXP	A-HEX	A-MAPNC	A-MAPPC
8	0.460547524	0.516897096	0.669189364	0.481913263	3.49160074
9	0.454371039	0.508113473	0.653169141	0.473217286	3.413958319
10	0.449602661	0.501326161	0.640887194	0.466486286	3.352603699
11	0.445811675	0.495926348	0.631180481	0.461123347	3.302897622
12	0.442726335	0.491529282	0.623320539	0.456750624	3.261808524
13	0.440166859	0.487880094	0.616828704	0.453117461	3.227271967
14	0.438009629	0.484803364	0.611377901	0.450051097	3.197833635
15	0.436166887	0.482174453	0.60673724	0.447428633	3.17243956
16	0.434574632	0.479902396	0.602739193	0.445160269	3.150307293
17	0.433185098	0.477919266	0.599259308	0.443178875	3.130843803

Table 2. Repair rate(ζ) vs E_{LP} - Exponential service

ζ	A-ERL	A-EXP	A-HEX	A-MAPNC	A-MAPPC
8	0.456043444	0.514280628	0.665052837	0.475131296	3.313513088
9	0.449591802	0.505385744	0.649263107	0.466351096	3.240449357
10	0.44463845	0.498538947	0.637169721	0.459580758	3.182683909
11	0.440718142	0.493109502	0.627622901	0.454203616	3.135862348
12	0.437539543	0.488700535	0.619901495	0.449831076	3.097138286
13	0.434911119	0.485050172	0.613531395	0.44620638	3.064572906
14	0.432701867	0.481978803	0.608188643	0.443153195	3.036800156
15	0.43081921	0.479359213	0.603644632	0.440546474	3.012829905
16	0.429195897	0.477098817	0.599733582	0.438295111	2.991927031
17	0.427781913	0.475128655	0.596332427	0.43633117	2.973534334

Table 3. Repair rate(ζ) vs E_{LP} - Hyper exponential service

ζ	A-ERL	A-EXP	A-HEX	A-MAPNC	A-MAPPC
8	0.445937718	0.497423054	0.627323153	0.441290722	2.653170641
9	0.438291338	0.48817811	0.613304699	0.432280204	2.5970312
10	0.43255103	0.481164227	0.602564249	0.425435933	2.552607298
11	0.428097432	0.475675406	0.5940964	0.420073485	2.516569203
12	0.424549231	0.471271301	0.587263425	0.415765823	2.486738534
13	0.421660147	0.467664172	0.58164269	0.412233779	2.461630586
14	0.419264672	0.464658634	0.576943802	0.409287694	2.440198777
15	0.417247805	0.462117676	0.572961095	0.40679449	2.421684267
16	0.41552733	0.459942551	0.5695451	0.4046582	2.405523507
17	0.414043013	0.458060389	0.566584744	0.402807961	2.391289261

Table 4. Service rate(μ_1) vs TC - Erlang service

μ_1	A-ERL	A-EXP	A-HEX	A-MAPNC	A-MAPPC
10	113.7481528	113.8225507	114.8148432	112.6056456	134.6013385
11	118.1093353	118.1283766	118.9931095	116.8722905	138.886483
12	122.5168769	122.4959578	123.2681932	121.2047844	143.2387109
13	126.9583123	126.9078982	127.6108436	125.5852576	147.6411124
14	131.4253122	131.352731	132.0023476	130.0017941	152.082002
15	135.9120911	135.8226022	136.4302286	134.4461405	156.5530485
16	140.4145004	140.3119585	140.8858742	138.9123963	161.0481534
17	144.9294843	144.8167658	145.3631621	143.3962296	165.5627511
18	149.4547407	149.334025	149.8576278	147.894387	170.0933562
19	153.9885008	153.8614619	154.3659443	152.4043765	174.637262

Table 5. Service rate(μ_1) vs TC - Exponential service

μ_1	A-ERL	A-EXP	A-HEX	A-MAPNC	A-MAPPC
10	113.6899184	113.8669063	114.9123723	112.5598755	133.1572135
11	118.0546662	118.1777644	119.1028028	116.8399283	137.4548261
12	122.4652905	122.5484034	123.3842475	121.1828788	141.814399
13	126.909388	126.9622231	127.7300924	125.5718322	146.22057
14	131.3786843	131.4082042	132.1230088	129.9954616	150.6626583
15	135.8674436	135.8787571	136.5512793	134.4458884	155.1330075
16	140.3715614	140.3684902	141.0067234	138.917462	159.6259863
17	144.8880211	144.8734719	145.4834706	143.406023	164.13736
18	149.4145541	149.3907699	149.9772075	147.9084408	168.6638826
19	153.9494206	153.9181543	154.4846981	152.4223132	173.2030242

Table 6. Service rate(μ_1) vs TC - Hyper exponential service

μ_1	A-ERL	A-EXP	A-HEX	A-MAPNC	A-MAPPC
10	113.7147727	114.0093165	115.070022	112.2976809	127.8355384
11	118.0951029	118.3603646	119.3485361	116.6355047	132.2080641
12	122.5177511	122.7594208	123.6897387	121.022944	136.6231068
13	126.9714677	127.1938328	128.0771446	125.4471985	141.0703578
14	131.4486648	131.655025	132.4995383	129.8995359	145.5426124
15	135.944043	136.1369845	136.949036	134.3737944	150.0346713
16	140.4537897	140.6353723	141.4199408	138.8655009	154.5426795
17	144.9750929	145.1469796	145.9080374	143.3713283	159.0637115
18	149.5058324	149.6693824	150.410142	147.8887496	163.595502
19	154.0443794	154.200716	154.9238064	152.4158097	168.1362652

9. Conclusion

In this study, two kinds of priority customers who come based on the MMAP, a generic version of MAP, are discussed in relation to the preemptive priority queueing model, with the Phase-type services, Phase-type vacations, Phase-type repairs, working breakdown, feedback is only for high priority customers, close-down, and balking. Additionally, we analyse busy periods and costs as part of our work. We calculated the mean number of low-priority customers in the system for various values of the repair rate and the total cost for various values of the high priority service rate using numerical arrivals and services. The 2D graph represents the mean number of low-priority customers in the system along with different values of the breakdown rate. The 3D graph represents the mean number of high-priority customers in the system along with different values of both the repair rate and the high priority service rate. All the tables and graphs demonstrate the stability of the system. Further, we extend our work with the inventory queueing system with multi-servers.

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Appendix

The following Figures 2 through 6 belong to Illustration 3, and Figures 7 through 21 belong to Illustration 4.

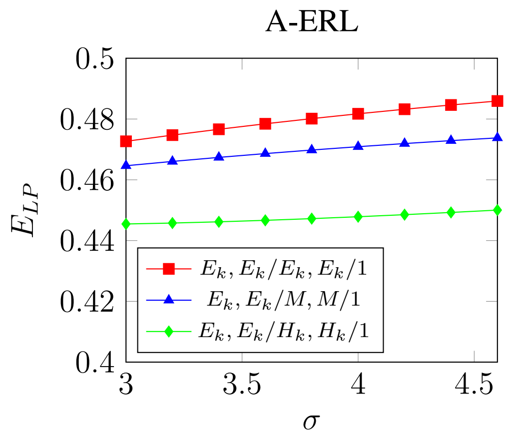

Figure 2. Breakdown rate (σ) vs E_{LP} - A-ERL

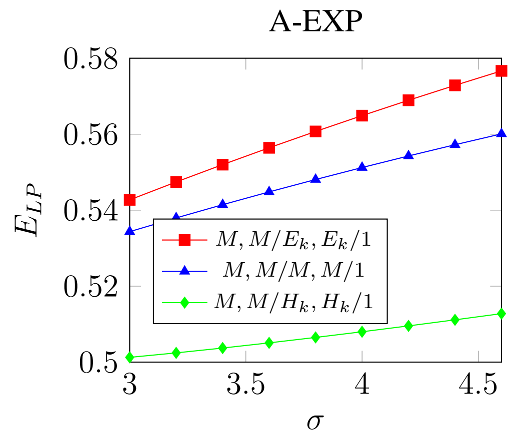

Figure 3. Breakdown rate (σ) vs E_{LP} - A-EXP

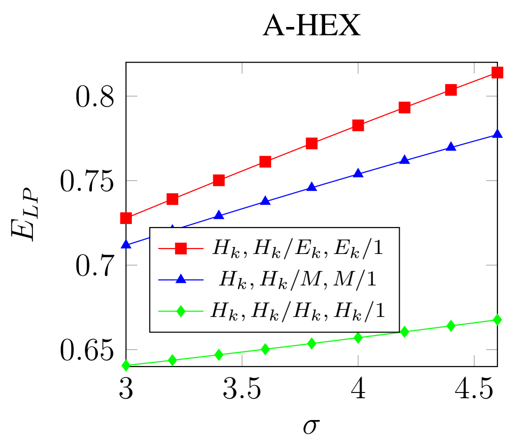

Figure 4. Breakdown rate (σ) vs E_{LP} - A-HEX

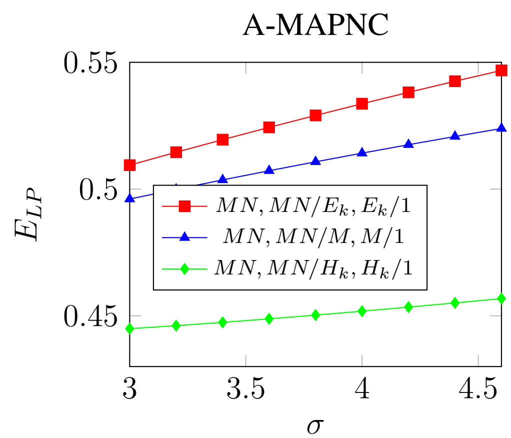

Figure 5. Breakdown rate (σ) vs E_{LP} - A-MAPNC

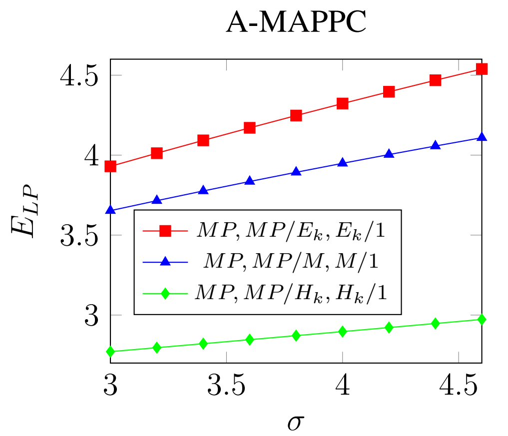

Figure 6. Breakdown rate (σ) vs E_{LP} - A-MAPPC

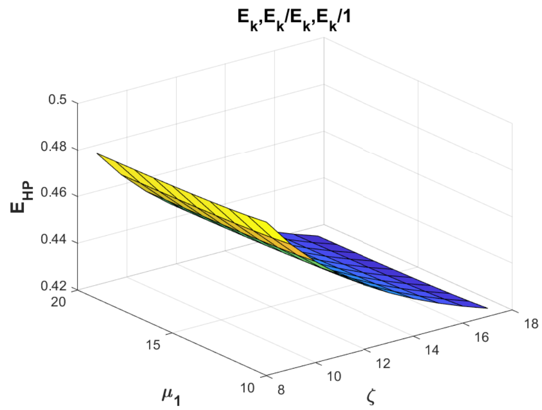

Figure 7. $E_k, E_k/E_k, E_k/1$ - Repair rate (ζ) and HP service rate (μ_1) vs E_{HP}

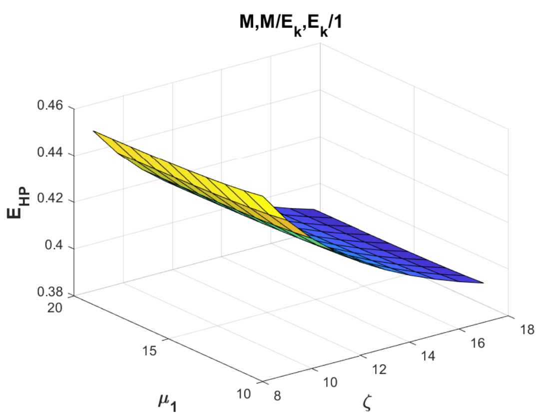

Figure 8. $M, M/E_k, E_k/1$ - Repair rate (ζ) and HP service rate (μ_1) vs E_{HP}

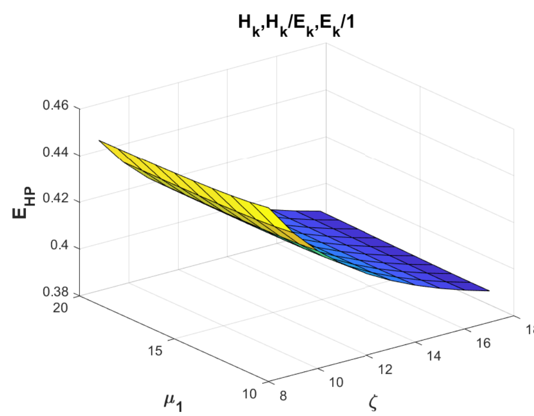

Figure 9. $H_k, H_k/E_k, E_k/1$ - Repair rate (ζ) and HP service rate (μ_1) vs E_{HP}

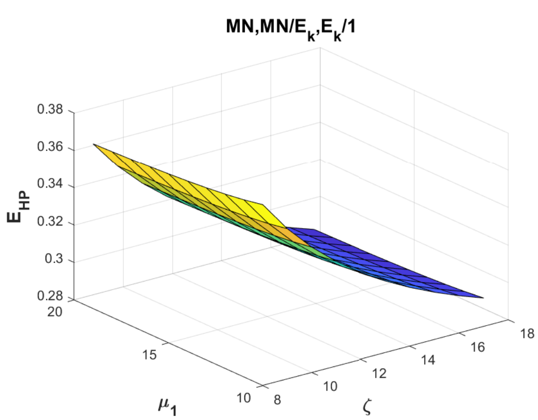

Figure 10. $MAPNC, MAPNC/E_k, E_k/1$ - Repair rate (ζ) and HP service rate (μ_1) vs E_{HP}

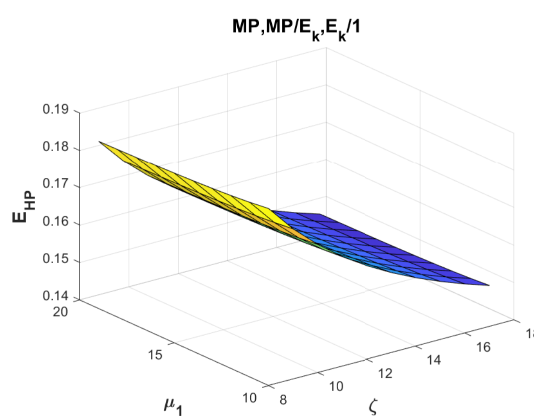

Figure 11. $MAPPC, MAPPC/E_k, E_k/1$ - Repair rate (ζ) and HP service rate (μ_1) vs E_{HP}

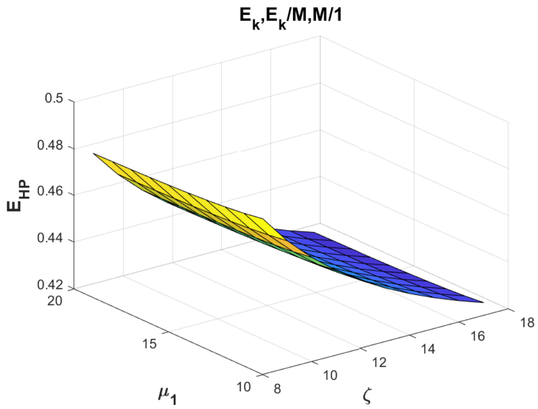

Figure 12. $E_k, E_k/M, M/1$ - Repair rate (ζ) and HP service rate (μ_1) vs E_{HP}

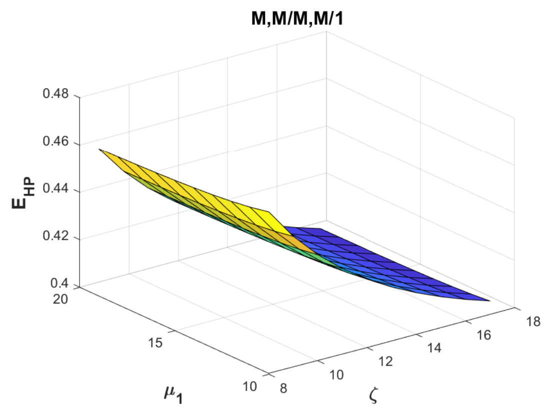

Figure 13. $M, M/M, M/1$ - Repair rate (ζ) and HP service rate (μ_1) vs E_{HP}

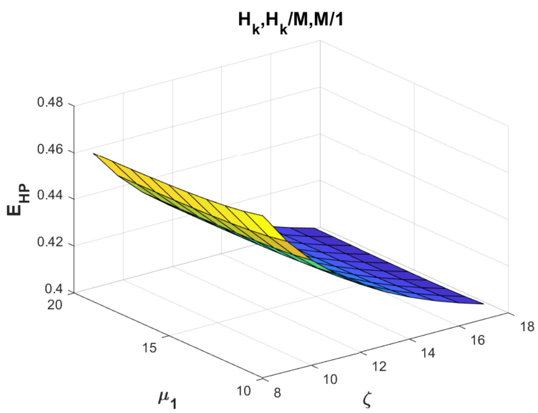

Figure 14. $H_k, H_k/M, M/1$ - Repair rate (ζ) and HP service rate (μ_1) vs E_{HP}

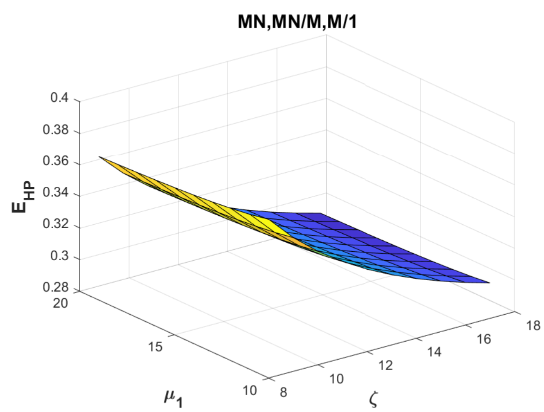

Figure 15. $MAPNC, MAPNC/M, M/1$ - Repair rate (ζ) and HP service rate (μ_1) vs E_{HP}

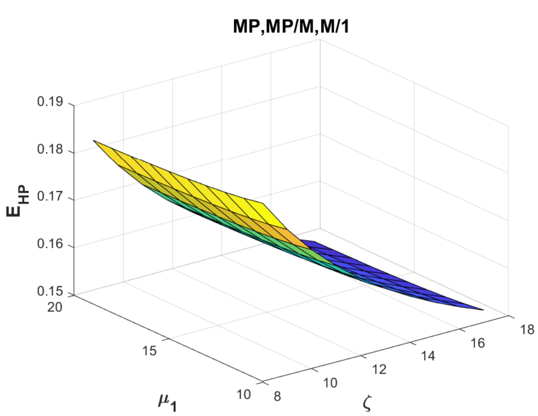

Figure 16. $MAPPC, MAPPC/M, M/1$ - Repair rate (ζ) and HP service rate (μ_1) vs E_{HP}

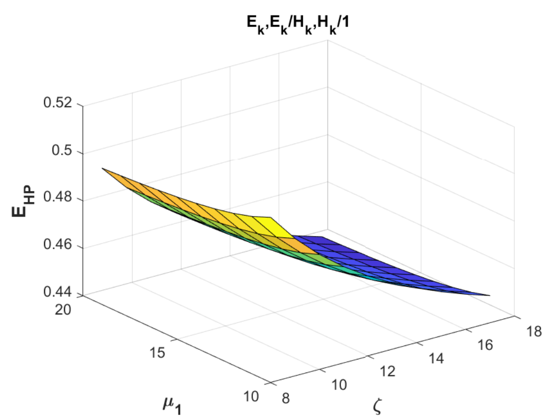

Figure 17. $E_k, E_k/H_k, H_k/1$ - Repair rate (ζ) and HP service rate (μ_1) vs E_{HP}

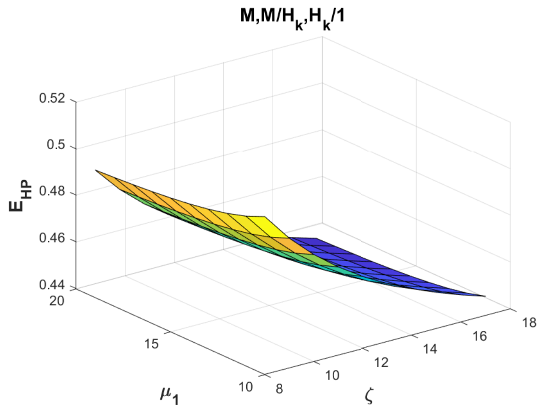

Figure 18. $M, M/H_k, H_k/1$ - Repair rate (ζ) and HP service rate (μ_1) vs E_{HP}

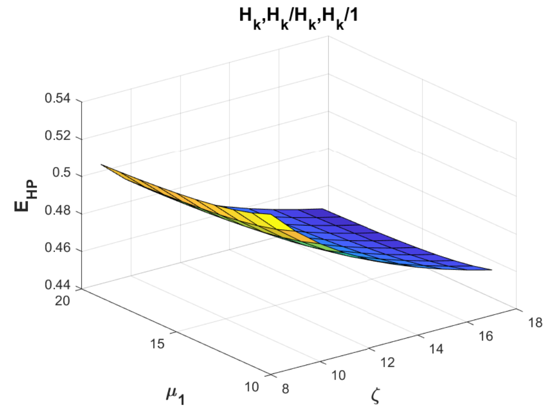

Figure 19. $H_k, H_k/H_k, H_k/1$ - Repair rate (ζ) and HP service rate (μ_1) vs E_{HP}

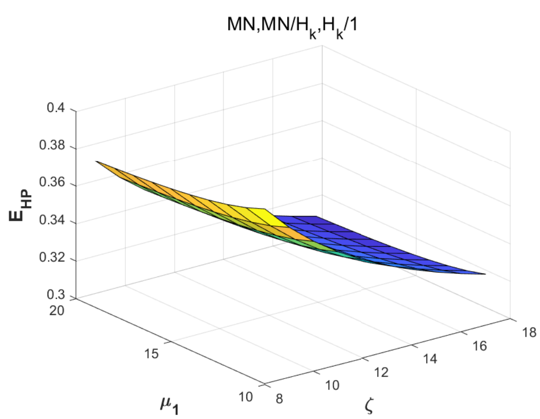

Figure 20. $MAPNC, MAPNC/H_k, H_k/1$ - Repair rate (ζ) and HP service rate (μ_1) vs E_{HP}

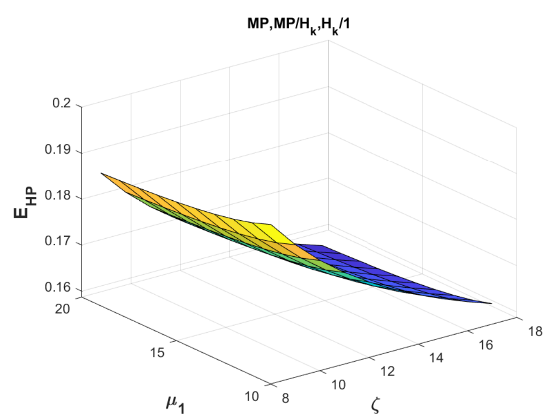

Figure 21. $MAPPC, MAPPC/H_k, H_k/1$ - Repair rate (ζ) and HP service rate (μ_1) vs E_{HP}