Applications and Applied Mathematics: An International Journal (AAM)

Volume 19 | Issue 1 Article 12

6-2024

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Sagar Sankeshwari
NMIMS Deemed to be University

Vinayak Kulkarni
University of Mumbai

Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466

Applications and Applied Mathematics:
An International Journal
(AAM)

Vol. 19, Issue 1 (June 2024), 20 pages

Solutions of Hyperbolic System of Time Fractional Partial Differential Equations for Heat Propagation

[1]Sagar Sankeshwari and [2,*]Vinayak Kulkarni

[1]School of Mathematics
Applied Statistics and Analytics
NMIMS Deemed to be University
Navi Mumbai - 410 210, Maharashtra, India
sagarsankeshwari1@gmail.com

[2]Department of Mathematics
University of Mumbai
Mumbai - 400 098, Maharashtra, India
drvinayaksk1@gmail.com

*Corresponding Author

Abstract

Hyperbolic linear theory of heat propagation has been established in the framework of a Caputo time fractional order derivative. The solution of a system of integer and fractional order initial value problems is achieved by employing the Adomian decomposition approach. The obtained solution is in convergent infinite series form, demonstrating the method's strengths in solving fractional differential equations. Moreover, the double Laplace transform method is employed to acquire the solution of a system of integer and fractional order boundary conditions in the Laplace domain. An inversion of double Laplace transforms has been achieved numerically by employing the Xiao algorithm in the space-time domain. Considering the non-Fourier effect of heat conduction, the finite speed of thermal wave propagation has been attained. The role of the fractional order parameter has been examined scientifically. The results obtained by considering the fractional order theory and the integer order theory perfectly coincide as a limiting case of fractional order parameter approaches one.

Keywords: System of partial differential equation; Adomian decomposition method; Fractional partial differential equations; Double Laplace transform method; Hyperbolic system

MSC 2010 No.: 26A33, 35L35, 44A10, 65R10, 74F05

1. Introduction

Adomian (1996) demonstrated how to get a solution for the system of coupled non-linear differential equations. Nonlinear PDEs serve an important role in describing and analyzing real-life processes and occurrences as shown by Nisar et al. (2021). Bougoffa and Bougouffa (2006) employed the Adomian decomposition method (ADM) to get a solution of coupled systems first-second-order differential equations. Gu and Li (2007) employed the modified ADM to get a solution for a system of non-linear differential equations and demonstrated that the method of calculating speed is faster than the original ADM. Nagy et al. (1994) investigated how hyperbolic heat equation solutions behaved at their parabolic limits.

Following Kai (2010), many scientists and engineers are interested in fractional differential equations (FDEs) because they have been used in different fields such as control engineering, bioengineering, mechanics, signal processing, viscoelasticity, and polymer networks. Duan et al. (2007) studied a system of differential equations based on Caputo and Riemann Liouville's fractional derivatives definitions. In a rectangular domain, Mamchuev (2008) demonstrated a closed-form solution for a system of fractional PDEs subject to boundary conditions. Parthiban and Balachandran (2013) employed ADM to get a solution of the system of FPDEs. Dhunde and Waghmare (2022) obtained a solution for a linear system of integer and FPDEs under initial and boundary conditions.

Nisar et al. (2023) demonstrated the existence and uniqueness of a solution to the functional integrodifferential equation with nonlocal conditions and a finite delay function. It includes the continuous dependency of the integral solution as well as the existence of the solution, which were demonstrated utilizing integrated resolvent operator theory with Lipschitz continuous support. Ravichandran et al. (2022) derived the existence of a solution for the neutral partial integrodifferential nonlocal system. Zada et al. (2021) implemented a new iterative method for the solutions of inhomogeneous FPDEs. The heat transfer properties and uses of various nanofluids have been explored by Kumar et al. (2020). In a finite Cartesian system, Kulkarni and Mittal (2021) introduced and investigated two temperature dual phase lag concepts in the framework of fractional order by considering thermal stress analysis.

In this paper, a linear hyperbolic system of equations for heat propagation has been established in the framework of Caputo (1967) fractional derivative of order $\alpha \in (0, 1]$. The solutions of a system of integer and FPDEs with respect to initial conditions are obtained by using the Adomian (1996) decomposition approach. The obtained solutions are in convergent infinite series form. Also, the method of DLT developed by Sneddon (1972) is implemented to acquire a solution for the system of integer and FPDEs with respect to initial and boundary conditions in the Laplace domain. An inversion of the double Laplace transforms has been achieved numerically by employing Xiao and Zhang (2011) algorithm in the space-time domain. Graphical representations are used to show numerical results.

The non-Fourier effect of heat conduction results in attaining the finite speed of thermal wave propagation. The fractional order parameter α is not only a mathematical parameter but it also plays an important role in the rate of heat transfer and hence categorizes the material as per its ability of heat conduction. As a limiting case $\alpha \to 1$, the results obtained by considering the fractional order theory and the integer order theory perfectly coincide.

The whole manuscript is classified into two sections viz system of integer order PDE and system of fractional order PDE. Both sections are again classified into two subsections viz the initial value problem and the boundary value problem. The convergence theorem of the infinite series solution is demonstrated mathematically. The manuscript is concluded by the illustrative example and the results obtained are shown graphically.

2. Basic Equations

Following Biot (1956), the conservation of heat energy is represented as

$$-\nabla \cdot \mathbf{q}(\mathbf{x}, t) = c\frac{dT}{dt}, \tag{1}$$

where T denotes temperature, $c > 0$ represents specific heat, **x** stands for a material point, **q** represents heat flux, and t is time.

By replacing the time derivative in Equation (1) by Caputo (1967) fractional derivative of order α, then the generalized heat energy equation is represented as

$$-\nabla \cdot \mathbf{q} = c\frac{\partial^\alpha T}{\partial t^\alpha}, \;\; 0 < \alpha \le 1. \tag{2}$$

Following Cattaneo (1948), short-tail memory with an exponential kernel is given by

$$\mathbf{q}(t) = -\frac{\kappa}{\tau_0}\int_0^t e^{-(\frac{t-\xi}{\tau_0})}\, \nabla T(\xi)\, d\xi, \tag{3}$$

where τ_0 is a non-negative constant and denotes the delay time translation in the heat flux.

The Cattaneo (1958) and Vernotte (1961) form is represented by

$$\mathbf{q} + \tau_0\frac{\partial \mathbf{q}}{\partial t} = -\kappa \text{ grad } T. \tag{4}$$

The generalization of Equation (4) in the context of the Caputo time fractional order derivative of order α is given by

$$\mathbf{q} + \tau_0\frac{\partial^\alpha \mathbf{q}}{\partial t^\alpha} = -\kappa \text{ grad } T, \;\; 0 < \alpha \le 1. \tag{5}$$

In one dimension, Equations (2) and (5) are in the form of

$$c\frac{\partial^\alpha T}{\partial t^\alpha} = -\frac{\partial q}{\partial x}, \;\; 0 < \alpha \le 1, \tag{6}$$

$$\tau_0\frac{\partial^\alpha q}{\partial t^\alpha} = -\kappa\, \frac{\partial T}{\partial x} - q, \;\; 0 < \alpha \le 1. \tag{7}$$

Dividing Equation (6) by c and Equation (7) by -κ, and considering $U = T$ and $V = -q/c$, the above equations become

$$\frac{\partial^\alpha U}{\partial t^\alpha} = \frac{\partial V}{\partial x},\ \ t > 0,\ 0 \leq x < \infty,\ 0 < \alpha \leq 1, \tag{8}$$

$$\epsilon^2 \frac{\partial^\alpha V}{\partial t^\alpha} = \frac{\partial U}{\partial x} - \frac{V}{a},\ \ t > 0,\ 0 \leq x < \infty,\ 0 < \alpha \leq 1, \tag{9}$$

where $\epsilon = \frac{1}{c^*} = \sqrt{\frac{\tau_0}{a}} > 0$ represents the reciprocal of the characteristic speed of the system, and $a = \frac{\kappa}{c}$ represents the thermal diffusivity.

Equations $(8) - (9)$ represent the system of fractional PDEs which is hyperbolic when $\alpha \to 1$.

3. System of Integer Order PDEs

3.1. Initial Value Problem

Consider the symmetric hyperbolic system developed by Nagy et al. (1994),

$$U_t = V_x, \tag{10}$$

$$\epsilon^2 V_t = U_x - \frac{V}{a}. \tag{11}$$

Assume initial conditions are

$$U(x,0) = f_1(x),\ V(x,0) = f_2(x),\ \ 0 \leq x < \infty, \tag{12}$$

where $f_1(x)$ and $f_2(x)$ are continuous functions.

The main objective is to analyze the initial value problem defined in the Equations $(10) - (12)$.

3.1.1. *General Solution*

Let U and V be the solutions of Equations (10) through (12) of the form

$$U = \sum_{n=0}^{\infty} U_n, \tag{13}$$

$$V = \sum_{n=0}^{\infty} V_n. \tag{14}$$

Using Equations (10) and (11), then one can express U and V as follows,

$$U = U(x,0) + \int_0^t \frac{\partial V(x,\xi)}{\partial x} d\xi = f_1 + \mathscr{L}^{-1}\big(\mathscr{R}V\big), \tag{15}$$

$$\begin{aligned} V &= V(x,0) + \frac{1}{\epsilon^2}\left[\int_0^t \frac{\partial U(x,\xi)}{\partial x} d\xi - \frac{1}{a}\int_0^t V(x,\xi)d\xi\right] \\ &= f_2 + \frac{1}{\epsilon^2}\left[\mathscr{L}^{-1}\big(\mathscr{R}U\big) - \frac{1}{a}\mathscr{L}^{-1}(V)\right], \end{aligned} \tag{16}$$

where $\mathscr{R}$ represents a linear operator involving partial derivative at spatial coordinate x and $\mathscr{L} \equiv \frac{d^k}{dt^k}$ implies that k fold integration as $\mathscr{L}^{-1}(\cdot) = \int_0^t \cdots \int_0^t (\cdot)\, d\xi$.

Take initial approximations as

$$U_0 = U(x,0) = f_1, \quad V_0 = V(x,0) = f_2. \tag{17}$$

By using the Adomian (1996) decomposition, one can write

$$U_{n+1}(x,t) = \int_0^t \frac{\partial V_n}{\partial x} d\xi, \;\; n \geq 0, \tag{18}$$

$$V_{n+1}(x,t) = \frac{1}{\epsilon^2}\left[\int_0^t \frac{\partial U_n}{\partial x} d\xi - \frac{1}{a}\int_0^t V_n d\xi\right], \;\; n \geq 0. \tag{19}$$

From Equation (18), one obtains

$$\begin{aligned} U_1 &= \dot{f}_2 t, \\ U_2 &= \left[\frac{\ddot{f}_1}{\epsilon^2} - \frac{\dot{f}_2}{a\epsilon^2}\right]\frac{t^2}{2!}, \\ U_3 &= \left[\frac{\dddot{f}_2}{\epsilon^2} - \frac{\ddot{f}_1}{a\epsilon^4} + \frac{\dot{f}_2}{a^2\epsilon^4}\right]\frac{t^3}{3!}, \\ U_4 &= \left[\frac{\ddddot{f}_1}{\epsilon^4} - \frac{2\dddot{f}_2}{a\epsilon^4} + \frac{\ddot{f}_1}{a^2\epsilon^6} - \frac{\dot{f}_2}{a^3\epsilon^6}\right]\frac{t^4}{4!}, \end{aligned} \tag{20}$$

and so on.

From Equation (19), one obtains

$$\begin{aligned} V_1 &= \left[\frac{\dot{f}_1}{\epsilon^2} - \frac{f_2}{a\epsilon^2}\right]t, \\ V_2 &= \left[\frac{\ddot{f}_2}{\epsilon^2} - \frac{\dot{f}_1}{a\epsilon^4} + \frac{f_2}{a^2\epsilon^4}\right]\frac{t^2}{2!}, \\ V_3 &= \left[\frac{\dddot{f}_1}{\epsilon^4} - \frac{2\ddot{f}_2}{a\epsilon^4} + \frac{\dot{f}_1}{a^2\epsilon^6} - \frac{f_2}{a^3\epsilon^6}\right]\frac{t^3}{3!}, \\ V_4 &= \left[\frac{\ddddot{f}_2}{\epsilon^4} - \frac{2\dddot{f}_1}{a\epsilon^6} + \frac{3\ddot{f}_2}{a^2\epsilon^6} - \frac{\dot{f}_1}{a^3\epsilon^8} + \frac{f_2}{a^4\epsilon^8}\right]\frac{t^4}{4!}, \end{aligned} \tag{21}$$

and so on.

Then, the solution is represented by

$$\begin{aligned} U(x,t) =& f_1 + \dot{f}_2 t + \left[\frac{\ddot{f}_1}{\epsilon^2} - \frac{\dot{f}_2}{a\epsilon^2}\right]\frac{t^2}{2!} + \left[\frac{\dddot{f}_2}{\epsilon^2} - \frac{\ddot{f}_1}{a\epsilon^4} + \frac{\dot{f}_2}{a^2\epsilon^4}\right]\frac{t^3}{3!} \\ &+ \left[\frac{\ddddot{f}_1}{\epsilon^4} - \frac{2\dddot{f}_2}{a\epsilon^4} + \frac{\ddot{f}_1}{a^2\epsilon^6} - \frac{\dot{f}_2}{a^3\epsilon^6}\right]\frac{t^4}{4!} + \cdots, \end{aligned} \tag{22}$$

and

$$\begin{aligned} V(x,t) =& f_2 + \left[\frac{\dot{f}_1}{\epsilon^2} - \frac{f_2}{a\epsilon^2}\right]t + \left[\frac{\ddot{f}_2}{\epsilon^2} - \frac{\dot{f}_1}{a\epsilon^4} + \frac{f_2}{a^2\epsilon^4}\right]\frac{t^2}{2!} + \left[\frac{\dddot{f}_1}{\epsilon^4} - \frac{2\ddot{f}_2}{a\epsilon^4} + \frac{\dot{f}_1}{a^2\epsilon^6} - \frac{f_2}{a^3\epsilon^6}\right]\frac{t^3}{3!} \\ &+ \left[\frac{\ddddot{f}_2}{\epsilon^4} - \frac{2\dddot{f}_1}{a\epsilon^6} + \frac{3\ddot{f}_2}{a^2\epsilon^6} - \frac{\dot{f}_1}{a^3\epsilon^8} + \frac{f_2}{a^4\epsilon^8}\right]\frac{t^4}{4!} + \cdots, \end{aligned} \tag{23}$$

where the dot represents the derivative at spatial coordinate x.

3.2. Boundary Value Problem

Consider the symmetric hyperbolic system developed by Nagy et al. (1994),

$$U_t = V_x, \tag{24}$$

$$\epsilon^2 V_t = U_x - \frac{V}{a}. \tag{25}$$

Assume initial conditions as

$$U(x,0) = f_1(x),\ V(x,0) = f_2(x),\ \ 0 \le x < \infty, \tag{26}$$

and boundary conditions as

$$U(0,t) = h(t),\ V(0,t) = k(t),\ \lim_{x\to\infty} U(x,t) = \lim_{x\to\infty} V(x,t) = 0,\ \ t > 0, \tag{27}$$

where $f_1(x), f_2(x), h(t)$ and $k(t)$ are continuous functions.

The main objective is to analyze the boundary value problem defined by the Equations (24) through (27).

3.2.1. *General Solution*

Applying the double Laplace transform on both sides of the equations (24) and (25), one obtains

$$\tilde{s}\bar{U}(s,\tilde{s}) - \bar{U}(s,0) = s\bar{V}(s,\tilde{s}) - \bar{V}(0,\tilde{s}), \tag{28}$$

$$\epsilon^2\left[\tilde{s}\bar{V}(s,\tilde{s}) - \bar{V}(s,0)\right] = s\bar{U}(s,\tilde{s}) - \bar{U}(0,\tilde{s}) - \frac{\bar{V}(s,\tilde{s})}{a}. \tag{29}$$

Applying a single Laplace transform to Equations (26) and (27), one obtains

$$\bar{U}(s,0) = F_1(s),\ \bar{V}(s,0) = F_2(s), \tag{30}$$

$$\bar{U}(0,\tilde{s}) = H(\tilde{s}),\ \bar{V}(0,\tilde{s}) = K(\tilde{s}), \tag{31}$$

where $F_1(s), F_2(s), H(\tilde{s})$ and $K(\tilde{s})$ are Laplace transform of functions $f_1(x), f_2(x), h(t)$ and $k(t)$, respectively.

From Equations (28) and (29) by using Equations (30) and (31), one obtains

$$\bar{U}(s,\tilde{s}) = \frac{\big[\epsilon^2\tilde{s} + \frac{1}{a}\big]\big\{s\big[K(\tilde{s}) - F_1(s)\big] + \tilde{s}\big[H(\tilde{s}) - \epsilon^2 F_2(s)\big]\big\}}{s\big[s^2 - (\epsilon\tilde{s})^2 - \frac{\tilde{s}}{a}\big]} + \frac{H(\tilde{s})}{s} - \frac{\epsilon^2 F_2(s)}{s}, \tag{32}$$

$$\bar{V}(s,\tilde{s}) = \frac{s\big[K(\tilde{s}) - F_1(s)\big] + \tilde{s}\big[H(\tilde{s}) - \epsilon^2 F_2(s)\big]}{\big[s^2 - (\epsilon\tilde{s})^2 - \frac{\tilde{s}}{a}\big]}. \tag{33}$$

Applying double inverse Laplace transform on both sides of equations (32) and (33), one obtains

$$U(x,t) = \mathscr{L}_s^{-1}\mathscr{L}_{\tilde{s}}^{-1}\left\langle \frac{\big[\epsilon^2\tilde{s} + \frac{1}{a}\big]\big\{s\big[K(\tilde{s}) - F_1(s)\big] + \tilde{s}\big[H(\tilde{s}) - \epsilon^2 F_2(s)\big]\big\}}{s\big[s^2 - (\epsilon\tilde{s})^2 - \frac{\tilde{s}}{a}\big]} + \frac{H(\tilde{s})}{s} - \frac{\epsilon^2 F_2(s)}{s} \right\rangle, \tag{34}$$

and

$$V(x,t) = \mathscr{L}_s^{-1}\mathscr{L}_{\tilde{s}}^{-1}\left\langle \frac{s\big[K(\tilde{s}) - F_1(s)\big] + \tilde{s}\big[H(\tilde{s}) - \epsilon^2 F_2(s)\big]}{\big[s^2 - (\epsilon\tilde{s})^2 - \frac{\tilde{s}}{a}\big]} \right\rangle, \tag{35}$$

provided the inverse double Laplace transform exists for all terms on the right side of Equations (34) and (35).

Equations (32) and (33) represent a general solution in the Laplace domain. To find a solution in the space-time domain, an inversion of the double Laplace transform of the solution obtained in Equations (34) and (35) has been performed numerically by employing Xiao and Zhang (2011) algorithm.

4. System of Fractional PDEs

4.1. Initial Value Problem

Consider the system of equations

$$\frac{\partial^\alpha U}{\partial t^\alpha} = \frac{\partial V}{\partial x},\ \ 0 < \alpha \leq 1, \tag{36}$$

$$\epsilon^2 \frac{\partial^\alpha V}{\partial t^\alpha} = \frac{\partial U}{\partial x} - \frac{V}{a},\ \ 0 < \alpha \leq 1. \tag{37}$$

Assume initial conditions as

$$U(x,0) = f_1(x),\ V(x,0) = f_2(x),\ \ 0 \leq x < \infty, \tag{38}$$

where $f_1(x)$ and $f_2(x)$ are continuous functions.

The main purpose is to analyze the initial value problem as defined by the equations (36) through (38).

4.1.1. ***General Solution***

From equations (36) and (37), one can write

$$D_t U = D_t^{1-\alpha}(V_x), \tag{39}$$

$$D_t V = D_t^{1-\alpha}\left(\frac{U_x}{\epsilon^2}\right) - D_t^{1-\alpha}\left(\frac{V}{a\epsilon^2}\right). \tag{40}$$

Let's denote U_x by $\mathscr{L}_x(U)$, V_x by $\mathscr{L}_x(V)$, and D_t by $\mathscr{L}_t$ is invertible in the above equations.

Then, $\mathscr{L}_t^{-1} = \int_0^t (\cdot)\, d\xi$. One obtains

$$U(x,t) = U(x,0) + \mathscr{L}_t^{-1}\left\{D_t^{1-\alpha}\left[\mathscr{L}_x\big(V\big)\right]\right\}, \tag{41}$$

$$V(x,t) = V(x,0) + \mathscr{L}_t^{-1}\left\{D_t^{1-\alpha}\left[\mathscr{L}_x\left(\frac{U}{\epsilon^2}\right)\right]\right\} - \mathscr{L}_t^{-1}\left\{D_t^{1-\alpha}\left(\frac{V}{a\epsilon^2}\right)\right\}. \tag{42}$$

Take initial approximations as

$$U_0 = U(x,0) = f_1(x),\ V_0 = V(x,0) = f_2(x). \tag{43}$$

By Adomian (1996) decomposition method, one can write

$$U_{n+1} = \mathscr{L}_t^{-1}\left\{D_t^{1-\alpha}\left[\mathscr{L}_x\big(V_n\big)\right]\right\},\ \ n \geq 0, \tag{44}$$

$$V_{n+1} = \mathscr{L}_t^{-1}\left\{D_t^{1-\alpha}\left[\mathscr{L}_x\left(\frac{U_n}{\epsilon^2}\right)\right]\right\} - \mathscr{L}_t^{-1}\left\{D_t^{1-\alpha}\left(\frac{V_n}{a\epsilon^2}\right)\right\},\ \ n \geq 0. \tag{45}$$

From Equation (44), one obtains

$$\begin{aligned}
U_1 &= \dot{f}_2 \frac{t^{\alpha}}{\Gamma(\alpha+1)},\\
U_2 &= \left[\frac{\ddot{f}_1}{\epsilon^2} - \frac{\dot{f}_2}{a\epsilon^2}\right]\frac{t^{2\alpha}}{\Gamma(2\alpha+1)},\\
U_3 &= \left[\frac{\dddot{f}_2}{\epsilon^2} - \frac{\ddot{f}_1}{a\epsilon^4} + \frac{\dot{f}_2}{a^2\epsilon^4}\right]\frac{t^{3\alpha}}{\Gamma(3\alpha+1)},\\
U_4 &= \left[\frac{\ddddot{f}_1}{\epsilon^4} - \frac{2\dddot{f}_2}{a\epsilon^4} + \frac{\ddot{f}_1}{a^2\epsilon^6} - \frac{\dot{f}_2}{a^3\epsilon^6}\right]\frac{t^{4\alpha}}{\Gamma(4\alpha+1)},
\end{aligned} \tag{46}$$

and so on.

From Equation (45), one obtains

$$
\begin{aligned}
V_1 &= \left[\frac{\dot{f}_1}{\epsilon^2} - \frac{f_2}{a\epsilon^2}\right]\frac{t^{\alpha}}{\Gamma(\alpha+1)},\\
V_2 &= \left[\frac{\ddot{f}_2}{\epsilon^2} - \frac{\dot{f}_1}{a\epsilon^4} + \frac{f_2}{a^2\epsilon^4}\right]\frac{t^{2\alpha}}{\Gamma(2\alpha+1)},\\
V_3 &= \left[\frac{\dddot{f}_1}{\epsilon^4} - \frac{2\ddot{f}_2}{a\epsilon^4} + \frac{\dot{f}_1}{a^2\epsilon^6} - \frac{f_2}{a^3\epsilon^6}\right]\frac{t^{3\alpha}}{\Gamma(3\alpha+1)},\\
V_4 &= \left[\frac{\ddddot{f}_2}{\epsilon^4} - \frac{2\dddot{f}_1}{a\epsilon^6} + \frac{3\ddot{f}_2}{a^2\epsilon^6} - \frac{\dot{f}_1}{a^3\epsilon^8} + \frac{f_2}{a^4\epsilon^8}\right]\frac{t^{4\alpha}}{\Gamma(4\alpha+1)},
\end{aligned} \tag{47}
$$

and so on.

Then, the solution is represented by

$$
\begin{aligned}
U(x,t) &= \sum_{n=0}^{\infty} U_n\\
&= f_1 + \dot{f}_2\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \left[\frac{\ddot{f}_1}{\epsilon^2} - \frac{\dot{f}_2}{a\epsilon^2}\right]\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \left[\frac{\dddot{f}_2}{\epsilon^2} - \frac{\ddot{f}_1}{a\epsilon^4} + \frac{\dot{f}_2}{a^2\epsilon^4}\right]\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}\\
&\quad + \left[\frac{\ddddot{f}_1}{\epsilon^4} - \frac{2\dddot{f}_2}{a\epsilon^4} + \frac{\ddot{f}_1}{a^2\epsilon^6} - \frac{\dot{f}_2}{a^3\epsilon^6}\right]\frac{t^{4\alpha}}{\Gamma(4\alpha+1)} + \cdots ,
\end{aligned} \tag{48}
$$

and

$$
\begin{aligned}
V(x,t) &= \sum_{n=0}^{\infty} V_n\\
&= f_2 + \left[\frac{\dot{f}_1}{\epsilon^2} - \frac{f_2}{a\epsilon^2}\right]\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \left[\frac{\ddot{f}_2}{\epsilon^2} - \frac{\dot{f}_1}{a\epsilon^4} + \frac{f_2}{a^2\epsilon^4}\right]\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\\
&\quad + \left[\frac{\dddot{f}_1}{\epsilon^4} - \frac{2\ddot{f}_2}{a\epsilon^4} + \frac{\dot{f}_1}{a^2\epsilon^6} - \frac{f_2}{a^3\epsilon^6}\right]\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}\\
&\quad + \left[\frac{\ddddot{f}_2}{\epsilon^4} - \frac{2\dddot{f}_1}{a\epsilon^6} + \frac{3\ddot{f}_2}{a^2\epsilon^6} - \frac{\dot{f}_1}{a^3\epsilon^8} + \frac{f_2}{a^4\epsilon^8}\right]\frac{t^{4\alpha}}{\Gamma(4\alpha+1)} + \cdots ,
\end{aligned} \tag{49}
$$

where dot represents the derivative at spatial coordinate x.

4.2. Boundary Value Problem

Consider the system of equations

$$\frac{\partial^{\alpha} U}{\partial t^{\alpha}} = \frac{\partial V}{\partial x}, \quad 0 < \alpha \le 1, \tag{50}$$

$$\epsilon^2 \frac{\partial^{\alpha} V}{\partial t^{\alpha}} = \frac{\partial U}{\partial x} - \frac{V}{a}, \quad 0 < \alpha \le 1. \tag{51}$$

Assume initial conditions as

$$U(x,0)=f_1(x),\ V(x,0)=f_2(x),\ \ 0\leq x<\infty, \tag{52}$$

and boundary conditions as

$$U(0,t)=h(t),\ V(0,t)=k(t),\ \lim_{x\to\infty}U(x,t)=\lim_{x\to\infty}V(x,t)=0,\ \ t>0, \tag{53}$$

where $f_1(x), f_2(x), h(t)$ and $k(t)$ are continuous functions.

The main objective is to analyze the boundary value problem defined by the equations (50) through (53).

4.2.1. ***General Solution***

Applying the double Laplace transform on both sides of the equations (50) and (51), one obtains

$$\tilde{s}^{\alpha}\bar{U}(s,\tilde{s})-\tilde{s}^{\alpha-1}\bar{U}(s,0)=s\bar{V}(s,\tilde{s})-\bar{V}(0,\tilde{s}), \tag{54}$$

$$\epsilon^2\big[\tilde{s}^{\alpha}\bar{V}(s,\tilde{s})-\tilde{s}^{\alpha-1}\bar{V}(s,0)\big]=s\bar{U}(s,\tilde{s})-\bar{U}(0,\tilde{s})-\frac{\bar{V}(s,\tilde{s})}{a}. \tag{55}$$

Applying a single Laplace transform to Equations (52) and (53), one obtains

$$\bar{U}(s,0)=F_1(s),\ \ \bar{V}(s,0)=F_2(s), \tag{56}$$

$$\bar{U}(0,\tilde{s})=H(\tilde{s}),\ \ \bar{V}(0,\tilde{s})=K(\tilde{s}), \tag{57}$$

where $F_1(s), F_2(s), H(\tilde{s})$ and $K(\tilde{s})$ are Laplace transform of functions $f_1(x), f_2(x), h(t)$ and $k(t)$, respectively.

From Equations (54) and (55) by using Equations (56) and (57), one obtains

$$\bar{U}(s,\tilde{s})=\frac{\big[\epsilon^2\tilde{s}^{\alpha}+\frac{1}{a}\big]\big\{s\big[K(\tilde{s})-\tilde{s}^{\alpha-1}F_1(s)\big]+\tilde{s}^{\alpha}\big[H(\tilde{s})-\epsilon^2\tilde{s}^{\alpha-1}F_2(s)\big]\big\}}{s\big[s^2-\epsilon^2\tilde{s}^{2\alpha}-\frac{\tilde{s}^{\alpha}}{a}\big]}+\frac{H(\tilde{s})-\epsilon^2F_2(s)\tilde{s}^{\alpha-1}}{s}, \tag{58}$$

$$\bar{V}(s,\tilde{s})=\frac{s\big[K(\tilde{s})-\tilde{s}^{\alpha-1}F_1(s)\big]+\tilde{s}^{\alpha}\big[H(\tilde{s})-\epsilon^2\tilde{s}^{\alpha-1}F_2(s)\big]}{\big[s^2-\epsilon^2\tilde{s}^{2\alpha}-\frac{\tilde{s}^{\alpha}}{a}\big]}. \tag{59}$$

Applying the double inverse Laplace transform on both sides of Equations (58) and (59), one obtains

$$\begin{aligned}U(x,t)=&\mathscr{L}_s^{-1}\mathscr{L}_{\tilde{s}}^{-1}\Bigg\langle\frac{\big[\epsilon^2\tilde{s}^{\alpha}+\frac{1}{a}\big]\big\{s\big[K(\tilde{s})-\tilde{s}^{\alpha-1}F_1(s)\big]+\tilde{s}^{\alpha}\big[H(\tilde{s})-\epsilon^2\tilde{s}^{\alpha-1}F_2(s)\big]\big\}}{s\big[s^2-\epsilon^2\tilde{s}^{2\alpha}-\frac{\tilde{s}^{\alpha}}{a}\big]}\\&+\frac{H(\tilde{s})}{s}-\frac{\epsilon^2F_2(s)\tilde{s}^{\alpha-1}}{s}\Bigg\rangle,\end{aligned} \tag{60}$$

and

$$V(x,t)=\mathscr{L}_s^{-1}\mathscr{L}_{\tilde{s}}^{-1}\Bigg\langle\frac{s\big[K(\tilde{s})-\tilde{s}^{\alpha-1}F_1(s)\big]+\tilde{s}^{\alpha}\big[H(\tilde{s})-\epsilon^2\tilde{s}^{\alpha-1}F_2(s)\big]}{\big[s^2-\epsilon^2\tilde{s}^{2\alpha}-\frac{\tilde{s}^{\alpha}}{a}\big]}\Bigg\rangle, \tag{61}$$

provided that the inverse double Laplace transform exists for all terms on the right side of Equations (60) and (61).

Equations (58) and (59) represent a general solution in the Laplace domain. To find a solution in the space-time domain, an inversion of the double Laplace transform of the solution obtained in Equations (60) and (61) has been performed numerically by employing the Xiao and Zhang (2011) algorithm.

5. Convergence Analysis

A sufficient condition for the convergence of the method according to the approach proposed in the previous section is proved.

Theorem 5.1.

Series solutions U and V are defined in Equations (13) and (14) of Equations (36) and (38) and converge if $0 < \theta_1, \theta_2, \theta_3 < 1$ and $||U_{\tilde{n}}||, ||V_{\tilde{m}}|| < \infty$.

Proof:

From Equations (8) and (9), one has

$$\mathscr{L}^{\alpha} U = \frac{\partial V}{\partial x}, \;\; 0 < \alpha \leq 1, \tag{62}$$

$$\mathscr{L}^{\alpha} V = \frac{1}{\epsilon^2}\frac{\partial U}{\partial x} - \frac{V}{a\epsilon^2}, \;\; 0 < \alpha \leq 1, \tag{63}$$

where $\mathscr{L}$ denotes fractional order derivative.

Define a sequence of partial sums of a sequences $\{C_{\tilde{n}}\}_{\tilde{n}=0}^{\infty}$ and $\{D_{\tilde{m}}\}_{\tilde{m}=0}^{\infty}$ are as follows

$$C_{\tilde{n}} = \sum_{i=0}^{\tilde{n}} U_i \text{ and } D_{\tilde{m}} = \sum_{j=0}^{\tilde{m}} V_j, \tag{64}$$

to show that $\{C_{\tilde{n}}\}_{\tilde{n}=0}^{\infty}$ and $\{D_{\tilde{m}}\}_{\tilde{m}=0}^{\infty}$ are Cauchy sequences in the Hilbert space $\mathscr{H}$.

Expanding function $\mathscr{R}U$ about U_0, one can write

$$\mathscr{R}U = \sum_{k_1=0}^{\infty} \bar{A}_{k_1}. \tag{65}$$

Therefore, it can be arranged as

$$\bar{A}_0 = \mathscr{R}(U_0) = \mathscr{R}(C_0),\; \bar{A}_0 + \bar{A}_1 = \mathscr{R}(U_0 + U_1) = \mathscr{R}(C_1),\; \cdots,\; \sum_{i=0}^{\tilde{n}-1} \bar{A}_i = \mathscr{R}(C_{\tilde{n}}) - \bar{A}_{\tilde{n}}. \tag{66}$$

Expanding function $\mathscr{R}V$ about V_0, one can write

$$\mathscr{R}V = \sum_{k_2=0}^{\infty} \bar{B}_{k_2}. \tag{67}$$

Therefore, it can be arranged as

$$\bar{B}_0 = \mathscr{R}(V_0) = \mathscr{R}(D_0),\ \bar{B}_0 + \bar{B}_1 = \mathscr{R}(V_0 + V_1) = \mathscr{R}(D_1),\ \cdots,\sum_{j=0}^{\tilde{m}-1} \bar{B}_j = \mathscr{R}(D_{\tilde{m}}) - \bar{B}_{\tilde{m}}. \tag{68}$$

Consider

$$\begin{aligned} ||C_{\tilde{n}+\tilde{p}} - C_{\tilde{n}}|| &= \max_{t\in I}\left\{\left|\mathscr{L}^{-\alpha}\left(\sum_{j=\tilde{m}+1}^{\tilde{m}+\tilde{p}} \mathscr{R}V_{j-1}\right)\right|\right\} \\ &= \max_{t\in I}\left\{\left|\mathscr{L}^{-\alpha}\left(\sum_{j=\tilde{m}}^{\tilde{m}+\tilde{p}-1} \mathscr{R}V_{j}\right)\right|\right\} \\ &= \max_{t\in I}\left\{\left|\mathscr{L}^{-\alpha}\left[\mathscr{R}\left(D_{\tilde{m}+\tilde{p}-1}\right) - \mathscr{R}\left(D_{\tilde{m}-1}\right)\right]\right|\right\} \\ &\le \max_{t\in I}\left\{\mathscr{L}^{-\alpha}\left[\left|\mathscr{R}\left(D_{\tilde{m}+\tilde{p}-1}\right) - \mathscr{R}\left(D_{\tilde{m}-1}\right)\right|\right]\right\}. \end{aligned} \tag{69}$$

Since $\mathscr{R}$ is the Lipschitzian function, one obtains

$$\le L_1 \max_{t\in I}\left\{\mathscr{L}^{-\alpha}\left[\left|D_{\tilde{m}+\tilde{p}-1} - D_{\tilde{m}-1}\right|\right]\right\}, \tag{70}$$

where L_1 is Lipschitz constant.

Hence,

$$||C_{\tilde{n}+\tilde{p}} - C_{\tilde{n}}|| \le \theta_1 ||D_{\tilde{m}+\tilde{p}-1} - D_{\tilde{m}-1}||, \tag{71}$$

where $\theta_1 = \frac{L_1 t^\alpha}{\alpha!}$.

Consider

$$\begin{aligned} ||D_{\tilde{m}+\tilde{p}} - D_{\tilde{m}}|| &= \max_{t\in I}\left\{\left|\frac{1}{\epsilon^2}\mathscr{L}^{-\alpha}\left(\sum_{i=\tilde{n}+1}^{\tilde{n}+\tilde{p}} \mathscr{R}U_i\right) - \frac{1}{a\epsilon^2}\mathscr{L}^{-\alpha}\left(\sum_{j=\tilde{m}+1}^{\tilde{m}+\tilde{p}} V_{j-1}\right)\right|\right\} \\ &= \max_{t\in I}\left\{\left|\frac{1}{\epsilon^2}\mathscr{L}^{-\alpha}\left(\sum_{i=\tilde{n}}^{\tilde{n}+\tilde{p}-1} \mathscr{R}U_{i+1}\right) - \frac{1}{a\epsilon^2}\mathscr{L}^{-\alpha}\left(\sum_{j=\tilde{m}}^{\tilde{m}+\tilde{p}-1} V_{j}\right)\right|\right\} \\ &= \max_{t\in I}\left\{\left|\frac{1}{\epsilon^2}\mathscr{L}^{-\alpha}\left[\mathscr{R}\left(C_{\tilde{n}+\tilde{p}-1}\right) - \mathscr{R}\left(C_{\tilde{n}-1}\right)\right] - \frac{1}{a\epsilon^2}\mathscr{L}^{-\alpha}\left[D_{\tilde{m}+\tilde{p}-1} - D_{\tilde{m}-1}\right]\right|\right\} \\ &\le \max_{t\in I}\left\{\frac{1}{\epsilon^2}\left|\mathscr{L}^{-\alpha}\left[\mathscr{R}\left(C_{\tilde{n}+\tilde{p}-1}\right) - \mathscr{R}\left(C_{\tilde{n}-1}\right)\right]\right| + \frac{1}{a\epsilon^2}\left|\mathscr{L}^{-\alpha}\left[D_{\tilde{m}+\tilde{p}-1} - D_{\tilde{m}-1}\right]\right|\right\}. \end{aligned} \tag{72}$$

Since $\mathscr{R}$ is the Lipschitzian function, one obtains

$$\le \max_{t\in I}\left\{\frac{L_2}{\epsilon^2}\mathscr{L}^{-\alpha}\left[\left|C_{\tilde{n}+\tilde{p}-1} - C_{\tilde{n}-1}\right|\right] + \frac{1}{a\epsilon^2}\mathscr{L}^{-\alpha}\left[\left|D_{\tilde{m}+\tilde{p}-1} - D_{\tilde{m}-1}\right|\right]\right\}, \tag{73}$$

where L_2 is Lipschitz constant;

$$\begin{aligned} &\leq \frac{L_2}{\epsilon^2}\max_{t\in I}\left\{\mathscr{L}^{-\alpha}\left[\left|C_{\tilde{n}+\tilde{p}-1}-C_{\tilde{n}-1}\right|\right]\right\}+\frac{1}{a\epsilon^2}\max_{t\in I}\left\{\mathscr{L}^{-\alpha}\left[\left|D_{\tilde{m}+\tilde{p}-1}-D_{\tilde{m}-1}\right|\right]\right\}, \\ &\leq \frac{L_2 t^\alpha}{\epsilon^2\alpha!}\max_{t\in I}\left\{\left|C_{\tilde{n}+\tilde{p}-1}-C_{\tilde{n}-1}\right|\right\}+\frac{t^\alpha}{a\epsilon^2\alpha!}\max_{t\in I}\left\{\left|D_{\tilde{m}+\tilde{p}-1}-D_{\tilde{m}-1}\right|\right\}. \end{aligned} \tag{74}$$

Hence,

$$||D_{\tilde{m}+\tilde{p}}-D_{\tilde{m}}|| \leq \theta_2||C_{\tilde{n}+\tilde{p}-1}-C_{\tilde{n}-1}||+\theta_3||D_{\tilde{m}+\tilde{p}-1}-D_{\tilde{m}-1}||, \tag{75}$$

where $\theta_2=\frac{L_2 t^\alpha}{\epsilon^2\alpha!}$ and $\theta_3=\frac{t^\alpha}{a\epsilon^2\alpha!}$.

Fix $p=1$, from Equations (71) and (75),

$$\begin{aligned} &||C_{\tilde{n}+1}-C_{\tilde{n}}|| \leq \theta_1||D_{\tilde{m}}-D_{\tilde{m}-1}|| \leq \theta_1||V_{\tilde{m}}||, \text{ and} \\ &||D_{\tilde{m}+1}-D_{\tilde{m}}|| \leq \theta_2||C_{\tilde{n}}-C_{\tilde{n}-1}||+\theta_3||D_{\tilde{m}}-D_{\tilde{m}-1}|| \leq \theta_2||U_{\tilde{n}}||+\theta_3||V_{\tilde{m}}||. \end{aligned} \tag{76}$$

Since solutions U and V are bounded, this implies that $||U_{\tilde{n}}||<\infty$, $||V_{\tilde{m}}||<\infty$ and $0<\theta_1,\ \theta_2,\ \theta_3<1$. As $\tilde{n},\tilde{m}\to\infty,\ ||C_{\tilde{n}+1}-C_{\tilde{n}}||\to 0$ and $||D_{\tilde{m}+1}-D_{\tilde{m}}||\to 0$.

Hence $\{C_{\tilde{n}}\}_{\tilde{n}=0}^{\infty}$ and $\{D_{\tilde{m}}\}_{\tilde{m}=0}^{\infty}$ are Cauchy sequences in Hilbert space $\mathscr{H}$. Therefore, series solutions U and V converge. Hence, proof of Theorem 5.1. ■

6. Illustrative Example

Consider a piece of iron having physical properties $a=2.1\times10^{-1}\ cm^2/s$ and $\epsilon=2.0\times10^{-6}\ s/cm$.

6.1. Initial Value Problem

Consider the system of fractional partial differential equations (8) and (9) subjected to the initial conditions,

$$U(x,0)=\mathrm{e}^{-x},\ V(x,0)=\mathrm{e}^{x},\ \ 0\leq x<\infty. \tag{77}$$

6.1.1. ***Integer Order***

The solution of the problem defined in Equation (77) is obtained for the integer order derivative by applying the equations (20) and (21). One obtains

$$\begin{aligned} U_1 &= \mathrm{e}^{x} t, \\ U_2 &= \left[\frac{\mathrm{e}^{-x}}{\epsilon^2} - \frac{\mathrm{e}^{x}}{a\epsilon^2}\right]\frac{t^2}{2!}, \\ U_3 &= \left[\frac{\mathrm{e}^{x}}{\epsilon^2} - \frac{\mathrm{e}^{-x}}{a\epsilon^4} + \frac{\mathrm{e}^{x}}{a^2\epsilon^4}\right]\frac{t^3}{3!}, \\ U_4 &= \left[\frac{\mathrm{e}^{-x}}{\epsilon^4} - \frac{2\mathrm{e}^{x}}{a\epsilon^4} + \frac{\mathrm{e}^{-x}}{a^2\epsilon^6} - \frac{\mathrm{e}^{x}}{a^3\epsilon^6}\right]\frac{t^4}{4!}, \end{aligned} \tag{78}$$

and so on.

From Equation (21), one obtains

$$\begin{aligned} V_1 &= \left[-\frac{\mathrm{e}^{-x}}{\epsilon^2} - \frac{\mathrm{e}^{x}}{a\epsilon^2}\right] t, \\ V_2 &= \left[\frac{\mathrm{e}^{x}}{\epsilon^2} + \frac{\mathrm{e}^{-x}}{a\epsilon^4} + \frac{\mathrm{e}^{x}}{a^2\epsilon^4}\right]\frac{t^2}{2!}, \\ V_3 &= \left[-\frac{\mathrm{e}^{-x}}{\epsilon^4} - \frac{2\mathrm{e}^{x}}{a\epsilon^4} - \frac{\mathrm{e}^{-x}}{a^2\epsilon^6} - \frac{\mathrm{e}^{x}}{a^3\epsilon^6}\right]\frac{t^3}{3!}, \\ V_4 &= \left[\frac{\mathrm{e}^{x}}{\epsilon^4} + \frac{2\mathrm{e}^{-x}}{a\epsilon^6} + \frac{3\mathrm{e}^{x}}{a^2\epsilon^6} + \frac{\mathrm{e}^{-x}}{a^3\epsilon^8} + \frac{\mathrm{e}^{x}}{a^4\epsilon^8}\right]\frac{t^4}{4!}, \end{aligned} \tag{79}$$

and so on.

From Equations (22) and (23), then the solution is written as

$$U(x,t) = \mathrm{e}^{-x} + \mathrm{e}^{x} t + \left[\frac{\mathrm{e}^{-x}}{\epsilon^2} - \frac{\mathrm{e}^{x}}{a\epsilon^2}\right]\frac{t^2}{2!} + \left[\frac{\mathrm{e}^{x}}{\epsilon^2} - \frac{\mathrm{e}^{-x}}{a\epsilon^4} + \frac{\mathrm{e}^{x}}{a^2\epsilon^4}\right]\frac{t^3}{3!} + \left[\frac{\mathrm{e}^{-x}}{\epsilon^4} - \frac{2\mathrm{e}^{x}}{a\epsilon^4} + \frac{\mathrm{e}^{-x}}{a^2\epsilon^6} - \frac{\mathrm{e}^{x}}{a^3\epsilon^6}\right]\frac{t^4}{4!} + \cdots, \tag{80}$$

and

$$\begin{aligned} V(x,t) = \mathrm{e}^{x} &+ \left[-\frac{\mathrm{e}^{-x}}{\epsilon^2} - \frac{\mathrm{e}^{x}}{a\epsilon^2}\right] t + \left[\frac{\mathrm{e}^{x}}{\epsilon^2} + \frac{\mathrm{e}^{-x}}{a\epsilon^4} + \frac{\mathrm{e}^{x}}{a^2\epsilon^4}\right]\frac{t^2}{2!} \\ &+ \left[-\frac{\mathrm{e}^{-x}}{\epsilon^4} - \frac{2\mathrm{e}^{x}}{a\epsilon^4} - \frac{\mathrm{e}^{-x}}{a^2\epsilon^6} - \frac{\mathrm{e}^{x}}{a^3\epsilon^6}\right]\frac{t^3}{3!} \\ &+ \left[\frac{\mathrm{e}^{x}}{\epsilon^4} + \frac{2\mathrm{e}^{-x}}{a\epsilon^6} + \frac{3\mathrm{e}^{x}}{a^2\epsilon^6} + \frac{\mathrm{e}^{-x}}{a^3\epsilon^8} + \frac{\mathrm{e}^{x}}{a^4\epsilon^8}\right]\frac{t^4}{4!} + \cdots. \end{aligned} \tag{81}$$

The numerical calculations have been done by considering values of $a, \epsilon, \alpha = 1$. The numerically computed results are hereby plotted and demonstrated in Figure 1 and Figure 2.

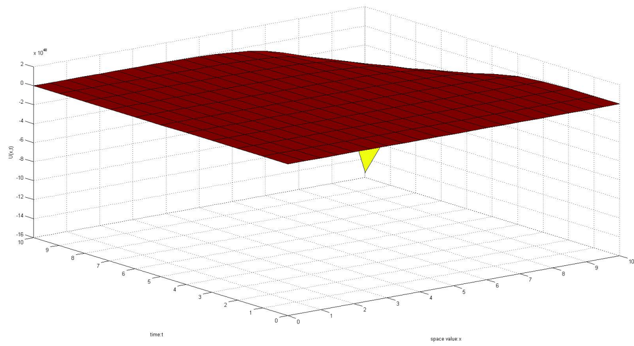

Figure 1. Temperature $U(x,t)$ at integer order $\alpha = 1$

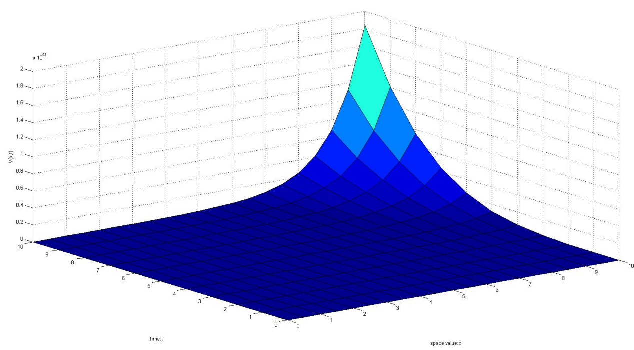

Figure 2. Heat flux $V(x,t)$ at integer order $\alpha = 1$

6.1.2. ***Fractional Order***

The solution of the problem defined in the equation (77) is obtained for the fractional order derivative by applying the equations (48) and (49), and one obtains

$$\begin{aligned} U(x,t) &= e^{-x} + e^{x}\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \left[\frac{e^{-x}}{\epsilon^2} - \frac{e^{x}}{a\epsilon^2}\right]\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \left[\frac{e^{x}}{\epsilon^2} - \frac{e^{-x}}{a\epsilon^4} + \frac{e^{x}}{a^2\epsilon^4}\right]\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} \\ &\quad + \left[\frac{e^{-x}}{\epsilon^4} - \frac{2e^{x}}{a\epsilon^4} + \frac{e^{-x}}{a^2\epsilon^6} - \frac{e^{x}}{a^3\epsilon^6}\right]\frac{t^{4\alpha}}{\Gamma(4\alpha+1)} + \cdots, \end{aligned} \tag{82}$$

and

$$\begin{aligned} V(x,t) &= e^{x} + \left[-\frac{e^{-x}}{\epsilon^2} - \frac{e^{x}}{a\epsilon^2}\right]\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \left[\frac{e^{x}}{\epsilon^2} + \frac{e^{-x}}{a\epsilon^4} + \frac{e^{x}}{a^2\epsilon^4}\right]\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \\ &\quad + \left[-\frac{e^{-x}}{\epsilon^4} - \frac{2e^{x}}{a\epsilon^4} - \frac{e^{-x}}{a^2\epsilon^6} - \frac{e^{x}}{a^3\epsilon^6}\right]\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} \\ &\quad + \left[\frac{e^{x}}{\epsilon^4} + \frac{2e^{-x}}{a\epsilon^6} + \frac{3e^{x}}{a^2\epsilon^6} + \frac{e^{-x}}{a^3\epsilon^8} + \frac{e^{x}}{a^4\epsilon^8}\right]\frac{t^{4\alpha}}{\Gamma(4\alpha+1)} + \cdots. \end{aligned} \tag{83}$$

The numerical calculations have been done by considering values of $a, \epsilon, \alpha = 0.5$. The numerically computed results are hereby plotted and demonstrated in Figure 3 and Figure 4.

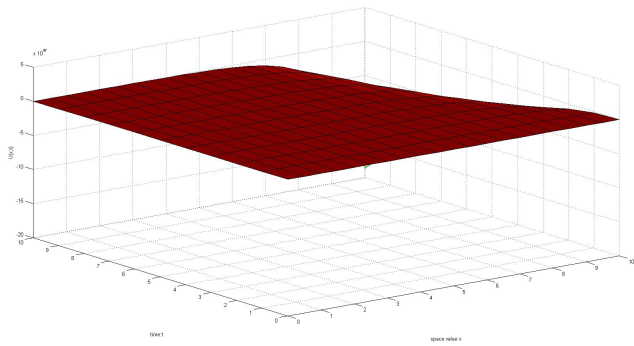

Figure 3. Temperature $U(x, t)$ at fractional order $\alpha = 0.5$

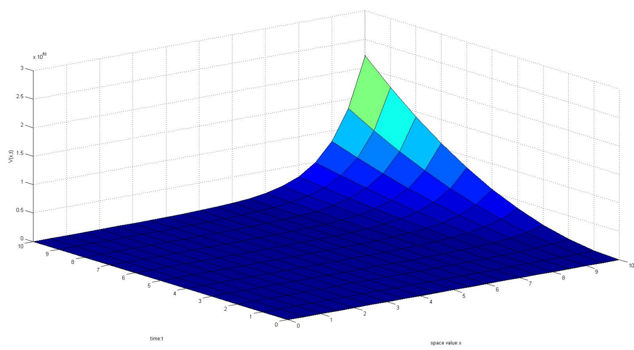

Figure 4. Heat flux $V(x, t)$ at fractional order $\alpha = 0.5$

Figure 1 and Figure 3 represent the temperature variations $U(x, t)$ with respect to $\alpha = 1$ and $\alpha = 0.5$. The temperature gradually decreases when parameter α increases along distance x. The temperature distributions are attained minimum for $\alpha = 0.5$ along distance x.

Figure 2 and Figure 4 represent the heat flux $V(x, t)$ with respect to $\alpha = 1$ and $\alpha = 0.5$. The heat flux gradually increases when parameter α increases along distance x. The heat flux along distance x attains maximum for $\alpha = 1$ and minimum for $\alpha = 0.5$.

6.2. Boundary Value Problem

Consider the system of fractional partial differential equations (8) and (9) subjected to the initial conditions

$$U(x, 0) = \sin(x),\ V(x, 0) = \cos(x),\ \ 0 \leq x < \infty, \tag{84}$$

and boundary conditions

$$U(0, t) = 0,\ V(0, t) = 0,\ \lim_{x\to\infty} U(x, t) = \lim_{x\to\infty} V(x, t) = 0,\ \ t > 0. \tag{85}$$

6.2.1. *Integer Order*

The Xiao (2023) algorithm has been implemented to Equations (34) and (35) using Equations (84) and (85) through MATLAB programming. The numerical calculations have been done by considering values of $a, \epsilon, \alpha = 1$. The numerically computed results are hereby plotted and demonstrated in Figure 5 and Figure 6.

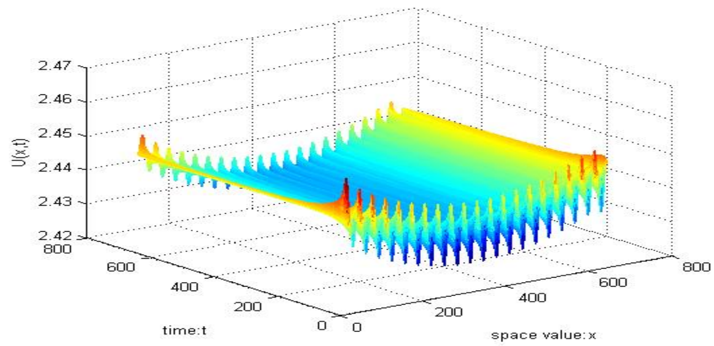

Figure 5. Temperature $U(x, t)$ at integer order $\alpha = 1$

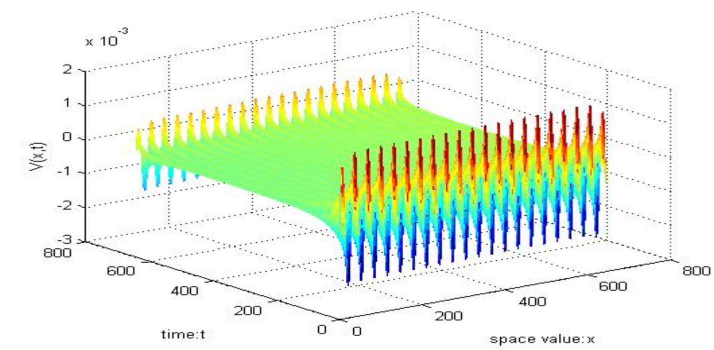

Figure 6. Heat flux $V(x, t)$ at integer order $\alpha = 1$

6.2.2. *Fractional Order*

The Xiao (2023) algorithm has been implemented to Equations (60) and (61) using Equations (84) and (85) through MATLAB programming. The numerical calculations have been done by considering values of $a, \epsilon, \alpha = 0.5$. The numerically computed results are hereby plotted and demonstrated in Figure 7 and Figure 8.

Figure 5 and Figure 7 represent the temperature variations $U(x, t)$ with respect to $\alpha = 1$ and $\alpha = 0.5$. The temperature gradually decreases when parameter α increases along distance x. One can be observed that the solution satisfies the boundary conditions. The temperature along distance x attains minimum for $\alpha = 1$ and maximum for $\alpha = 0.5$.

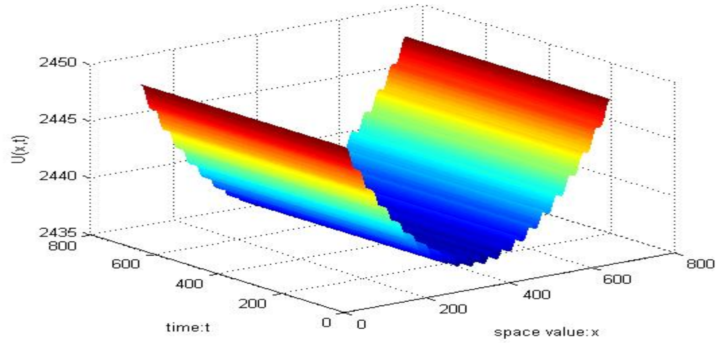

Figure 7. Temperature $U(x,t)$ at fractional order $\alpha = 0.5$

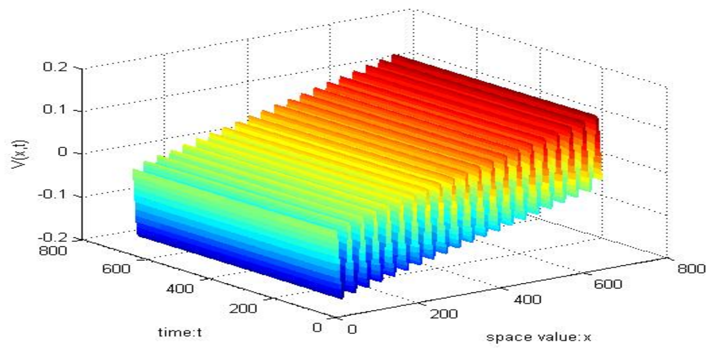

Figure 8. Heat flux $V(x,t)$ at fractional order $\alpha = 0.5$

Figure 6 and Figure 8 represent the heat flux variations $V(x,t)$ with respect to $\alpha = 1$ and $\alpha = 0.5$. The heat flux along distance x attains maximum for $\alpha = 0.5$ and minimum for $\alpha = 1$. The heat flux increases when parameter α decreases along distance x. It has been found that the solution satisfies the boundary conditions.

7. Concluding Remarks

The main outcomes are as follows.

(1) The initial and boundary value problem for a linear hyperbolic system of equations for heat propagation in the sense of Caputo fractional time derivative of order $0 < \alpha \leq 1$ is presented in the one-dimensional case.
(2) A system of integer and fractional order PDEs have been solved with respect to initial conditions by employing the method of Adomian decomposition. The obtained solutions are in convergent infinite series form.
(3) The solution of the initial and boundary value problem of the system of integer and fractional order PDEs is acquired by employing the double Laplace transform method.

(4) The results computed for the temperature U and heat flux V for the integer and fractional order case satisfies all imposed conditions in this proposed model.
(5) The convergence theorem for the infinite series solution obtained by the Adomian Decomposition Method has been proved mathematically.
(6) The finite speed of thermal wave propagation has been attained by introducing the non-Fourier effect of heat conduction in the context of delay time translation τ_0.
(7) According to numerical results, the fractional parameter α has evolved into a new measure of its ability to conduct thermal energy.

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Nomenclature

T	Absolute temperature
$\mathbf{x}$	Material point
x	Spatial coordinate
a	Thermal diffusivity
$\mathbf{q}$	Heat flux vector
t	Time
c	Specific heat of the body
ϵ	Reciprocal of the characteristic speed
c^*	Characteristic speed
τ_0	Constant relaxation time
α	Fractional order parameter
$s, \tilde{s}$	Laplace transform parameters
∇	Gradient operator