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Poisson-Exponentiated Weibull Distribution: Properties, Applications and Extension

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Abstract

In this article, we introduce the Poisson-exponentiated Weibull distribution, a novel member of Poisson-X family. The statistical as well as distributional aspects of the distribution are studied, and simulation method is opted to verify the accuracy of the method of maximum likelihood estimation. The flexibility of the distribution is illustrated by real data. A reliability test plan based on the new distribution is developed and with suitable real data its feasibility is confirmed. Later, we introduce an extension of the Poisson-exponentiated Weibull distribution called the Farlie-Gumbel-Morgenstern bivariate Poisson-exponentiated Weibull distribution and consider the concomitants that arise from this bivariate distribution.

Keywords: Poisson-X family; Poisson-expoentiated Weibull distribution; Reliability test plan; Farlie-Gumbel-Morgenstern family; Concomitants of order statistics

MSC 2010 No.: 62E05, 62N05, 62E15

1. Introduction

Poisson distribution is one of the well-known distribution for modelling count data sets, and now there has been a renewed interest in modelling these count data sets. The equality of mean and variance of this prominent distribution is considered as the major factor behind its fitness in analysing count data sets. In several fields like medical, economics, biological studies, and finance, count data plays a vital role. Since it has influence in different fields, each count data may possess different characteristics. Thus, it cannot be model with a specific distribution. Though Poisson distribution is considered to be suitable for modelling count data, it is not compatible with modelling the datasets that are not equidispersed. In order to deal with such situations, many new distributions from the Poisson family have been put forward by researchers in recent years. A lifetime distribution from exponetial family called Poisson-exponential distribution is developed by Cancho et al. (2011). For analyzing motor vehicle crash data, Cheng et al. (2013) developed Poisson-Weibull generalized linear model. Al-Zahrani and Sagor (2014) introduced Poisson-Lomax distribution. Poisson-transmuted exponential distribution is a count model given by Bhati et al. (2017). Altun (2019) proposed a regression model from Poisson quasi-Lindley distribution. A review of distributions arises from the Poisson family is shown by Maurya and Nadarajah (2021). George and George (2022) introduced the Poisson-Uniform distribution and illustrated its flexibility using different real data sets. Recently, a generalization of weighted Poisson distribution is proposed by Diafouka et al. (2023) and George and George (2023) designed Poisson-power Lindley distribution in the context of acceptance sampling plan.

Here we present a new distribution, namely, the Poisson-exponentiated Weibull (PEW) distribution from the Poisson-X family by Tahir et al. (2016). The improved version of Weibull distribution known as the exponentiated Weibull (EW) family is presented by Mudholkar and Srivastava (1993). Besides the class of monotone failure rates, distributions with unimodal and bathtub-shaped failure rates are available from the exponentiated Weibull family, which is not the case of distributions from the Weibull family. In this paper, the remaining sections are unfolded as follows. We provide Poisson-exponentiated Weibull distribution and examine its statistical characteristics, including parameter estimation, in Section 2. In Section 3, we explore the new model's feasibility using real data. A reliability test plan for the PEW distribution and underline the corresponding results through a real data set are given in Section 4. We propose the Farlie-Gumbel-Morgenstern bivariate Poisson-exponentiated Weibull (FGMBPEW) distribution and establish the distributional aspects of concomitants of order statistics in Section 5. Finally, Section 6 gives the concluding remarks.

2. Poisson-expoentiated Weibull Distribution (PEW)

If the parent distribution of a Poisson-X family of distributions is exponentiated Weibull, then the newly generated distribution is named as Poisson-expoentiated Weibull distribution with cdf,

$$F(x;\tau,\theta,m,\zeta) = (1-e^{-1})^{-1}\left[1-e^{-\left[1-e^{-\left(\frac{x}{\zeta}\right)^{\tau}}\right]^{m\theta}}\right], \quad x>0,\ \tau,\theta,m,\zeta>0, \tag{1}$$

and the corresponding pdf becomes

$$f(x;\tau,\theta,m,\zeta) = \frac{m}{(1-e^{-1})}\frac{\tau\theta}{\zeta^{\tau}}x^{\tau-1}e^{-\left(\frac{x}{\zeta}\right)^{\tau}}\left[1-e^{-\left(\frac{x}{\zeta}\right)^{\tau}}\right]^{m\theta-1}e^{-\left[1-e^{-\left(\frac{x}{\zeta}\right)^{\tau}}\right]^{m\theta}}, x>0,$$
$$\tau,\theta,m,\zeta>0. \tag{2}$$

For various values of the parameters, the pdf plots of PEW distribution are displayed in Figure 1. It can be seen that for a fixed value of τ, θ and ζ with increasing the value of m, the peakness of the pdf curve decreases. We can also find that by fixing the parameter θ and increasing the values of other parameters, the curves become more positively skewed.

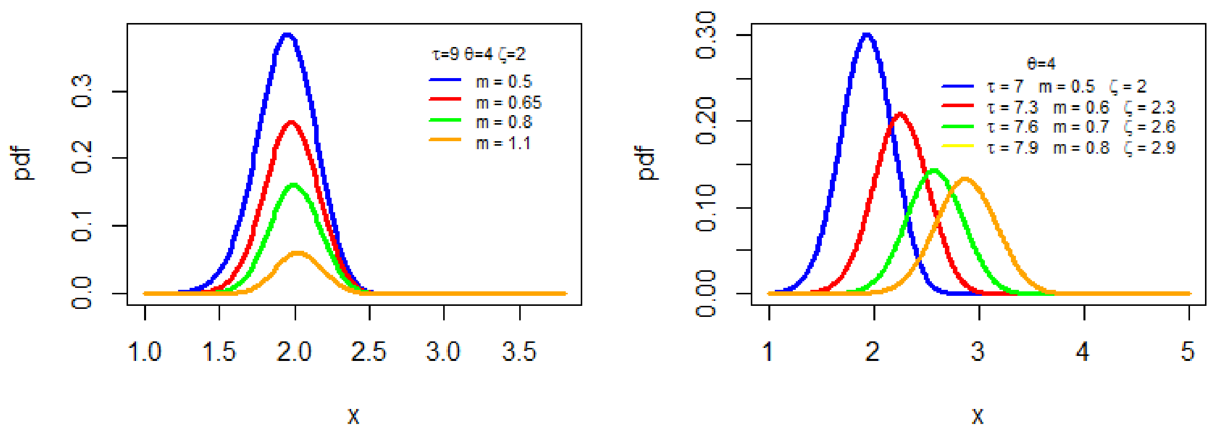

Figure 1. pdf graphs of Poisson-exponentiated Weibull distribution for different values of parameters

A comparison with exponentiated Weibull distribution

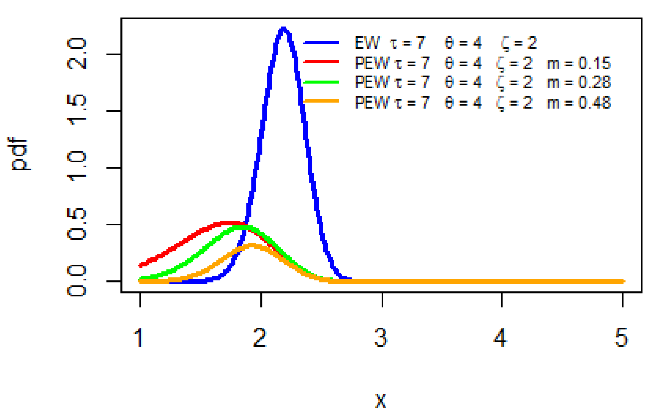

Figure 2. pdf graphs of Poisson-exponentiated Weibull distribution vs exponentiated Weibull

The graphs showing the pdfs of the PEW distribution along with the EW distribution are given in Figure 2. From the figure, we can see that for the EW distribution, the curve is in leptokurtic nature. But when considering PEW distribution for the same parameters ($\tau = 7, \theta = 4, \zeta = 2$), with increasing the values of the additional parameter m, the peakness of the curves decreases and the negatively skewed distribution comes to be symmetric or positively skewed.

Properties of PEW Distribution

The survival function, hazard rate function and cumulative hazard rate function of the PEW distribution are, respectively,

$$S(x) = \frac{e^{-\left[1-e^{-\left(\frac{x}{\zeta}\right)^{\tau}}\right]^{m\theta}} - e^{-1}}{1-e^{-1}}, \tag{3}$$

$$n(x) = \frac{m\tau\theta\zeta^{-\tau}x^{\tau-1}e^{-\left(\frac{x}{\zeta}\right)^{\tau}}\left[1-e^{-\left(\frac{x}{\zeta}\right)^{\tau}}\right]^{m\theta-1}e^{-\left[1-e^{-\left(\frac{x}{\zeta}\right)^{\tau}}\right]^{m\theta}}}{e^{-\left[1-e^{-\left(\frac{x}{\zeta}\right)^{\tau}}\right]^{m\theta}} - e^{-1}}, \quad \text{and} \tag{4}$$

$$N(x) = -\log\left[\frac{e^{-\left[1-e^{-\left(\frac{x}{\zeta}\right)^{\tau}}\right]^{m\theta}} - e^{-1}}{1-e^{-1}}\right]. \tag{5}$$

The reversed hazard rate function, residual life time at time t and the corresponding survival function are respectively given by

$$q(x) = \frac{m\tau\theta\zeta^{-\tau}x^{\tau-1}e^{-\left(\frac{x}{\zeta}\right)^{\tau}}\left[1-e^{-\left(\frac{x}{\zeta}\right)^{\tau}}\right]^{m\theta-1}e^{-\left[1-e^{-\left(\frac{x}{\zeta}\right)^{\tau}}\right]^{m\theta}}}{1-e^{-\left[1-e^{-\left(\frac{x}{\zeta}\right)^{\tau}}\right]^{m\theta}}}, \tag{6}$$

$$r_{x_t}(x) = \frac{m\tau\theta\zeta^{-\tau}x^{\tau-1}e^{-\left(\frac{x}{\zeta}\right)^{\tau}}\left[1-e^{-\left(\frac{x}{\zeta}\right)^{\tau}}\right]^{m\theta-1}e^{-\left[1-e^{-\left(\frac{x}{\zeta}\right)^{\tau}}\right]^{m\theta}}}{e^{-\left[1-e^{-\left(\frac{t}{\zeta}\right)^{\tau}}\right]^{m\theta}} - e^{-1}}, \quad \text{and} \tag{7}$$

$$R_{x_t}(x) = \frac{e^{-\left[1-e^{-\left[1-e^{-\left(\frac{x}{\zeta}\right)^{\tau}}\right]^{m\theta}}\right]} - e^{-1}}{e^{-\left[1-e^{-\left[1-e^{-\left(\frac{t}{\zeta}\right)^{\tau}}\right]^{m\theta}}\right]} - e^{-1}}. \tag{8}$$

The past life time and corresponding distribution function of the PEW distribution are

$$d_{t_x}(x) = \frac{m\tau\theta\zeta^{-\tau}x^{\tau-1}e^{-\left(\frac{x}{\zeta}\right)^{\tau}}\left[1-e^{-\left(\frac{x}{\zeta}\right)^{\tau}}\right]^{m\theta-1}e^{-\left[1-e^{-\left(\frac{x}{\zeta}\right)^{\tau}}\right]^{m\theta}}}{1-e^{-\left[1-e^{-\left(\frac{t}{\zeta}\right)^{\tau}}\right]^{m\theta}}} \tag{9}$$

and

$$D_{t_x}(x) = \frac{1 - e^{-\left[1 - e^{-\left(\frac{x}{\zeta}\right)^{\tau}}\right]^{m\theta}}}{1 - e^{-\left[1 - e^{-\left(\frac{t}{\zeta}\right)^{\tau}}\right]^{m\theta}}}. \tag{10}$$

Figure 3 presents the hrf graphs of the PEW distribution. It can be note that, for a fixed value of ζ, with changing the values of other parameters we obtain reverse J-shaped curves. We can see an increasing failure rate (IFR) by fixing the values of ζ, m and changing τ, θ values. Again, on setting the values of m, θ as fixed and making variations in the τ, ζ values, we can see a J-shaped curve. Also, we obtain a decreasing failure rate (DFR) for some fixed values of ζ, m and θ with different τ.

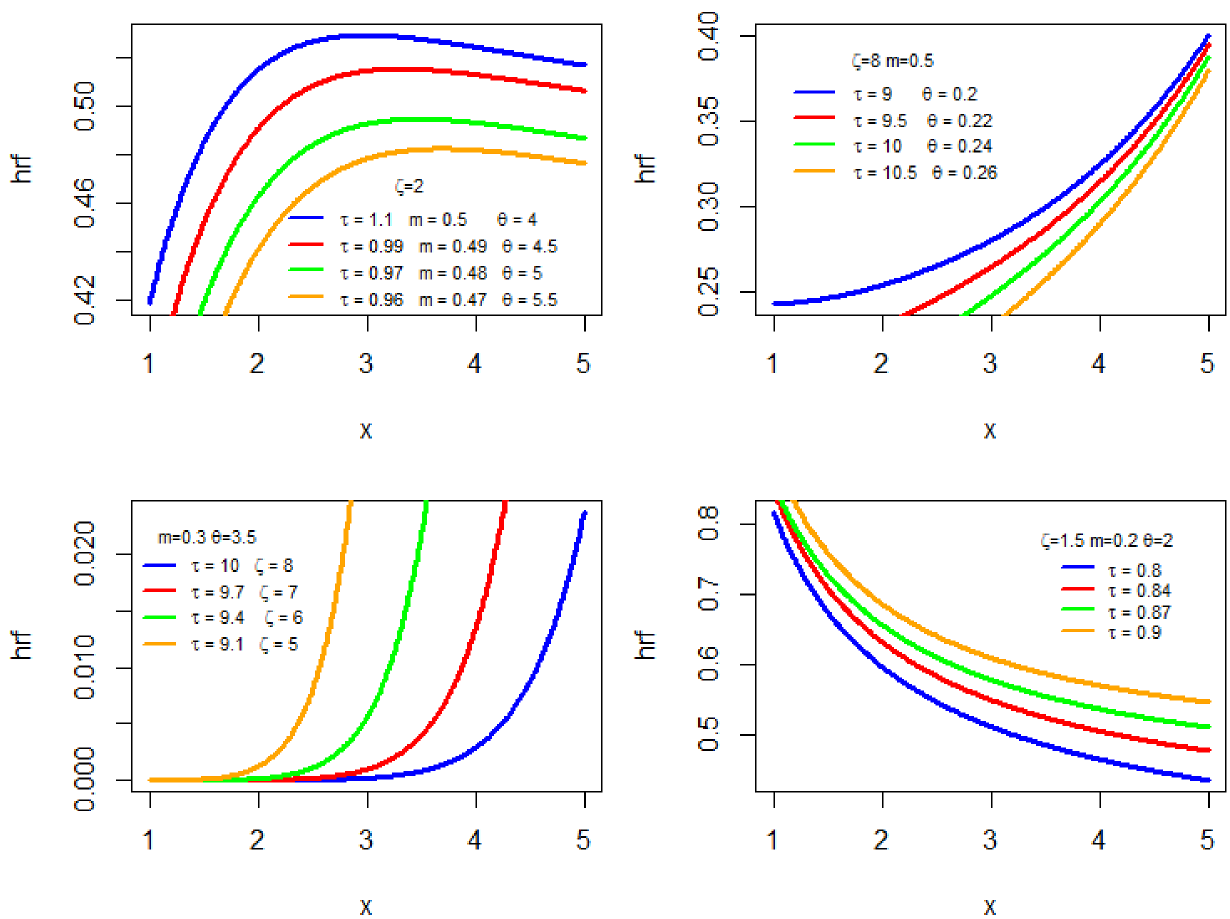

Figure 3. hrf graphs of Poisson-exponentiated Weibull distribution for different values of parameters

Linear Representation

The concept of exponentiated distribution is used to derive some important expansions for (1) and (2). We can say, a baseline cdf G(x) has the exp-G distribution if its cdf is $L_{\gamma}(x) = G^{\gamma}(x) \quad \gamma > 0$.

On expanding (1),

$$F(x;\tau,\theta,m,\zeta)=\sum_{i=0}^{\infty} w_{i+1}\left[1-e^{-\left(\frac{x}{\zeta}\right)^{\tau}}\right]^{\theta m(i+1)}, \tag{11}$$

where $w_{i+1}=\frac{((-1)^i)}{[(i+1)!(1-e^{-1})]}$ (for $i\geq 0$) and thereby the pdf,

$$f(x;\tau,\theta,m,\zeta)=\sum_{i=0}^{\infty} w_{i+1}\,(i+1)\,m\frac{\tau\theta}{\zeta^{\tau}}x^{\tau-1}e^{-\left(\frac{x}{\zeta}\right)^{\tau}}\left[1-e^{-\left(\frac{x}{\zeta}\right)^{\tau}}\right]^{\theta m(i+1)-1}. \tag{12}$$

(12) can have the other form

$$f(x;\tau,\theta,m,\zeta)=\sum_{i=0}^{\infty} w_{i+1}l_{(i+1)m}(x). \tag{13}$$

Here $l_{(i+1)m}(x)$ is the density of exponentiated extension of exponentiated Weibull distribution with $(i+1)m$ as power parameter. (13) reveals that a linear representation of the exponentiated form of exponentiated Weibull density can be obtained from Poisson-exponentiated Weibull density.

Lemma 2.1.

If $Y\sim$ exponentiated Weibull (τ,θ,ζ) distribution, then

$$X=-\zeta\left\{\log\left(1+\left\{\log\left[1-Y\left(1-e^{-1}\right)\right]\right\}^{\frac{1}{m\theta}}\right)\right\}^{\frac{1}{\tau}}, \tag{14}$$

follows the Poisson-exponentiated Weibull (τ,θ,m,ζ) distribution.

Quantile Function

The quantile function of Poisson-exponentiated Weibull distribution is as below,

$$x_p=-\zeta\left\{\log\left(1+\left\{\log\left[1-p\left(1-e^{-1}\right)\right]\right\}^{\frac{1}{m\theta}}\right)\right\}^{\frac{1}{\tau}}. \tag{15}$$

Hence, the median is

$$x_p=-\zeta\left\{-\log\left[1+(0.16498)^{\frac{1}{m\theta}}\right]\right\}^{\frac{1}{\tau}}. \tag{16}$$

Moments

In Poisson-exponentiated Weibull (τ,θ,m,ζ) distribution, the k^{th} moment is

$$\mu_k'=E(X^k)=E_t\left[-\zeta\left\{\left(1-t^{\frac{1}{\theta}}\right)\right\}^{\frac{1}{\tau}}\right]. \tag{17}$$

Order Statistics

Consider the Poisson-exponentiated Weibull random variables $X_1,...X_n$. The pdf of r^{th} order statistics have the form,

$$f_r(x) = \frac{r!}{(i-1)!\,(r-i)!} m\tau\theta\zeta^{-\tau}x^{\tau-1}e^{-\left(\frac{x}{\zeta}\right)^\tau}\left[1-e^{-\left(\frac{x}{\zeta}\right)^\tau}\right]^{m\theta-1} e^{-\left[1-e^{-\left(\frac{x}{\zeta}\right)^\tau}\right]^{m\theta}}$$
$$\sum_{j=0}^{r-i}(-1)^j\binom{r-i}{j}(1-e^{-1})^{-(i+j)}\left[1-e^{-\left[1-e^{-\left(\frac{x}{\zeta}\right)^\tau}\right]^{m\theta}}\right]^{i+j-1}. \tag{18}$$

Now the cdf $F_n(x)$ of the largest and $F_1(x)$ of the smallest order statistics are respectively given by,

$$F_n(x) = (1-e^{-1})^{-n}\left[1-e^{-\left[1-e^{-\left(\frac{x}{\zeta}\right)^\tau}\right]^{m\theta}}\right]^n, \tag{19}$$

and

$$F_1(x) = 1-\left[\frac{e^{-\left[1-e^{-\left(\frac{x}{\zeta}\right)^\tau}\right]^{m\theta}}-e^{-1}}{1-e^{-1}}\right]^n. \tag{20}$$

Corresponding pdfs are given by,

$$f_n(x) = (1-e^{-1})^{-n}\left[1-e^{-\left[1-e^{-\left(\frac{x}{\zeta}\right)^\tau}\right]^{c\theta}}\right]^{n-1} nm\tau\theta\zeta^{-\tau}x^{\tau-1}e^{-\left(\frac{x}{\zeta}\right)^\tau}$$
$$\left[1-e^{-\left(\frac{x}{\beta}\right)^\tau}\right]^{m\theta-1} e^{-\left[1-e^{-\left(\frac{x}{\zeta}\right)^\tau}\right]^{m\theta}}, \tag{21}$$

and

$$f_1(x) = (1-e^{-1})^{-n}\left[e^{-\left[1-e^{-\left(\frac{x}{\zeta}\right)^\tau}\right]^{m\theta}}-e^{-1}\right]^{n-1} nm\tau\theta\zeta^{-\tau}x^{\tau-1}e^{-\left(\frac{x}{\zeta}\right)^\tau}$$
$$\left[1-e^{-\left(\frac{x}{\zeta}\right)^\tau}\right]^{m\theta-1} e^{-\left[1-e^{-\left(\frac{x}{\zeta}\right)^\tau}\right]^{m\theta}}. \tag{22}$$

Parameter Estimation

Here, we use maximum likelihood method of estimation. Let $X_1, X_2...X_n$ be Poisson-exponentiated Weibull random variables. The log likelihood function is,

$$l(\tau,\theta,m,\zeta;x) = n\log\left[m\left(1-e^{-1}\right)^{-1}\right] + n\log\tau + n\log\theta - n\tau\log\zeta + (\tau-1)$$
$$\sum_{i=1}^n \log x_i - \sum_{i=1}^n\left(\frac{x_i}{\zeta}\right)^\tau + (m\theta-1)\sum_{i=1}^n\log\left[1-e^{-\left(\frac{x_i}{\zeta}\right)^\tau}\right]$$
$$-\sum_{i=1}^n\left[1-e^{-\left(\frac{x_i}{\zeta}\right)^\tau}\right]^{m\theta}. \tag{23}$$

The computations are implemented using the nlm function of R software.

Simulation

The exactness of the MLEs of PEW distribution is studied using the Monte Carlo method of simulation. We consider two set of parameter values given by,

$$(\tau = 1, \theta = 2, m = 0.5, \zeta = 2.9) \text{ and } (\tau = 1, \theta = 4, m = 0.25, \zeta = 0.5).$$

We simulate data from the PEW distribution, for different sample sizes $n = 20$, $n = 30$, $n = 50$ and $n = 100$ and is repeated for 10000 times. Table 1 along with Table 2 reveals the outcomes.

Table 1. Average estimates(AEs), Biases and MSEs for set I ($\tau = 1, \theta = 2, m = 0.5, \zeta = 2.9$)

n	AEs	Bias	MSE
	1.1424	0.1424	0.1615
	1.9465	-0.0568	0.05684
$n = 20$	0.1422	-0.3577	0.1395
	2.7076	-0.1924	0.1519
	1.1004	0.1003	0.1463
	1.9754	-0.0245	0.0010
$n = 30$	0.2692	-0.2307	0.1071
	2.7379	-0.1629	0.1395
	1.0981	0.0981	0.0391
	1.9865	-0.01347	0.0002
$n = 50$	0.5604	0.0604	0.01798
	2.8055	-0.0944	0.0336
	1.0499	0.0499	0.0096
	2.003	0.0029	0.0002
$n = 100$	0.5285	0.0285	0.0019
	2.8837	-0.0163	0.0017

Table 2. Average estimates(AEs), Biases and MSEs for set II ($\tau = 1, \theta = 4, m = 0.25, \zeta = 0.5$)

n	AEs	Bias	MSE
	1.2845	0.2845	0.0811
	4.3534	0.3533	0.0.1972
$n = 20$	0.4388	0.1888	0.0.0896
	0.1523	-0.3476	0.1694
	1.2146	0.2146	0.0523
	4.2770	0.2770	0.1298
$n = 30$	0.3332	-0.0832	0.0134
	0.2220	-0.2779	0.1058
	1.1119	0.1119	0.0138
	4.1450	0.1450	0.0282
$n = 50$	0.2627	0.0127	0.0029
	0.3942	-0.1057	0.0157
	1.0372	0.0372	0.0028
	4.0200	0.0200	0.0024
$n = 100$	0.2511	0.0011	0.0001
	0.4911	-0.0088	0.0004

From these tables, it is clear that bias and MSE seems declining as increasing the sample size. Which means that the estimation method that we considered performs admirable.

3. Real Data Using PEW Distribution

Here, the flexibility of the Poisson-exponentiated Weibull distribution is illustrated by real data set. The MLEs of the parameters are obtained using the nlm function of R software and the well-relevant statistics are also calculated for establishing its goodness of fit. The data includes the failure times of Aircraft winshield having 85 observations given in Tahir et al. (2015).

We plot the histogram of the observed data and the embedded pdf plot of PEW disribution. It is seen that the PEW distribution fit well for the observations. We also compare the proposed distribution with Exponentiated Weibull (EW) (Nassar and Eissa (2003)), Exponentiated Weibull poisson (EWP) (Mahmoudi and Sepahdar (2013)), and Poisson-Weibull (PW) (Bereta et al. (2011)) distributions. The embedded density plots of compared ones along with the proposed is given in Figure 4.

The numerical values of the statistics of the examined models for the dataset are offered in Table 3. We note that p value for PEW distribution is higher, and the values of -logl, AIC, BIC, along with K-S are lower as compared to EW, EWP and PW distributions. Therefore, PEW distribution can be considered as a finer model for the given data set than the other considered distributions.

Table 3. The MLE, -logl, AIC, BIC, K-S and p-value for the fitted models to the Dataset

Distribution	Parameters	-logl	AIC	BIC	K-S	p-value
PEW	$\tau = 1.748$ $\zeta = 2.461$ $\theta = 3.6$ $m = 0.56$	2928.483	5864.966	5874.737	0.056734	0.949
EW	$\alpha = 4.102$ $\beta = 1.231$ $\theta = 1.8$	3110.325	6226.65	6233.978	0.52549	0.2996
EWP	$\alpha = 2.311$ $\beta = 0.366$ $\gamma = 1.60$ $\theta = 0.498$	3461.192	6930.384	6940.155	0.56176	0.1211
PW	$\alpha = 2.451$ $\beta = 0.512$ $\gamma = 1.692$	2981.483	5968.966	5976.294	0.49412	0.3685

4. Reliability Test Plan

A reliability test plan enables us to make the accept-reject decision of a lot of products under inspection. Suppose we want to select a lot of products. Let the lot contain N products. It is not possible to check each of the N items of the lot to know whether it has a defect or not. In such situations we can adopt the methodologies of acceptance sampling plan. In this testing procedure, a sample of size n, is considered for testing and is truncated at a fixed time t. Here the decision regarding the lot is drawn as follows: if the number of defects, d, at the end of time t, is not more than b, a pre-assigned number, the lot is termed as an acceptable one. On the other hand, if the $(b+1)^{th}$ defected item is found before the time t, then the test is terminated and the lot is moved to the rejected category.

Here, the lifetime of the product, the random variable T, have PEW distribution with cdf (1). ζ is the average life time of the product and ζ_0 is the specified mean life time. Then, $F(t, \tau, \theta, m, \zeta) \leq F(t, \tau, \theta, m, \zeta_0) \Leftrightarrow \zeta \geq \zeta_0$. Since the testing procedure is based on a sample, there may exist

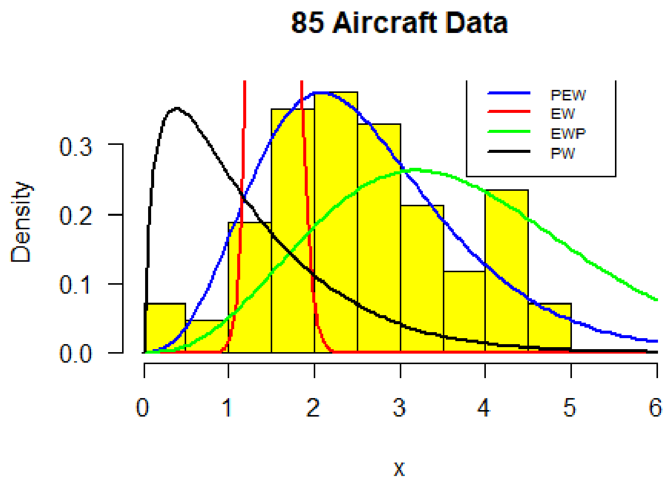

Figure 4. Fitted pdf plot for aircraft data

corresponding risks. The probability of rejecting a good lot is termed as producers risk where as consumers risk is the probability of accepting a bad lot. It is essential to keep the risk not to exceed $1-p^*$. Thus, the sampling plan based on PEW distribution is characterized by $(n, b, \frac{t}{\zeta_0})$. We assume N is large enough, so the Binomial distribution is used. We have to determine the minimum sample size n for which (24) holds:

$$\sum_{i=0}^{b} \binom{n}{i} p_0^i \left(1-p_0\right)^{n-i} \leq 1-p^*. \tag{24}$$

Table 4 gives the minimum values of n satisfying (24) for $\tau = 1, \theta = 1, m = 1, p^* = 0.75, 0.90, 0.95, 0.99$ and $\frac{t}{\zeta_0} = 0.628, 0.942, 1.257, 1.571, 2.356, 3.141, 3.927$ and 4.712. Suppose if p_0 is small and n is large. A Poisson approach with $\lambda = np_0$ can adopt instead of Binomial distribution. Table 5 gives the values of n holds (25) for the same values of τ, θ, m, $\frac{t}{\zeta_0}$ and p^*,

$$L_1\left(p_0\right) = \sum_{i=0}^{b} \frac{\lambda^i}{i!} e^{-\lambda} \leq 1-p^*. \tag{25}$$

If we consider a comparison, for these selected values of p^* and $\frac{t}{\zeta_0}$ the n values obtained are found to be less than or equal to the n values given by Rosaiah et al. (2006) for exponentiated log-logistic distribution.

When we moves to the operating characteristic function,

$$L(p) = \sum_{i=0}^{b} \binom{n}{i} p_0^i \left(1-p_0\right)^{n-i}, \tag{26}$$

which gives the probability of accepting the lot. Where $p = F\left(\frac{t}{\zeta_0} / \frac{\zeta_0}{\zeta}\right)$, the values of L(p) for the sampling plan $(n, b, \frac{t}{\zeta_0})$ is given in Table 6. The table values revels that as $\frac{\zeta}{\zeta_0}$ increases, OC values are also found to be increasing, which indicates that, for the given sampling plan, when the mean life time of the product is greater than the specified one, the consumers risk is become diminished.

For a specified value of producer's risk, say 0.05, it is interested to know that which values of $\frac{\zeta}{\zeta_0}$ will ensure a producer's risk less than or equal to 0.05. Table 7 presents that value of $\frac{\zeta}{\zeta_0}$ that holds, (27)

$$\sum_{i=0}^{b} \binom{n}{i} p_0^i \left(1 - p_0\right)^{n-i} \geq 0.95. \tag{27}$$

See the Appendix for the tables.

Application

Here, the above suggested test plan is verified through data of measurements includeing the number of revolutions in millions, taken before the failure of ball bearings, obtained from Rao (2013a).

Let the experimenter wish to establish the mean life time is 70 million revolutions, with $p^* = 0.90$ and test truncation at 44 million revolutions. This gives the ratio $\frac{t}{\zeta_0} = \frac{44}{70} = 0.628$, corresponding $n = 23$ and $b = 10$ are obtained from Table 4. Now the designed reliability test plan is $\left(n = 23, b = 10, \frac{t}{\zeta_0} = 0.628\right)$ is suitable only if the life time of the products follows PEW distribution. Using the MLEs, $\tau = 0.6469, \zeta = 8.314, \theta = 5.3022$ and $m = 7.51$, the obtained p-value = 0.9015 and fitted pdf plot for the data set given in Figure 5, confirm the adoption of the PEW distribution. From the observations it is clear that, since there are only 5 failures before the time 45, we accept the lot.

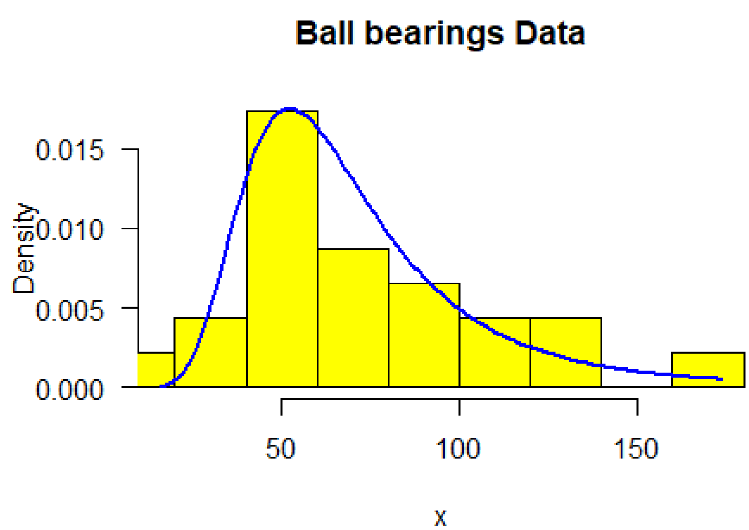

Figure 5. Fitted pdf plot for ball bearings data

5. Farlie–Gumbel–Morgenstern Bivariate Poisson-exponentiated Weibull (FGMBPEW) Distribution

Morgenstern (1956) proposed the Farlie–Gumbel–Morgenstern (FGM) bivariate distributions for modeling bivariate data. Let X and Y be two univariate PEW random variables, Then the joint pdf of Farlie–Gumbel– Morgenstern bivariate Poisson-exponentiated Weibull distribution is given as

$$\begin{aligned} f(x,y) = & \frac{m_1}{(1-e^{-1})}\frac{\tau_1\theta_1}{\zeta_1^{\tau_1}}x^{\tau_1-1}e^{-\left(\frac{x}{\zeta_1}\right)^{\tau_1}}\left[1-e^{-\left(\frac{x}{\zeta_1}\right)^{\tau_1}}\right]^{(m_1-1)\theta_1}e^{-\left[1-e^{-\left(\frac{x}{\zeta_1}\right)^{\tau_1}}\right]^{m_1\theta_1}} \\ & \frac{m_2}{(1-e^{-1})}\frac{\tau_2\theta_2}{\zeta_2^{\tau_2}}y^{\tau_2-1}e^{-\left(\frac{y}{\zeta_2}\right)^{\tau_2}}\left[1-e^{-\left(\frac{y}{\zeta_2}\right)^{\tau_2}}\right]^{(m_2-1)\theta_2}e^{-\left[1-e^{-\left(\frac{y}{\zeta_2}\right)^{\tau_2}}\right]^{m_2\theta_2}} \\ & \left[1+\epsilon\left(1-2(1-e^{-1})^{-1}\left[1-e^{-\left[1-e^{-\left(\frac{x}{\zeta_1}\right)^{\tau_1}}\right]^{m_1\theta_1}}\right]\right)\right. \\ & \left.\left(1-2(1-e^{-1})^{-1}\left[1-e^{-\left[1-e^{-\left(\frac{y}{\zeta_2}\right)^{\tau}_2}\right]^{m_2\theta_2}}\right]\right)\right] \\ & x>0, \tau_1, \theta_1, m_1, \zeta_1, \tau_2, \theta_2, m_2, \zeta_2 > 0, -1 \leqslant \epsilon \leqslant 1, \end{aligned} \tag{28}$$

and the corresponding cdf as

$$\begin{aligned} F(x,y) = & (1-e^{-1})^{-2}\left[1-e^{-\left[1-e^{-\left(\frac{x}{\zeta_1}\right)^{\tau_1}}\right]^{m_1\theta_1}}\right]\left[1-e^{-\left[1-e^{-\left(\frac{y}{\zeta_2}\right)^{\tau_2}}\right]^{m_2\theta_2}}\right] \\ & \left[1+\epsilon\left(1-(1-e^{-1})^{-1}\left[1-e^{-\left[1-e^{-\left(\frac{x}{\zeta_1}\right)^{\tau_1}}\right]^{m_1\theta_1}}\right]\right)\right. \\ & \left.\left(1-(1-e^{-1})^{-1}\left[1-e^{-\left[1-e^{-\left(\frac{y}{\zeta_2}\right)^{\tau_2}}\right]^{m_2\theta_2}}\right]\right)\right]. \end{aligned} \tag{29}$$

For the Farlie–Gumbel–Morgenstern bivariate Poisson exponentiated Weibull distribution, the conditional distribution of $Y|X=x$ can be obtained as

$$\begin{aligned} F_{Y|X}(y|x) = & (1-e^{-1})^{-1}\left[1-e^{-\left[1-e^{-\left(\frac{y}{\zeta_2}\right)^{\tau_2}}\right]^{m_2\theta_2}}\right] \\ & \left[1+\epsilon\left(1-2(1-e^{-1})^{-1}\left[1-e^{-\left[1-e^{-\left(\frac{x}{\beta_1}\right)^{\tau_1}}\right]^{m_1\theta_1}}\right]\right)\right. \\ & \left.\left(1-2(1-e^{-1})^{-1}\left[1-e^{-\left[1-e^{-\left(\frac{y}{\zeta_2}\right)^{\tau_2}}\right]^{m_2\theta_2}}\right]\right)\right], \quad y>0. \end{aligned} \tag{30}$$

ML Estimate of FGMPEW Distribution

Let (X_i, Y_i), i = 1, 2,...,n be FGMBPEW random variables. Then, the unknown parameters of FGMBPEW distribution can be extracted using log likelihood function,

$$\begin{aligned}\log L = & n\log\left[m_1\left(1-e^{-1}\right)^{-1}\right] + n\log\tau_1 + n\log\zeta_1 - n\tau_1\log\zeta_1 + (\tau_1-1)\sum_{i=1}^{n}\log x_i \\ & -\sum_{i=1}^{n}\left(\frac{x_i}{\zeta_1}\right)^{\tau_1} + (m_1\theta_1-1)\sum_{i=1}^{n}\log\left[1-e^{-\left(\frac{x_i}{\zeta_1}\right)^{\tau_1}}\right] - \sum_{i=1}^{n}\left[1-e^{-\left(\frac{x_i}{\zeta_1}\right)^{\tau_1}}\right]^{m_1\theta_1} \\ & n\log\left[m_2\left(1-e^{-1}\right)^{-1}\right] + n\log\tau_2 + n\log\zeta_2 - n\tau_2\log\zeta_2 + (\tau_2-1)\sum_{i=1}^{n}\log y_i \\ & -\sum_{i=1}^{n}\left(\frac{y_i}{\zeta_2}\right)^{\tau_2} + (m_2\theta_2-1)\sum_{i=1}^{n}\log\left[1-e^{-\left(\frac{y_i}{\zeta_2}\right)^{\tau_2}}\right] - \sum_{i=1}^{n}\left[1-e^{-\left(\frac{y_i}{\zeta_2}\right)^{\tau_2}}\right]^{m_2\theta_2} \\ & +\sum_{i=1}^{n}A_i \end{aligned} \tag{31}$$

where

$$\begin{aligned} A_i = & \left[1+\epsilon\left(1-2(1-e^{-1})^{-1}\left[1-e^{-\left[1-e^{-\left(\frac{x}{\zeta_1}\right)^{\tau_1}}\right]^{m_1\theta_1}}\right]\right)\right. \\ & \left.\left(1-2(1-e^{-1})^{-1}\left[1-e^{-\left[1-e^{-\left(\frac{y}{\zeta_2}\right)^{\tau_2}}\right]^{m_2\theta_2}}\right]\right)\right]. \end{aligned} \tag{32}$$

Distribution of Concomitants

Let $(X_i, Y_i), i = 1, 2..., n$ be the random variables from an absolutely continuous bivariate distribution. The concomitant of r^{th} order statistics $X_{r:n}$ is the Y-variate affiliated with $X_{r:n}$ and is marked by $Y_{[r:n]}$. Concomitants have their most important application in selection strategy, that is, when $k(< n)$ individuals are selected based on their X-values. Then, the corresponding y-values denotes the result of an associated feature. The imperative properties regarding concomitant of order statistics (COS) are given by Tahir et al. (2016). For details see Veena and Thomas (2008), Philip and Thomas (2015) and Arun et al. (2023). Lately, inference based on COS from FGM family can be seen on Arun et al. (2023). For a FGMBPEW distribution, the cdf of $Y_{[r:n]}$, for $1 \leqslant r \leqslant n$ is noted as,

$$\begin{aligned} F_{Y[r:n]}(y) = & (1-e^{-1})^{-1}\left[1-e^{-\left[1-e^{-\left(\frac{y}{\zeta_2}\right)^{\tau_2}}\right]^{m_2\theta_2}}\right]\left[1+\frac{n-2r+1}{n+1}\epsilon\right. \\ & \left.\left(1-\frac{1}{(1-e^{-1})}\left[1-e^{-\left[1-e^{-\left(\frac{y}{\zeta_2}\right)^{\tau_2}}\right]^{m_2\theta_2}}\right]\right)\right], \end{aligned} \tag{33}$$

the pdf corresponding to (33) is

$$f_{Y[r:n]}(y) = \frac{m_2}{(1-e^{-1})}\frac{\tau_2\theta_2}{\zeta_2^{\tau_2}}y^{\tau_2-1}e^{-\left(\frac{y}{\zeta_2}\right)^{\tau_2}}\left[1-e^{-\left(\frac{y}{\zeta_2}\right)^{\tau_2}}\right]^{(m_2-1)\theta_2} e^{-\left[1-e^{-\left(\frac{y}{\zeta_2}\right)^{\tau_2}}\right]^{m_2\theta_2}} \left[1+\frac{n-2r+1}{n+1}\epsilon\left(1-2(1-e^{-1})^{-1}\left[1-e^{-\left[1-e^{-\left(\frac{y}{\zeta_2}\right)^{\tau_2}}\right]^{m_2\theta_2}}\right]\right)\right]. \quad (34)$$

In particular, we have for $r = 1$,

$$f_{Y[1:n]}(y) = \frac{m_2}{(1-e^{-1})}\frac{\tau_2\theta_2}{\zeta_2^{\tau_2}}y^{\tau_2-1}e^{-\left(\frac{y}{\zeta_2}\right)^{\tau_2}}\left[1-e^{-\left(\frac{y}{\zeta_2}\right)^{\tau_2}}\right]^{(m_2-1)\theta_2} e^{-\left[1-e^{-\left(\frac{y}{\zeta_2}\right)^{\tau_2}}\right]^{m_2\theta_2}} \left[1+\frac{n-1}{n+1}\epsilon\left(1-2(1-e^{-1})^{-1}\left[1-e^{-\left[1-e^{-\left(\frac{y}{\zeta_2}\right)^{\tau_2}}\right]^{m_2\theta_2}}\right]\right)\right], \quad (35)$$

where for $r = n$,

$$f_{Y[n:n]}(y) = \frac{m_2}{(1-e^{-1})}\frac{\tau_2\theta_2}{\zeta_2^{\tau_2}}y^{\tau_2-1}e^{-\left(\frac{y}{\zeta_2}\right)^{\tau_2}}\left[1-e^{-\left(\frac{y}{\zeta_2}\right)^{\tau_2}}\right]^{(m_2-1)\theta_2} e^{-\left[1-e^{-\left(\frac{y}{\zeta_2}\right)^{\tau_2}}\right]^{m_2\theta_2}} \left[1-\frac{n-1}{n+1}\epsilon\left(1-2(1-e^{-1})^{-1}\left[1-e^{-\left[1-e^{-\left(\frac{y}{\zeta_2}\right)^{\tau_2}}\right]^{m_2\theta_2}}\right]\right)\right]. \quad (36)$$

Again, the joint pdf of $f_{r,s:n}(y_1, y_2)$ of $Y_{[r:n]}, Y_{[s:n]}$ for $1 \leqslant r \leqslant s \leqslant n$ is as follows,

$$
\begin{aligned}
f_{[r,s:n]}(y_1,y_2) = {} & \frac{m_1}{(1-e^{-1})}\frac{\tau_1\theta_1}{\zeta_1^{\tau_1}}y_1^{\tau_1-1}e^{-\left(\frac{y_1}{\zeta_1}\right)^{\tau_1}}\left[1-e^{-\left(\frac{y_1}{\zeta_1}\right)^{\tau_1}}\right]^{(m_1\theta_1-1)}e^{-\left[1-e^{-\left(\frac{y_1}{\zeta_1}\right)^{\tau_1}}\right]^{m_1\theta_1}} \\
& \frac{m_2}{(1-e^{-1})}\frac{\tau_2\theta_2}{\zeta_2^{\tau_2}}y_2^{\tau_2-1}e^{-\left(\frac{y_1}{\zeta_2}\right)^{\tau_2}}\left[1-e^{-\left(\frac{y_2}{\zeta_2}\right)^{\tau_2}}\right]^{(m_2\theta_2-1)}e^{-\left[1-e^{-\left(\frac{y_2}{\zeta_2}\right)^{\tau_2}}\right]^{m_2\theta_2}} \\
& \left[1+\epsilon\left[1-2(1-e^{-1})^{-1}\left[1-e^{-\left[1-e^{-\left(\frac{y_1}{\zeta_1}\right)^{\tau_1}}\right]^{m_1\theta_1}}\right]\right]\frac{n-2r+1}{n+1}\right. \\
& +\epsilon\left[1-2(1-e^{-1})^{-1}\left[1-e^{-\left[1-e^{-\left(\frac{y_2}{\zeta_2}\right)^{\tau_2}}\right]^{m_2\theta_2}}\right]\right]\frac{n-2s+1}{n+1} \\
& +\epsilon^2\left[1-2(1-e^{-1})^{-1}\left[1-e^{-\left[1-e^{-\left(\frac{y_1}{\zeta_1}\right)^{\tau_1}}\right]^{m_1\theta_1}}\right]\right] \\
& \left[1-2(1-e^{-1})^{-1}\left[1-e^{-\left[1-e^{-\left(\frac{y_2}{\zeta_2}\right)^{\tau_2}}\right]^{m_2\theta_2}}\right]\right] \\
& \left.\left\{\frac{n-2s+1}{n+1}-\frac{2r(n-2s)}{(n+1)(n+2)}\right\}\right] \quad y_1,y_2>0. \qquad (37)
\end{aligned}
$$

6. Conclusion

A new member of Poisson-X family, called Poisson-exponentiated Weibull distribution, is introduced and studied. The performance of the ML estimators are established by simulation study. A real data application of the distribution is also considered using aircraft data. When compared with exponentiated Weibull, exponentiated Weibull Poisson and Poisson-Weibull distributions, it is seen that Poisson-exponentiated Weibull is an appropriate model for the given data, more than the other two. Again, a reliability test plan proceeds, as life time follows the proposed distribution. When compared with exponentiated Log-Logistic model, it is seen that in acceptance sampling under similar conditions, the Poisson-exponentiated Weibull model draws smaller sample sizes than the later one. A real data analysis is also conducted and it found to give a better decision regarding the acceptability of the product. Also, by applying the Farlie–Gumbel–Morgenstern methodology, a bivariate variant of the Poisson-exponentiated Weibull distribution along with its COS is developed.

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Appendix

Table 4. Minimum sample sizes using Binomial probabilities

		t/ζ_0							
p^*	b	0.628	0.942	1.257	1.571	2.356	3.141	3.927	4.712
0.75	0	2	2	1	1	1	1	1	1
	1	4	3	3	3	2	2	2	2
	2	6	5	4	4	3	3	3	3
	3	8	6	6	5	4	4	4	4
	4	10	8	7	6	6	5	5	5
	5	12	9	8	7	7	6	6	6
	6	14	11	9	9	8	7	7	7
	7	15	12	11	10	9	8	8	8
	8	17	14	12	11	10	9	9	9
	9	19	15	13	12	11	10	10	10
	10	21	17	15	14	12	11	11	11
0.90	0	3	2	2	2	1	1	1	1
	1	5	4	3	3	3	2	2	2
	2	8	6	5	4	4	3	3	3
	3	10	7	6	6	5	4	4	4
	4	12	9	8	7	6	6	5	5
	5	14	11	9	8	7	7	6	6
	6	16	12	11	10	8	8	7	7
	7	18	14	12	11	9	9	8	8
	8	20	15	13	12	11	10	9	9
	9	22	17	15	13	12	11	11	10
	10	23	18	16	15	13	12	12	11
0.95	0	4	3	2	2	2	1	1	1
	1	7	5	4	4	3	3	2	2
	2	9	7	6	5	4	4	3	3
	3	11	9	7	6	5	5	5	4
	4	17	11	8	7	6	5	5	5
	5	15	12	10	9	8	7	7	6
	6	18	14	12	10	9	8	8	7
	7	20	15	13	12	10	9	9	8
	8	22	17	14	13	11	10	10	10
	9	24	18	16	14	12	11	11	11
	10	26	20	17	16	13	12	12	12
0.99	0	6	4	3	3	2	2	2	1
	1	9	6	5	4	3	3	3	3
	2	11	8	7	6	5	4	4	4
	3	18	11	8	7	5	4	4	4
	4	16	12	10	9	7	6	6	6
	5	18	14	11	10	8	8	7	7
	6	20	15	13	12	10	9	8	8
	7	22	17	14	13	11	10	9	9
	8	25	19	16	14	12	11	10	10
	9	27	20	17	15	13	12	11	11
	10	29	22	19	17	14	13	12	12

Table 5. Minimum sample sizes using Poisson probabilities

		t/ζ_0							
p^*	b	0.628	0.942	1.257	1.571	2.356	3.141	3.927	4.712
0.75	0	3	2	2	2	2	2	2	2
	1	5	4	4	4	3	3	3	3
	2	7	6	5	5	5	5	5	5
	3	9	8	7	6	6	6	6	6
	4	11	9	8	8	7	7	7	7
	5	13	11	10	9	8	8	8	8
	6	15	12	11	10	10	9	9	9
	7	17	14	13	12	11	11	10	10
	8	19	16	14	13	12	12	12	11
	9	21	17	15	14	13	13	13	13
	10	23	19	17	16	14	14	14	14
0.90	0	4	4	3	3	3	3	3	3
	1	7	6	5	5	5	4	4	4
	2	10	8	7	7	6	6	6	6
	3	12	10	9	8	8	7	7	7
	4	14	12	10	10	9	9	9	9
	5	16	13	12	11	10	10	10	10
	6	18	15	14	13	12	11	11	11
	7	20	17	15	14	13	13	12	12
	8	23	18	17	16	14	14	14	14
	9	25	20	18	17	16	15	15	15
	10	27	22	20	18	17	16	16	16
0.95	0	6	5	4	4	4	4	4	4
	1	9	7	7	6	6	6	6	5
	2	12	10	9	8	7	7	7	7
	3	14	12	11	10	9	9	9	9
	4	17	14	12	11	10	10	10	10
	5	19	16	14	13	12	12	11	11
	6	21	17	16	15	13	13	13	13
	7	23	19	17	16	15	14	14	14
	8	26	21	19	18	16	16	15	15
	9	28	23	20	19	18	17	17	17
	10	30	25	22	21	19	18	18	18
0.99	0	8	7	6	6	5	5	5	5
	1	12	10	9	8	8	7	7	7
	2	15	12	11	10	9	9	9	9
	3	18	14	13	12	11	11	11	11
	4	20	17	15	14	13	12	12	12
	5	23	19	17	16	14	14	14	14
	6	25	21	19	17	16	15	15	15
	7	28	23	20	19	17	17	17	17
	8	30	25	22	21	19	18	18	18
	9	32	26	24	22	20	20	20	19
	10	35	28	25	24	22	21	21	21

Table 6. *OC* values for the plan $(n, b, t/\zeta_0)$

$p^\star$	n	b	$\frac{t}{\zeta_0}$	ζ/ζ_0					
				2	4	6	8	10	12
0.75	6	2	0.628	0.59872	0.8833	0.9529	0.9767	0.9869	0.9919
	5	2	0.942	0.5087	0.8376	0.9308	0.9647	0.9796	0.9872
	4	2	1.257	0.5422	0.8495	0.9356	0.9671	0.9809	0.9881
	4	2	1.571	0.4124	0.7734	0.8957	0.9444	0.9671	0.9789
	3	2	2.356	0.5066	0.8164	0.9156	0.9548	0.9731	0.9828
	3	2	3.141	0.3517	0.7946	0.8515	0.9156	0.9478	0.9655
	3	2	3.927	0.2407	0.6021	0.7803	0.8684	0.9156	0.9428
	3	2	4.712	0.1637	0.5066	0.7075	0.8164	0.8783	0.9156
0.90	8	2	0.628	0.3726	0.7656	0.8954	0.9454	0.9681	0.9798
	6	2	0.942	0.3524	0.7453	0.8832	0.9381	0.9634	0.9767
	5	2	1.257	0.3351	0.7266	0.8714	0.9307	0.9587	0.9735
	4	2	1.571	0.4124	0.7734	0.8957	0.9444	0.9671	0.9789
	3	2	2.356	0.1961	0.5787	0.7734	0.8675	0.9166	0.9444
	3	2	3.141	0.3417	0.7945	0.8515	0.9156	0.9477	0.9655
	3	2	3.927	0.2407	0.6021	0.7803	0.8684	0.9156	0.9428
	3	2	4.712	0.1637	0.5066	0.7075	0.8183	0.8783	0.9156
0.95	9	2	0.628	0.2849	0.7008	0.85989	0.9249	0.9554	0.9715
	7	2	0.942	0.23422	0.6484	0.8272	0.9047	0.9424	0.9627
	6	2	1.257	0.1934	0.5983	0.7935	0.8831	0.9281	0.9529
	5	2	1.571	0.2141	0.6137	0.8016	0.8875	0.9307	0.9545
	4	2	2.356	0.1961	0.5787	0.7734	0.8675	0.9166	0.9444
	4	2	3.141	0.0901	0.5419	0.6419	0.7735	0.8497	0.8958
	3	2	3.927	0.2407	0.60213	0.7803	0.8684	0.9156	0.9428
	3	2	4.712	0.1637	0.5066	0.7075	0.8164	0.8783	0.9156
0.99	11	2	0.628	0.1579	0.5712	0.7798	0.8758	0.9239	0.95029
	8	2	0.942	0.1507	0.5532	0.7654	0.8658	0.9169	0.9454
	7	2	1.257	0.1064	0.4778	0.7085	0.8269	0.8903	0.9265
	6	2	1.571	0.1029	0.4644	0.6957	0.8172	0.8831	0.9212
	5	2	2.356	0.0657	0.3735	0.6138	0.7551	0.8375	0.8875
	4	2	3.141	0.0900	0.5419	0.6419	0.7735	0.8497	0.8958
	4	2	3.927	0.04093	0.2864	0.5189	0.6744	0.7734	0.8374
	4	2	4.712	0.0186	0.1961	0.4125	0.5787	0.6942	0.7734

Table 7. Minimum ratio of true and specified mean life for the acceptability of a lot with $\alpha = 0.05$

		t/ζ_0							
p^*	b	0.628	0.942	1.257	1.571	2.356	3.141	3.927	4.712
0.75	0	0.2523	0.3784	0.5113	0.6390	0.9583	1.2777	1.5974	1.9167
	1	0.2541	0.3787	0.5053	0.6324	0.9484	1.2645	1.5809	1.8969
	2	0.2546	0.3775	0.5044	0.6304	0.9486	1.2647	1.5812	1.8973
	3	0.2551	0.3842	0.5127	0.6356	0.9505	1.2672	1.5843	1.9010
	4	0.2525	0.3779	0.5127	0.6341	0.9509	1.2616	1.5777	1.8926
	5	0.2528	0.3795	0.5077	0.6313	0.9467	1.2731	1.5917	1.9098
	6	0.2542	0.3778	0.5076	0.6345	0.9468	1.2593	1.5744	1.8892
	7	0.2516	0.3775	0.5046	0.6323	0.9490	1.2604	1.5758	1.8908
	8	0.2538	0.3788	0.5102	0.6300	0.9541	1.2777	1.5974	1.9167
	9	0.2554	0.3809	0.5055	0.6332	0.9541	1.2641	1.5805	1.8964
	10	0.2534	0.3807	0.5059	0.6135	0.9474	1.2631	1.5857	1.9027
0.90	0	0.2524	0.3786	0.5053	0.6315	0.9446	1.2594	1.5745	1.8893
	1	0.2524	0.3785	0.5040	0.6299	0.9447	1.2645	1.5809	1.8969
	2	0.2535	0.3795	0.5056	0.6332	0.9496	1.2647	1.5812	1.8974
	3	0.2526	0.3789	0.5040	0.6299	0.9451	1.2881	1.6104	1.9323
	4	0.2575	0.3788	0.5057	0.6346	0.9509	1.2678	1.5773	1.8926
	5	0.2556	0.3841	0.5038	0.6284	0.9467	1.2622	1.5856	1.9026
	6	0.2543	0.3792	0.5042	0.6349	0.9426	1.2567	1.5615	1.8736
	7	0.2495	0.3773	0.5011	0.6273	0.9401	1.2533	1.5758	1.8908
	8	0.2502	0.3785	0.5063	0.6271	0.9443	1.2532	1.5743	1.8891
	9	0.2498	0.3801	0.5051	0.6312	0.9496	1.2608	1.5763	1.8914
	10	0.2492	0.3770	0.4982	0.6323	0.9444	1.2493	1.5620	1.8764
0.95	0	0.2535	0.3778	0.5042	0.6302	0.9451	1.2604	1.5758	1.8908
	1	0.2520	0.3758	0.5036	0.6293	0.9436	1.2581	1.5712	1.8853
	2	0.2522	0.3746	0.5031	0.6285	0.9415	1.2552	1.5699	1.8837
	3	0.2514	0.3781	0.5027	0.6253	0.9378	1.2503	1.5632	1.8734
	4	0.2502	0.3788	0.5025	0.6320	0.9383	1.2678	1.5851	1.8927
	5	0.2497	0.3793	0.5020	0.6296	0.9425	1.2622	1.5781	1.8953
	6	0.2517	0.3786	0.5029	0.6282	0.9422	1.2623	1.5782	1.8736
	7	0.2514	0.3774	0.5011	0.6263	0.9435	1.2534	1.5670	1.8993
	8	0.2523	0.3751	0.5023	0.6317	0.9444	1.2533	1.5669	1.8801
	9	0.2506	0.3785	0.5002	0.6294	0.9496	1.2543	1.5681	1.8816
	10	0.2506	0.3774	0.5046	0.6247	0.9445	1.2494	1.5620	1.8743
0.99	0	0.31410	0.47158	0.62932	0.78653	1.17517	1.57161	1.96488	2.35766
	1	0.2506	0.3781	0.5062	0.6335	0.9440	1.2586	1.5735	1.8881
	2	0.2522	0.3770	0.5046	0.6329	0.9477	1.2660	1.5828	1.8992
	3	0.2525	0.3775	0.5042	0.6283	0.9470	1.2657	1.5824	1.8987
	4	0.2513	0.3758	0.5021	0.6261	0.9383	1.2678	1.5851	1.9019
	5	0.2533	0.3772	0.5022	0.6311	0.9381	1.5507	1.5781	1.8935
	6	0.2534	0.3753	0.5034	0.6285	0.9471	1.2501	1.5781	1.8936
	7	0.2502	0.3771	0.5035	0.6302	0.9407	1.2642	1.5670	1.8802
	8	0.2506	0.3776	0.5059	0.6277	0.9509	1.2590	1.5746	1.8894
	9	0.2499	0.3776	0.5072	0.6272	0.9420	1.2591	1.5681	1.8816
	10	0.2534	0.3766	0.5069	0.6264	0.9476	1.2592	1.5620	1.8742