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Analytical Approximations in Short Times of Exact Operational Solutions to Reaction–Diffusion Problems on Bounded Intervals

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Abstract

This paper aims to provide an exact solution in the Laplace domain and related analytic approximations in short time limits for the class of boundary value problems of the one-dimensional linear parabolic equation with constant coefficients. The problem's most general form involves a parameterized equation on a bounded interval, with unified specification of the three classical types of boundary conditions: Dirichlet, Neumann, and Robin. Under certain integrability assumptions, we have proven that a unique solution exists in the Laplace domain. This operational solution can be obtained in a closed form by using classical integral transforms. Four distinct cases have been identified based on the operational solution. Innovative formulas have been derived from these cases, which provide precise approximations within short timescales. These time-domain expressions are particularly useful for understanding the behavior of the solution at the boundaries. The formulas consist of elementary functions obtained from asymptotic expansions, and the estimation error can be minimized to the desired order of magnitude. The analytical approximations in short time limits can open up new perspectives and applications. Improved numerical efficiency in simulations of reaction-diffusion problems and of one-dimensional Stefan models are envisaged.

Keywords: Reaction–diffusion equation; Fourier and Laplace transforms; Existence and uniqueness; Exact operational solution; Asymptotic expansions; Time step; Error estimate; Series solutions; Numerical efficiency

MSC 2020 No.: 35A09, 35K05, 44A10

1. Introduction

Initial-boundary value problems for linear parabolic equations are still much used as first model approximations of nonlinear and time-dependent problems in bounded domains (see, for examples, Kacur (1999), Kumar and Umavathi (2013), and Brenn (2017)). As shown in Chapter 6 of Henner et al. (2019), any one-dimensional linear parabolic equation with constant coefficients and containing a convection term can be reduced to an equation without a convection term. We will therefore consider the most general form of the reaction-diffusion equation with constant coefficients, i.e., the nonhomogenous equation with no convection term. Until now, exact analytical solutions to such linear boundary value problems are mainly obtained as infinite series of functions, as in Luikov (2012) or in Henner et al. (2019). These series solutions are generally obtained via the Fourier decomposition method, by means of separation of variables and using the Sturm-Liouville theory, as presented in many textbooks of which Han (2016) and Dobrushkin (2017) can be cited among others. But, accurate analytic approximations in short time limits are hardly derivable from such infinite series solutions (see Chapter 8 in Brenn (2017) and Carr and March (2018)).

Compared to classical numeric schemes as reported in Minkowycz et al. (2006), recent numerical approaches to boundary value problems for heat transfer equations have much gained in sophistication and accuracy as it can be seen in Lin et al. (2020), Tassaddiq et al. (2021), and Li et al. (2022) for examples. However, apart from the Fourier decomposition method mentioned above, there is no other established method for finding exact analytical solutions to boundary value problems involving linear reaction-diffusion equations, even with constant coefficients. According to Luikov (2012), the Laplace integral transform and its inversion formula are found to be not appropriate for solving boundary value problems with a non-uniform function as initial condition. Moreover, the classical Fourier integral transform applied to space coordinates is validated only for infinite and semi-infinite solids. Concerning other analytical and semi-approximate methods, some powerful techniques have been used to handle heat transfer problems in finite domains, but without exhibiting new exact solution. One can cite the Adomian's decomposition method as in Bougoffa et al. (2015) and Turkyilmazoglu (2019a), the variational iteration method as in Liu and Zhao (2010), the homotopy perturbation method (see Ghasemia and Kajani (2010) and Turkyilmazoglu (2019b)), Bessel collocation method as in Yüzbaşı and Şahin (2013), etc. More details on analytical methods applicable to all types of equations are available in the book by Zheng and Zhang (2017).

In this paper, an exact solution in the Laplace domain and analytical approximations in short time limits are obtained for the boundary value problem of the linear reaction-diffusion equation with constant coefficients. In Section 2, the general form of the problem is introduced and the assumptions are briefly commented. Section 3 presents the application of the Fourier integral transform to the unified problem, and a resulting equation of the solution in a form of integrals. In Section 4, the exact solution is found in the Laplace domain from the previous relation. Then, analytical solutions in short time limits are obtained in Section 5, especially with respect to the behavior of the solution at the ends points of the domain. Each of these analytical approximations is important and useful in its own way, and its error estimate can be improved. However, analytical approximations are only obtained in the short term, not the long term, as they result from asymptotic developments of the exact operational solution. Applications of the results to specific examples of initial-boundary value problems for parabolic equations are shown in Section 6, and the unified solution in the Laplace domain is also extended to unbounded domains. Finally, Section 7 outlines the conclusion.

2. The reaction–diffusion equation

We generalize the boundary value problem referenced by Equations (6.43), (6.44), and (6.65) in Henner et al. (2019) to the nonhomogenous linear equation now expressed on a generic bounded interval. That is, for space variable x, $l_1 \leq x \leq l_2$, and for time t, $0 \leq t \leq T$, (we'll assume that all intervals are open at T in the case where $T = +\infty$), the reaction-diffusion equation:

$$\frac{\partial u}{\partial t}(x,t) - a^2 \frac{\partial^2 u}{\partial x^2}(x,t) + bu(x,t) = f(x,t), \quad l_1 < x < l_2, \quad 0 < t \leq T, \tag{1}$$

is subject to the initial condition

$$u(x,0) = \varphi(x), \quad l_1 \leq x \leq l_2, \tag{2}$$

and to the boundary conditions:

$$\alpha_1 u(l_1,t) + \beta_1 \frac{\partial u}{\partial x}(l_1,t) = g_1(t), \quad \alpha_1, \beta_1 \in \mathbb{R}, \quad \alpha_1^2 + \beta_1^2 \neq 0, \tag{3}$$

$$\alpha_2 u(l_2,t) + \beta_2 \frac{\partial u}{\partial x}(l_2,t) = g_2(t), \quad \alpha_2, \beta_2 \in \mathbb{R}, \quad \alpha_2^2 + \beta_2^2 \neq 0, \tag{4}$$

for $0 \leq t \leq T$. The four terms in Equation (1) represent respectively transient, diffusion, reaction and source terms. The function $u(x,t)$ is to be determined, and may represent species concentration for mass transfer or temperature for heat transfer, while the functions $f(x,t), g_1(t)$ and $g_2(t)$ are given. The coefficient $a > 0$ is related to the constant diffusivity of the mass or heat transfer. The reaction term (linear term in u) indicates the possibility of mass or heat exchange with the environment through the lateral surface of the body, at some rates proportional to the concentration or to the temperature (Henner et al. (2009)). In a process of mass diffusion for example, b is the coefficient of disintegration ($b < 0$) or multiplication ($b > 0$). In this paper we will limit ourselves to the case of the multiplication coefficient, and from now on we will assume that $b \geq 0$.

In order to consider homogeneous as well as nonhomogeneous equations, the source term $f(x,t)$ may be null or not, while the function $\varphi(x)$ may indicate zero or non-zero initial condition. The Dirichlet, Neumann and Robin boundary conditions are expressed in a unified way by Equations (3) and (4), that is, both homogeneous and nonhomogeneous forms of those three types of boundary conditions can be taken into account. It is sufficient to give some acceptable values to the real parameters $\alpha_1, \beta_1, \alpha_2, \beta_2$, and some convenient expressions to the time-dependent functions $g_1(t)$ and $g_2(t)$. Thus, Dirichlet conditions are satisfied when $\alpha_1 = \alpha_2 = 1$ and $\beta_1 = \beta_2 = 0$, while Neumann conditions can be obtained on the boundaries if $\alpha_1 = \alpha_2 = 0$ and $\beta_1 = \beta_2 = 1$. Likewise, Robin boundary conditions are obtained when $\alpha_1 = \alpha_2 = 1$ and $\beta_1 \neq 0, \ \beta_2 \neq 0$. As it can be noted, a given combination of these three classical types of boundary conditions can also be obtained.

In order to make physical sense of the problem (1)-(4), certain restrictions on the sign of the coefficients in the boundary conditions (3)-(4) are necessary. Most often, physical limitations lead to the restriction $\beta_1/\alpha_1 < 0$ and $\beta_2/\alpha_2 > 0$, as discussed in Henner et al. (2019) for example (see Chapters 4 and 6). The classical solution of the initial-boundary value problem (1)-(4) is defined as any function $u(x,t)$ continuous on $[l_1, l_2] \times [0, T]$, such that $u(x, .)$ is continuously differentiable for $x \in (l_1, l_2)$, and $u(., t)$ is twice continuously differentiable for $t \in (0, T]$, and u satisfies (1)-(4) pointwise. It can be proven (see Cioranescu et al. (2018) and references therein) that the problem admits at most a unique classical solution under the following assumptions: the source term $f(x,t)$ and the function $\varphi(x)$ are respectively continuous on $(l_1, l_2)\times(0, T]$ and $[l_1, l_2]$; the two related-to-boundary expressions $g_1(t)$ and $g_2(t)$ are continuous on $[0, T]$; the initial and boundary conditions are compatible, that is, $u(l_1, 0) = \varphi(l_1)$, and $u(l_2, 0) = \varphi(l_2)$. In the case where the compatibility condition is not satisfied, the initial and boundary conditions are not consistent (they are contradictory) and only a weak or generalized solution can be obtained. The weak solution, especially in two- or multi-dimensional spaces, when defined on sufficiently regular bounded open sets, can be called a classical solution depending on the so-called regularity conditions.

Practically, the problem (1)-(4) is first solved for homogeneous equation and boundary conditions ($f(x,t) = 0$ and $g_1(t) = g_2(t) = 0$), using the separation of variables and Fourier decomposition methods, together with the Sturm-Liouville theory of self-adjoint operators. Then, the principle of Duhamel intervenes in addition, when accounting for a non-zero source term $f(x,t)$. Likewise, in the case of nonhomogeneous boundary conditions, the problem (1)-(4) is first reduced to a problem with boundary conditions equal to zero by the means of the so-called auxiliary functions (see Henner et al. (2019) for example). The application of the above-mentioned techniques to the problem (1)-(4) will lead to an exact series solution, which converges uniformly as well as the series obtained by differentiating twice by x and once by t. For this, it is sufficient that $\varphi(x)$ and $f(x,t)$ are continuous respectively on $[l_1, l_2]$ and on $(l_1, l_2) \times (0, T]$, and $g_1(t)$ and $g_2(t)$ are continuously differentiable on $[0, T]$. The uniqueness of the series solution can be proven by using the maximum principle. In one or more dimensional spaces, existence and uniqueness theorems of classical, as well as generalized solutions for such initial-boundary value problems, have been proven only for specific classes of functions under certain assumptions (see Marin and Öchsner (2018) for example). In our present approach, we admit here that the source term $f(x,t)$ is continuous on $(l_1, l_2) \times (0, T]$, and the function $\varphi(x)$ satisfies at least the so-called Dirichlet conditions relatively to the space variable x on $[l_1, l_2]$. That is, this function is piece-wise continuous or can be expressed in a unique way as a convergent series of eigenfunctions forming a complete basis of the related Sturm-Liouville problem. Similarly, the time-dependent functions $g_1(t)$ and $g_2(t)$ are assumed here to be at least once piece-wise differentiable. Thus, they are continuous and their derivatives are piece-wise continuous on $[0, T]$. In addition, all the involved functions are assumed to be integrable with respect to the time and space variables. The source term $f(x,t)$ is considered to be of exponential order with respect to the time variable t, as are the functions $g_1(t)$ and $g_2(t)$ and their respective derivatives. Standing on these assumptions, we propose a novel approach that combines Fourier and Laplace integral transformations to obtain an exact operational solution for

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the boundary value problem (1)-(4), and accurate approximations in short time limits. Under the hypotheses specified above, the operational solution is also proven to be unique. The use of integral transform methods in the solution of partial differential equations can be found in Debnath and Bhatta (2014) and Meddahi et al. (2021), among others.

3. Method of Fourier integral transform

In this section, the Fourier integral transform (abbreviated FIT from now on) and its inverse are used in relation to the space variable x. We first recall that for any absolutely integrable function $\phi(x)$, i.e., $\int_{-\infty}^{+\infty} |\phi(x)|dx < \infty$, the FIT Φ is defined as:

$$\Phi(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \phi(x) \exp(-i\lambda x) dx, \tag{5}$$

where $i^2 = -1$, $\lambda \in \mathbb{R}$ and $x \in \mathbb{R}$. The inverse Fourier transform of Φ is:

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \Phi(\lambda) \exp(i\lambda x) d\lambda. \tag{6}$$

As a basic property, the FIT tends to 0 when $|\lambda|$ goes to ∞.

The homogeneous form of equation (1) reads:

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} - bu. \tag{7}$$

Let $u_1(x,t)$ be a solution for the boundary value problem formed by the homogeneous equation (7) together with the initial condition (2), and the boundary conditions (3) and (4). Assuming that the functions $u_1(x,t)$, $\frac{\partial u_1}{\partial t}(x,t)$, and $\frac{\partial^2 u_1}{\partial x^2}(x,t)$ are absolutely integrable with respect to the variables x and t, they can be identified to their extension by 0 outside the rectangle $[l_1, l_2] \times [0, T]$, without lost of generality. The Fourier integral transform (FIT) with respect to the space variable x can be applied to $u_1(x,t)$ and will give:

$$\begin{aligned} F(\lambda, t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} u_1(x,t) \exp(-i\lambda x) dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{l_1}^{l_2} u_1(x,t) \exp(-i\lambda x) dx. \end{aligned}$$

The FIT applied to the transient term $\frac{\partial u_1}{\partial t}(x,t)$ leads to:

$$\begin{aligned} A(\lambda, t) &= \frac{1}{\sqrt{2\pi}} \int_{l_1}^{l_2} \frac{\partial u_1}{\partial t}(x,t) \exp(-i\lambda x) dx \\ &= \frac{1}{\sqrt{2\pi}} \frac{\partial}{\partial t} \int_{l_1}^{l_2} u_1(x,t) \exp(-i\lambda x) dx \\ &= \frac{\partial F}{\partial t}(\lambda, t). \end{aligned}$$

Similarly, the FIT of the diffusion term $a^2 \frac{\partial^2 u_1}{\partial x^2}(x,t)$ can be written as:

$$B(\lambda, t) = \frac{a^2}{\sqrt{2\pi}} \int_{l_1}^{l_2} \frac{\partial^2 u_1}{\partial x^2}(x,t) \exp(-i\lambda x) dx.$$

Using two successive integration by parts in which $\frac{\partial^2 u_1}{\partial x^2}(x,t)$ and $\frac{\partial u_1}{\partial t}(x,t)$ are respectively considered as derivatives, and $\exp(-i\lambda x)$ as primitive, $B(\lambda,t)$ can be expressed as:

$$\begin{aligned} B(\lambda,t) =& - a^2\lambda^2 F(\lambda,t) + \frac{a^2}{\sqrt{2\pi}}\left[\frac{\partial u_1}{\partial x}(l_2,t)\exp(-i\lambda l_2) - \frac{\partial u_1}{\partial x}(l_1,t)\exp(-i\lambda l_1)\right] \\ &+ \frac{a^2}{\sqrt{2\pi}}\left[i\lambda u_1(l_2,t)\exp(-i\lambda l_2) - i\lambda u_1(l_1,t)\exp(-i\lambda l_1)\right]. \end{aligned} \tag{8}$$

The FIT of the linear term $-bu_1(x,t)$ is simply

$$C(\lambda,t) = -bF(\lambda,t).$$

Now, according to the application of the FIT to the terms of the homogeneous equation, if $u_1(x,t)$ considered to be null outside $[l_1,l_2]\times[0,T]$ is the solution of Equation (7), then $F(\lambda,t)$ is the solution of the following equation: $A(\lambda,t) - B(\lambda,t) - C(\lambda,t) = 0$; i.e.,

$$\begin{aligned} \frac{\partial F}{\partial t}(\lambda,t) + (b + a^2\lambda^2)F(\lambda,t) =& \frac{a^2}{\sqrt{2\pi}}\left[\frac{\partial u_1}{\partial x}(l_2,t)\exp(-i\lambda l_2) - \frac{\partial u_1}{\partial x}(l_1,t)\exp(-i\lambda l_1)\right] \\ &+ \frac{a^2}{\sqrt{2\pi}}\left[i\lambda u_1(l_2,t)\exp(-i\lambda l_2) - i\lambda u_1(l_1,t)\exp(-i\lambda l_1)\right], \end{aligned} \tag{9}$$

for $\lambda \in \mathbb{R}$ and $0 \le t \le T$.

In order to integrate Equation (9), we multiply each member by $\exp[(b + a^2\lambda^2)t]$ and obtain:

$$\begin{aligned} &\frac{\partial}{\partial t}(F(\lambda,t)\exp[(b+a^2\lambda^2)t]) = \frac{a^2}{\sqrt{2\pi}}\exp[(b+a^2\lambda^2)t] \\ &\times\left[\frac{\partial u_1}{\partial x}(l_2,t)\exp(-i\lambda l_2) - \frac{\partial u_1}{\partial x}(l_1,t)\exp(-i\lambda l_1)\right] \\ &+\frac{a^2}{\sqrt{2\pi}}\exp[(b+a^2\lambda^2)t]\left[i\lambda u_1(l_2,t)\exp(-i\lambda l_2) - i\lambda u_1(l_1,t)\exp(-i\lambda l_1)\right]. \end{aligned} \tag{10}$$

Proceeding by integration of Equation (10) relatively to the time variable, from $\eta = 0$ to $\eta = t \le T$, we obtain:

$$\begin{aligned} &F(\lambda,t)\exp[(b+a^2\lambda^2)t] - F(\lambda,0) = \frac{a^2}{\sqrt{2\pi}} \\ &\times\int_0^t\left[\frac{\partial u_1}{\partial x}(l_2,\eta)\exp(-i\lambda l_2) - \frac{\partial u_1}{\partial x}(l_1,\eta)\exp(-i\lambda l_1)\right]\exp[(b+a^2\lambda^2)\eta]d\eta \\ &+\frac{a^2}{\sqrt{2\pi}}\int_0^t\left[i\lambda u_1(l_2,\eta)\exp(-i\lambda l_2) - i\lambda u_1(l_1,\eta)\exp(-i\lambda l_1)\right]\exp[(b+a^2\lambda^2)\eta]d\eta. \end{aligned} \tag{11}$$

Due to the initial condition (2), $F(\lambda,0) = F(\lambda,t=0)$ can be calculated as

$$F(\lambda,0) = \frac{1}{\sqrt{2\pi}}\int_{l_1}^{l_2}\varphi(\xi)\exp(-i\lambda\xi)d\xi, \tag{12}$$

where the variable of integration is replaced by ξ in order to avoid confusion. Equation (11) can then be rewritten as:

$$\begin{aligned}
&F(\lambda,t)=\frac{a^2}{\sqrt{2\pi}}\\
&\times\int_0^t\left[\frac{\partial u_1}{\partial x}(l_2,\eta)\exp(-i\lambda l_2)-\frac{\partial u_1}{\partial x}(l_1,\eta)\exp(-i\lambda l_1)\right]\exp[-(b+a^2\lambda^2)(t-\eta)]d\eta\\
&+\frac{a^2}{\sqrt{2\pi}}\int_0^t[i\lambda u_1(l_2,\eta)\exp(-i\lambda l_2)-i\lambda u_1(l_1,\eta)\exp(-i\lambda l_1)]\exp[-(b+a^2\lambda^2)(t-\eta)]d\eta\\
&+\frac{1}{\sqrt{2\pi}}\exp[-(b+a^2\lambda^2)t]\int_{l_1}^{l_2}\varphi(\xi)\exp(-i\lambda\xi)d\xi.
\end{aligned}\tag{13}$$

In order to obtain $u_1(x,t)$, the inversion formula (6) will be applied to the above function $F(\lambda,t)$ expressed by Equation (13), the Fourier variable λ running from $-\infty$ to $+\infty$. First, by the definition of the convolution of two functions, the three terms of the right hand side of Equation (13) can be respectively rewritten as:

$$\begin{aligned}
&F_1(\lambda,t)=\frac{a^2}{\sqrt{2\pi}}\\
&\times\int_0^t\left[\frac{\partial u_1}{\partial x}(l_2,t-\eta)\exp(-i\lambda l_2)-\frac{\partial u_1}{\partial x}(l_1,t-\eta)\exp(-i\lambda l_1)\right]\exp[-(b+a^2\lambda^2)\eta]d\eta,
\end{aligned}$$

$$\begin{aligned}
&F_2(\lambda,t)=\frac{a^2}{\sqrt{2\pi}}\\
&\times\int_0^t[i\lambda u_1(l_2,t-\eta)\exp(-i\lambda l_2)-i\lambda u_1(l_1,t-\eta)\exp(-i\lambda l_1)]\exp[-(b+a^2\lambda^2)\eta]d\eta,
\end{aligned}$$

and

$$F_3(\lambda,t)=\frac{1}{\sqrt{2\pi}}\exp[-(b+a^2\lambda^2)t]\int_{l_1}^{l_2}\varphi(\xi)\exp(-i\lambda\xi)d\xi.$$

Changing the order of integration due to the convergence of the integrals involved, the inverse I_1 of the first term F_1 is calculated as:

$$\begin{aligned}
I_1(x,t)=&\frac{a^2}{2\pi}\int_0^t\frac{\partial u_1}{\partial x}(l_2,t-\eta)\int_{-\infty}^{\infty}\exp(-i\lambda l_2)\exp(i\lambda x)\exp[-(b+a^2\lambda^2)\eta]d\lambda d\eta\\
&-\frac{a^2}{2\pi}\int_0^t\frac{\partial u_1}{\partial x}(l_1,t-\eta)\int_{-\infty}^{\infty}\exp(-i\lambda l_1)\exp(i\lambda x)\exp[-(b+a^2\lambda^2)\eta]d\lambda d\eta.
\end{aligned}$$

By means of computations, I_1 is reduced to:

$$\begin{aligned}
I_1(x,t)=&7\frac{a}{2\sqrt{\pi}}\int_0^t\frac{\partial u_1}{\partial x}(l_2,t-\eta)\left[\exp\left(-\frac{(l_2-x)^2}{4a^2\eta}\right)\right]\frac{\exp(-b\eta)}{\sqrt{\eta}}d\eta\\
&-\frac{a}{2\sqrt{\pi}}\int_0^t\frac{\partial u_1}{\partial x}(l_1,t-\eta)\left[\exp\left(-\frac{(l_1-x)^2}{4a^2\eta}\right)\right]\frac{\exp(-b\eta)}{\sqrt{\eta}}d\eta.
\end{aligned}\tag{14}$$

Similarly, I_2 is obtained as:

$$\begin{aligned}
I_2(x,t)=&\frac{a^2}{2\pi}\int_0^t u_1(l_2,t-\eta)\int_{-\infty}^{\infty}i\lambda\exp(-i\lambda l_2)\exp(i\lambda x)\exp[-(b+a^2\lambda^2)\eta]d\lambda d\eta\\
&-\frac{a^2}{2\pi}\int_0^t u_1(l_1,t-\eta)\int_{-\infty}^{\infty}i\lambda\exp(-i\lambda l_1)\exp(i\lambda x)\exp[-(b+a^2\lambda^2)\eta]d\lambda d\eta,
\end{aligned}$$

i.e.,

$$\begin{aligned} I_2(x,t) =& \frac{1}{4a\sqrt{\pi}}\int_0^t u_1(l_2,t-\eta)\left[(l_2-x)\exp\left(-\frac{(l_2-x)^2}{4a^2\eta}\right)\right]\frac{\exp(-b\eta)}{\eta^{3/2}}d\eta \\ &-\frac{1}{4a\sqrt{\pi}}\int_0^t u_1(l_1,t-\eta)\left[(l_1-x)\exp\left(-\frac{(l_1-x)^2}{4a^2\eta}\right)\right]\frac{\exp(-b\eta)}{\eta^{3/2}}d\eta. \end{aligned} \tag{15}$$

And, the inverse I_3 of F_3 is calculated as:

$$I_3(x,t) = \frac{\exp(-bt)}{2a\sqrt{\pi t}}\int_{l_1}^{l_2}\varphi(\xi)\left[\exp\left(-\frac{(\xi-x)^2}{4a^2t}\right)\right]d\xi. \tag{16}$$

From the calculations above, the expression of u_1 is deduced as $u_1(x,t) = I_1(x,t) + I_2(x,t) + I_3(x,t)$, that is:

$$\begin{aligned} u_1(x,t) =& \frac{a}{2\sqrt{\pi}}\int_0^t \frac{\partial u_1}{\partial x}(l_2,t-\eta)\left[\exp\left(-\frac{(l_2-x)^2}{4a^2\eta}\right)\right]\frac{\exp(-b\eta)}{\sqrt{\eta}}d\eta \\ &-\frac{a}{2\sqrt{\pi}}\int_0^t \frac{\partial u_1}{\partial x}(l_1,t-\eta)\left[\exp\left(-\frac{(l_1-x)^2}{4a^2\eta}\right)\right]\frac{\exp(-b\eta)}{\sqrt{\eta}}d\eta \\ &+\frac{1}{4a\sqrt{\pi}}\int_0^t u_1(l_2,t-\eta)\left[(l_2-x)\exp\left(-\frac{(l_2-x)^2}{4a^2\eta}\right)\right]\frac{\exp(-b\eta)}{\eta^{3/2}}d\eta \\ &-\frac{1}{4a\sqrt{\pi}}\int_0^t u_1(l_1,t-\eta)\left[(l_1-x)\exp\left(-\frac{(l_1-x)^2}{4a^2\eta}\right)\right]\frac{\exp(-b\eta)}{\eta^{3/2}}d\eta \\ &+\frac{\exp(-bt)}{2a\sqrt{\pi t}}\int_{l_1}^{l_2}\varphi(\xi)\left[\exp\left(-\frac{(\xi-x)^2}{4a^2t}\right)\right]d\xi. \end{aligned} \tag{17}$$

By hypothesis, $u_1(x,t)$ as expressed by the integral form (17), satisfies the homogeneous equation (7). It can also be verified that the initial condition (2) is satisfied by the right-hand side of the expression (17). Indeed, when $t \to 0$, $I_1(x,t)$ and $I_2(x,t)$ vanish, and $\lim_{t\to 0} u_1(x,t)$ reduces to $\lim_{t\to 0} I_3(x,t)$. Now, if we write

$$G(x,\xi,t) = \frac{\exp(-bt)}{2a\sqrt{\pi t}}\exp\left(-\frac{(\xi-x)^2}{4a^2t}\right),$$

then, for t and ξ given, $G(x,\xi,t)$ is related to the probability density of the normal or Gaussian distribution and:

$$\lim_{t\to 0}\int_{-\infty}^{\infty} G(x,\xi,t)d\xi = \lim_{t\to 0}\exp(-bt) = 1.$$

Moreover, $G(x,\xi,t) \to 0$ as $t \to 0$ at all points $(x,\xi) \in \mathbb{R}^2$, with the exception of the diagonal $x = \xi$, where it becomes infinitely large. Thus, $G(x,\xi,t)$ is analogous to Green's function and

$$\lim_{t\to 0} G(x,\xi,t) = \delta(x-\xi),$$

where $\delta(x-\xi)$ is the Dirac delta function. If $\mathbb{I}_{[l_1,l_2]}$ denotes the indicator function of interval $[l_1,l_2]$, one has:

$$\begin{aligned}
\lim_{t\to 0} u_1(x,t) &= \lim_{t\to 0} I_3(x,t), \\
&= \lim_{t\to 0} \int_{l_1}^{l_2} \varphi(\xi) G(x,\xi,t) d\xi, \\
&= \lim_{t\to 0} \int_{-\infty}^{\infty} \mathbb{I}_{[l_1,l_2]}(\xi)\varphi(\xi) G(x,\xi,t) d\xi, \\
&= \int_{-\infty}^{\infty} \mathbb{I}_{[l_1,l_2]}(\xi)\varphi(\xi)\delta(x-\xi) d\xi, \\
&= \varphi(x),
\end{aligned} \tag{18}$$

and the initial condition (2) is satisfied by the expression (17) of $u_1(x,t)$. In fact, this has just proved that the function I_3 satisfies the non-zero initial condition (2). It can be also checked that I_3 is an exact solution for the homogeneous equation (7). According to the Duhamel's principle, a solution of the nonhomogeneous equation (1) with zero initial condition can be written as:

$$u_2(x,t) = \int_0^t d\theta \int_{l_1}^{l_2} G(x,\xi,t-\theta) f(\xi,\theta) d\xi.$$

Finally, using the superposition principle, a solution u of the nonhomogeneous equation (1) subject to the initial condition (2) can be expressed as $u = u_1 + u_2$, namely by the equation:

$$\begin{aligned}
u(x,t) =& \frac{a}{2\sqrt{\pi}} \int_0^t \frac{\partial u}{\partial x}(l_2, t-\eta) \left[\exp\left(-\frac{(l_2-x)^2}{4a^2\eta}\right)\right] \frac{\exp(-b\eta)}{\sqrt{\eta}} d\eta \\
&- \frac{a}{2\sqrt{\pi}} \int_0^t \frac{\partial u}{\partial x}(l_1, t-\eta) \left[\exp\left(-\frac{(l_1-x)^2}{4a^2\eta}\right)\right] \frac{\exp(-b\eta)}{\sqrt{\eta}} d\eta \\
&+ \frac{1}{4a\sqrt{\pi}} \int_0^t u(l_2, t-\eta) \left[(l_2-x)\exp\left(-\frac{(l_2-x)^2}{4a^2\eta}\right)\right] \frac{\exp(-b\eta)}{\eta^{3/2}} d\eta \\
&- \frac{1}{4a\sqrt{\pi}} \int_0^t u(l_1, t-\eta) \left[(l_1-x)\exp\left(-\frac{(l_1-x)^2}{4a^2\eta}\right)\right] \frac{\exp(-b\eta)}{\eta^{3/2}} d\eta \\
&+ r(x,t),
\end{aligned} \tag{19}$$

where

$$\begin{aligned}
r(x,t) =& \frac{\exp(-bt)}{2a\sqrt{\pi t}} \int_{l_1}^{l_2} \varphi(\xi) \left[\exp\left(-\frac{(\xi-x)^2}{4a^2 t}\right)\right] d\xi \\
&+ \frac{1}{2a\sqrt{\pi}} \int_0^t d\theta \int_{l_1}^{l_2} \frac{\exp(-b(t-\theta))}{\sqrt{(t-\theta)}} \exp\left(-\frac{(\xi-x)^2}{4a^2(t-\theta)}\right) f(\xi,\theta) d\xi.
\end{aligned} \tag{20}$$

Note that u_1 is replaced by u under the integrals of Equation (19) according to the first assumption on u_1. Since as solutions, both functions verified the same boundary conditions (3)-(4), it can be assumed that their values and derivatives with respect to x coincide at the two boundaries l_1 and l_2.

In brief, the FIT method has allowed us to determine an expression in integrals form of the temperature field $u(x,t)$, that satisfies equation (1) and the initial condition (2). But, this expression (19) of u depends on the values of the same function u at the boundaries. The boundary conditions (3) and (4) will now be taken into account via the Laplace integral transform (abbreviated LIT).

4. Exact solution in the Laplace domain

If $f(t)$ is a function defined for $t \geq 0$, then its unilateral Laplace integral transform (LIT) is given in the complex p-plane by (see Herron and Foster (2008)):

$$F(p) = \mathcal{L}\{f(t)\} = \int_0^{\infty} f(t)\mathrm{e}^{-pt}dt, \tag{21}$$

provided that $f(t)$ be of exponential order, that is, there are constants C and σ so that $|f(t)| < C\mathrm{e}^{\sigma t}$, when t is sufficiently large. The inversion, from the Laplace domain p to the time domain t is given by the complex integral:

$$f(t) = \mathcal{L}^{-1}\{F(p)\} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} F(p)\mathrm{e}^{pt}dp, \tag{22}$$

where $\gamma > \sigma$ is chosen so that $F(p)$ converges absolutely on the real part of the p-line $\Re(p) = \gamma$, and $F(p)$ is analytic at the right of this line. Also, analytical expressions of LIT and inverses can be obtained for many usual functions by using some tables of transforms as in Poularikas (2018). In the case where p is real, as considered in the sequel, the inequality $p \geq \sigma$ needs to be satisfied. An important property relatively to the LIT is the convolution theorem (see Debnath and Bhatta (2014)):
Let $f(t)$ and $g(t)$ be functions defined for $t \geq 0$. If $\mathcal{L}\{f(t)\} = F(p)$ and $\mathcal{L}\{g(t)\} = G(p)$, then

$$\mathcal{L}\{f(t) * g(t)\} = F(p)G(p),$$

where $f(t) * g(t)$ is called the convolution of $f(t)$ and $g(t)$, and is defined by the integral

$$f(t) * g(t) = \int_0^t f(t-\eta)g(\eta)d\eta.$$

For the application of the LIT, all time-dependent functions involved in the problem (1)-(4) are assumed to be original, i.e., the transforms of these functions exist. Thus, the LIT of temperature distribution $u(x,t)$, source term $f(x,t)$, boundary functions $g_1(t)$ and $g_2(t)$ are respectively denoted in the Laplace domain by $U(x,p)$, $F(x,p)$, $G_1(p)$ and $G_2(p)$. Under these assumptions, the analog of the initial-boundary value problem (1)-(4) in the Laplace domain can be written in the form of an ordinary differential equation as follows:

$$-a^2\frac{d^2U}{dx^2}(x,p) + (b+p)U(x,p) = F(x,p) + \varphi(x), \tag{23}$$

since the Laplace transform of the time derivative of u is equal to:

$$\mathcal{L}\left\{\frac{\partial u}{\partial t}(x,t)\right\} = pU(x,p) - \varphi(x). \tag{24}$$

Equation (23) is subjected to the boundary conditions:

$$\alpha_1 U(l_1,p) + \beta_1\frac{dU}{dx}(l_1,p) = G_1(p), \quad \alpha_1, \beta_1 \in \mathbb{R}, \quad \alpha_1^2 + \beta_1^2 \neq 0, \tag{25}$$

$$\alpha_2 U(l_2,p) + \beta_2\frac{dU}{dx}(l_2,p) = G_2(p), \quad \alpha_2, \beta_2 \in \mathbb{R}, \quad \alpha_2^2 + \beta_2^2 \neq 0. \tag{26}$$

Due to the convolution theorem, the integral form expressed by Equation (19) can also be transformed by the LIT to be written as:

$$\begin{aligned} U(x,p) =& \frac{a}{2\sqrt{b+p}} \\ & \times \left[U_x(l_2,p) \exp\left(\frac{-(l_2-x)\sqrt{b+p}}{a} \right) - U_x(l_1,p) \exp\left(\frac{-(x-l_1)\sqrt{b+p}}{a} \right) \right] \\ & + \frac{1}{2} \left[U(l_2,p) \exp\left(\frac{-(l_2-x)\sqrt{b+p}}{a} \right) + U(l_1,p) \exp\left(\frac{-(x-l_1)\sqrt{b+p}}{a} \right) \right] \\ & + R(x,p), \end{aligned} \tag{27}$$

where U_x denotes the derivative $\frac{dU}{dx}$, and $R(x,p)$ stands for the Laplace transform of the remaining term $r(x,t)$ expressed by Equation (20), namely, $R(x,p) = \mathcal{L}\{r(x,t)\}$. Substituting respectively $x = l_1$ and then $x = l_2$ in the above expression of $U(x,p)$, we have the following equations:

$$\begin{aligned} & \frac{1}{2} U(l_1,p) + \frac{a}{2\sqrt{b+p}} U_x(l_1,p) - \frac{1}{2} \exp\left(\frac{-(l_2-l_1)\sqrt{b+p}}{a} \right) U(l_2,p) \\ & - \frac{a}{2\sqrt{b+p}} \exp\left(\frac{-(l_2-l_1)\sqrt{b+p}}{a} \right) U_x(l_2,p) = R(l_1,p), \end{aligned} \tag{28}$$

and

$$\begin{aligned} & -\frac{1}{2} \exp\left(\frac{-(l_2-l_1)\sqrt{b+p}}{a} \right) U(l_1,p) + \frac{a}{2\sqrt{b+p}} \exp\left(\frac{-(l_2-l_1)\sqrt{b+p}}{a} \right) U_x(l_1,p) \\ & + \frac{1}{2} U(l_2,p) - \frac{a}{2\sqrt{b+p}} U_x(l_2,p) = R(l_2,p), \end{aligned} \tag{29}$$

Now, Equations (25), (26), (28) and (29) form a system $(\mathcal{S})$ of four linear equations with four unknown functions that are $U(l_1,p)$, $U_x(l_1,p)$, $U(l_2,p)$ and $U(l_2,p)$. Therefore, the function $U(x,p)$ is the solution of the analog problem (23)-(26) in the Laplace domain, or equivalently, $u(x,t)$ is the solution of the problem (1)-(4) in the time domain, if and only if the system $(\mathcal{S})$ admits a unique solution. The determinant of the system is calculated as:

$$\det(\mathcal{S}) = \begin{vmatrix} \alpha_1 & \beta_1 & 0 & 0 \\ 0 & 0 & \alpha_2 & \beta_2 \\ \frac{1}{2} & \frac{a}{2\sqrt{b+p}} & \frac{-\chi(l_2-l_1,p)}{2} & \frac{-a\chi(l_2-l_1,p)}{2\sqrt{b+p}} \\ \frac{-\chi(l_2-l_1,p)}{2} & \frac{a\chi(l_2-l_1,p)}{2\sqrt{b+p}} & \frac{1}{2} & \frac{-a}{2\sqrt{b+p}} \end{vmatrix}, \tag{30}$$

where the function $\chi(x,p)$ is defined as $\chi(x,p) = \exp\left(\frac{-x\sqrt{b+p}}{a} \right)$. This determinant is reduced by computations to:

$$\begin{aligned} \det(\mathcal{S}) = & -\frac{1}{4}(a^2\alpha_1\alpha_2 - a\alpha_1\beta_2 + a\alpha_2\beta_1) \frac{\chi(2(l_2-l_1),p)}{b+p} + \frac{1}{4}\beta_1\beta_2\chi(2(l_2-l_1),p) \\ & + \frac{1}{4} \frac{a^2\alpha_1\alpha_2 + a\alpha_1\beta_2 - a\alpha_2\beta_1}{b+p} - \frac{1}{4}\beta_1\beta_2. \end{aligned} \tag{31}$$

Thus, the determinant of the system is null if and only if, all the coefficients of the functions of p appearing in Equation (31) are null. This will imply:

$$\begin{cases} a^2\alpha_1\alpha_2 - a\alpha_1\beta_2 + a\alpha_2\beta_1 = 0, \\ a^2\alpha_1\alpha_2 + a\alpha_1\beta_2 - a\alpha_2\beta_1 = 0, \\ \beta_1\beta_2 = 0. \end{cases} \tag{32}$$

Using the third equation $\beta_1\beta_2 = 0$, the system (32) can be split into two systems, since it is equivalent to:

$$\begin{cases} a^2\alpha_1\alpha_2 - a\alpha_1\beta_2 = 0, \\ a^2\alpha_1\alpha_2 + a\alpha_1\beta_2 = 0, \\ \beta_1 = 0, \end{cases} \tag{33}$$

or

$$\begin{cases} a^2\alpha_1\alpha_2 + a\alpha_2\beta_1 = 0, \\ a^2\alpha_1\alpha_2 - a\alpha_2\beta_1 = 0, \\ \beta_2 = 0. \end{cases} \tag{34}$$

Now $a \neq 0$, and by adding and subtracting its two first equations, the sub-system (33) can be shown equivalent to:

$$\begin{cases} \alpha_1\alpha_2 = 0, \\ \alpha_1\beta_2 = 0, \\ \beta_1 = 0. \end{cases} \tag{35}$$

The system (35) leads to a contradiction with the hypotheses on the coefficients $\alpha_1,\ \beta_1,\ \alpha_2$ and β_2, since it implies that $\alpha_1 = \beta_1 = 0$ or $\alpha_2 = \beta_2 = 0$. The same contradiction is reached when trying to solve the sub-system (34). So, $\det(\mathcal{S}) \neq 0$ in all cases. Consequently, the analog boundary value problem in the Laplace domain (Equations (23)-(26)), or equivalently the problem in the time domain (Equations (1)-(4)), admits a unique solution whenever $\alpha_1^2+\beta_1^2 \neq 0$, and $\alpha_2^2+\beta_2^2 \neq 0$. The exact solution in the Laplace domain of the problem (1)-(4) is $U(x,p)$, given by Equation (27), with the functions $U(l_1,p)$, $U(l_2,p)$, $U_x(l_1,p)$ and $U_x(l_2,p)$ expressed by using determinants as:

$$U(l_1,p) = \frac{\begin{vmatrix} G_1(p) & \beta_1 & 0 & 0 \\ G_2(p) & 0 & \alpha_2 & \beta_2 \\ R(l_1,p) & \dfrac{a}{2\sqrt{b+p}} & \dfrac{-\chi(l_2-l_1,p)}{2} & \dfrac{-a\chi(l_2-l_1,p)}{2\sqrt{b+p}} \\ R(l_2,p) & \dfrac{a\chi(l_2-l_1,p)}{2\sqrt{b+p}} & \dfrac{1}{2} & \dfrac{-a}{2\sqrt{b+p}} \end{vmatrix}}{\det(\mathcal{S})}, \tag{36}$$

$$U_x(l_1,p) = \frac{\begin{vmatrix} \alpha_1 & G_1(p) & 0 & 0 \\ 0 & G_2(p) & \alpha_2 & \beta_2 \\ \dfrac{1}{2} & R(l_1,p) & \dfrac{-\chi(l_2-l_1,p)}{2} & \dfrac{-a\chi(l_2-l_1,p)}{2\sqrt{b+p}} \\ \dfrac{-\chi(l_2-l_1,p)}{2} & R(l_2,p) & \dfrac{1}{2} & \dfrac{-a}{2\sqrt{b+p}} \end{vmatrix}}{\det(\mathcal{S})}, \tag{37}$$

$$U(l_2,p)=\frac{\begin{vmatrix}\alpha_1 & \beta_1 & G_1(p) & 0\\ 0 & 0 & G_2(p) & \beta_2\\ \dfrac{1}{2} & \dfrac{a}{2\sqrt{b+p}} & R(l_1,p) & \dfrac{-a\chi(l_2-l_1,p)}{2\sqrt{b+p}}\\ \dfrac{-\chi(l_2-l_1,p)}{2} & \dfrac{a\chi(l_2-l_1,p)}{2\sqrt{b+p}} & R(l_2,p) & \dfrac{-a}{2\sqrt{b+p}}\end{vmatrix}}{\det(\mathcal{S})}, \tag{38}$$

and

$$U_x(l_x,p)=\frac{\begin{vmatrix}\alpha_1 & \beta_1 & 0 & G_1(p)\\ 0 & 0 & \alpha_2 & G_2(p)\\ \dfrac{1}{2} & \dfrac{a}{2\sqrt{b+p}} & \dfrac{-\chi(l_2-l_1,p)}{2} & R(l_1,p)\\ \dfrac{-\chi(l_2-l_1,p)}{2} & \dfrac{a\chi(l_2-l_1,p)}{2\sqrt{b+p}} & \dfrac{1}{2} & R(l_2,p)\end{vmatrix}}{\det(\mathcal{S})}. \tag{39}$$

In brief, the exact solution in the Laplace domain for the boundary value problem (1)-(4) is expressed in a unified way by the function $U(x,p)$ given by Equation (27) together with those given in Equations (36)-(39). Exact series solutions in the time domain of such linear boundary value problems are well established by using the Fourier decomposition method, as in Anani (2022) for example. Thus, the exact Laplace transform of those series solutions performed via the Sturm-Liouville theory, can be recovered in a closed form by the expression of the function $U(x,p)$.

5. Analytical approximations in short time limits

In this section, we aim to derive from the exact operational solution (27), approximate analytical solutions to the problem (1)-(4) at the earliest times of the process. As in many schemes in numerical analysis (see Fu et al. (2018) and Izadi and Yuzbasi (2022) among others), we assume a subdivision of the time interval $[0,T]$, such that the magnitude of the dimensionless time step Δt is sufficiently small, for example $\Delta t \leq 10^{-2}$. These approximate analytical solutions are valid during the first time step of the reaction–diffusion process, namely for $t\in[0,\Delta t]$.

The limiting case of a short time duration (Δt tending to 0) corresponds to a very large value of the Laplace domain variable (p tending to $+\infty$). In order to minimize calculations, some prior simplifications can be done on the determinant of the system $\det(\mathcal{S})$ before deriving the asymptotic expansions of the solution, and the related truncated expansions in the time domain. For $y>0$, the inverse Laplace transform of $\chi(y,p)$ is written as:

$$\begin{aligned}\mathcal{L}^{-1}\{\chi(y,p)\}&=\mathcal{L}^{-1}\left\{\exp\left(\frac{-y\sqrt{b+p}}{a}\right)\right\}\\ &=\frac{1}{2}\frac{y\exp\left(-bt-\frac{y^2}{4a^2t}\right)}{a\sqrt{\pi}t^{3/2}}\\ &=\mu(t).\end{aligned} \tag{40}$$

When $p \to \infty$, $\chi(y,p)$ is negligible compared to $1/p^n$ for all positive integers $n \geq 1$, whereas when $t \to 0$, the inverse Laplace transform $\mu(t)$ is negligible compared to t^n. Indeed,

$$\lim_{p\to\infty} p^n \chi(y,p) = \lim_{t\to 0} \frac{\mu(t)}{t^n} = 0,$$

implying $\chi(y,p) = o(1/p^n)$ at $p = \infty$, and $\mu(t) = o(t^n)$ at $t = 0$, where the Little-o is the asymptotic notation. Again, these relations are valid for arbitrary order $n \geq 1$. When $p \to \infty$, the term $\chi(l_2 - l_1, p)$ and its asymptotic expansions are negligible in the Laplace domain, as well as their inverses in the time domain when $t \to 0$. Therefore, when p is sufficiently large, the asymptotic solution can be obtained by using the following determinant of the system formed by Equations (25), (26), (28) and (29):

$$\det(\mathcal{S}^a) = \begin{vmatrix} \alpha_1 & \beta_1 & 0 & 0 \\ 0 & 0 & \alpha_2 & \beta_2 \\ \dfrac{1}{2} & \dfrac{a}{2\sqrt{b+p}} & 0 & 0 \\ 0 & 0 & \dfrac{1}{2} & \dfrac{-a}{2\sqrt{b+p}} \end{vmatrix}. \tag{41}$$

Assuming that the function $\chi(y,p)$ is negligible for all $y > 0$ when p is sufficiently large, we can deduce an asymptotic expansion of the exact solution $U(x,p)$ from Equation (27) for $l_1 < x < l_2$ to any order $n \geq 1$:

$$U^a(x,p) = R(x,p) + o\left(\frac{1}{p^n}\right). \tag{42}$$

For $0 < t < \Delta t$, Equation (42) corresponds to the following truncated expansion $u^a(x,t)$ of the solution $u(x,t)$ in the time domain:

$$u^a(x,t) = r(x,t) + o\left((\Delta t)^n\right), \tag{43}$$

where $r(x,t)$ is given by equation (20) and $o\left((\Delta t)^n\right)$ can be considered as an upper bound of the truncation error $o\left(t^n\right)$. But, at $x = l_1$ and $x = l_2$, some refined asymptotic expansions in the Laplace domain $U^a(l_1,p)$, $U^a(l_2,p)$, $U^a_x(l_1,p)$, and $U^a_x(l_2,p)$ can be obtained for the boundary-related functions $U(l_1,p)$, $U(l_2,p)$, $U_x(l_1,p)$, and $U_x(l_2,p)$. We distinguish four different cases with respect to the values of the two coefficients β_1 and β_2. For each case, the analytical approximations in the time domain, namely $u^a(l_1,t)$, $u^a(l_2,t)$, $u^a_x(l_1,t)$, and $u^a_x(l_2,t)$ are also given for $t \in [0, \Delta t]$. Although this can be improved, the asymptotic expansions retained here are at the second order, while the corresponding approximations in the time domain are given at the first order.

- Case $\beta_1\beta_2 \neq 0$
  (1) For the approximation of $U(l_1,p)$ and $u(l_1,t)$, the system is solved by using the reduced determinant $\det(\mathcal{S}^a)$ expressed in formula (41) leads to:

$$U(l_1,p) = -\frac{a}{\beta_1\sqrt{b+p} - a\alpha_1} G_1(p) + \frac{2\sqrt{b+p}\beta_1}{\beta_1\sqrt{b+p} - a\alpha_1} R(l_1,p),$$

and if the Big-O denotes the asymptotic notation, one has for the above coefficient of $G_1(p)$:

$$-\frac{a}{\beta_1\sqrt{b+p}-a\alpha_1}=U^{1l_1}(p)+O\left(\frac{1}{p^2}\right),$$

and for that of $R(l_1,p)$:

$$\frac{2\sqrt{b+p}\beta_1}{\beta_1\sqrt{b+p}-a\alpha_1}=2+U^{2l_1}(p)+O\left(\frac{1}{p^2}\right),$$

where

$$U^{1l_1}(p)=-\frac{a}{\beta_1}\sqrt{\frac{1}{p}}-\frac{a^2\alpha_1}{{\beta_1}^2p}-\frac{1}{\beta_1}\left(-\frac{1}{2}\,ab+\frac{a^3{\alpha_1}^2}{{\beta_1}^2}\right)\left(\frac{1}{p}\right)^{3/2},$$

and

$$U^{2l_1}(p)=2\,\frac{a\alpha_1}{\beta_1}\sqrt{\frac{1}{p}}+2\,\frac{a^2{\alpha_1}^2}{{\beta_1}^2p}+\,\frac{2}{\beta_1}\left(-\frac{1}{2}\,a\alpha_1b+\frac{a^3{\alpha_1}^3}{{\beta_1}^2}\right)\left(\frac{1}{p}\right)^{3/2}.$$

According to the properties of the LIT, $G_1(p)$ and $R(l_1,p)$ are bounded functions in the Laplace domain for $p>0$, and an asymptotic expansion of the solution $U(l_1,p)$ can then be written as:

$$U^a(l_1,p)=U^{1l_1}(p)G_1(p)+U^{2l_1}(p)R(l_1,p)+2R(l_1,p)+O\left(\frac{1}{p^2}\right).$$

The corresponding analytical approximation in the time domain during a short time step $t\in[0,\Delta t]$ reads

$$u^a(l_1,t)=u^{1l_1}(t)*g_1(t)+u^{2l_1}(t)*r(l_1,t)+2r(l_1,t)+O\left(\Delta t\right),$$

where $O\left(\Delta t\right)$ instead of $O\left(t\right)$ is considered as an upper bound of the truncation error of the approximation, $*$ denotes the convolution product, $r(l_1,t)$ is calculated through equation (20), $g_1(t)$ is the given function related to the boundary $x=l_1$,

$$u^{1l_1}(t)=\mathcal{L}^{-1}\{U^{1l_1}(p)\}=-\frac{a^2\alpha_1}{{\beta_1}^2}-\frac{a}{\sqrt{\pi\,t}\beta_1}+\frac{\left(-2\,a^2{\alpha_1}^2+b{\beta_1}^2\right)a}{{\beta_1}^3}\sqrt{\frac{t}{\pi}},$$

and

$$u^{2l_1}(t)=\mathcal{L}^{-1}\{U^{2l_1}(p)\}=2\,\frac{a\alpha_1}{\sqrt{\pi\,t}\beta_1}+2\,\frac{a^2{\alpha_1}^2}{{\beta_1}^2}+2\,\frac{a\alpha_1\left(2\,a^2{\alpha_1}^2-b{\beta_1}^2\right)}{{\beta_1}^3}\sqrt{\frac{t}{\pi}}.$$

(2) For the approximation of $U(l_2,p)$ and $u(l_2,t)$, similar calculations as above give:

$$U(l_2,p)=\frac{G_2\left(p\right)a}{\beta_2\sqrt{b+p}+\alpha_2a}+2\,\frac{R\left(l_2,p\right)\sqrt{b+p}\beta_2}{\beta_2\sqrt{b+p}+\alpha_2a},$$

and

$$U^a(l_2,p)=U^{1l_2}(p)G_2(p)+U^{2l_2}(p)R(l_2,p)+2R(l_2,p)+O\left(\frac{1}{p^2}\right),$$

where

$$U^{1l_2}(p)=\frac{a}{\beta_2}\sqrt{\frac{1}{p}}-\frac{a^2\alpha_2}{{\beta_2}^2p}+\frac{1}{\beta_2}\left(-\frac{1}{2}\,ab+\frac{a^3{\alpha_2}^2}{{\beta_2}^2}\right)\left(\frac{1}{p}\right)^{3/2},$$

and

$$U^{2l_2}(p) = -2\,\frac{\alpha_2 a}{\beta_2}\sqrt{\frac{1}{p}} + 2\,\frac{a^2{\alpha_2}^2}{{\beta_2}^2 p} + \frac{2}{\beta_2}\left(\frac{1}{2}\,\alpha_2 ab - \frac{{\alpha_2}^3 a^3}{{\beta_2}^2}\right)\left(\frac{1}{p}\right)^{3/2}.$$

The corresponding approximation in the time domain during a short time step $t \in [0, \Delta t]$ is:

$$u^a(l_2,t) = u^{1l_2}(t) * g_2(t) + u^{2l_2}(t) * r(l_2,t) + 2r(l_2,t) + O\left(\Delta t\right),$$

where

$$u^{1l_2}(t) = \mathcal{L}^{-1}\{U^{1l_2}(p)\} = -\frac{a^2\alpha_2}{{\beta_2}^2} + \frac{a}{\sqrt{\pi\,t}\beta_2} + \frac{a\left(2\,a^2{\alpha_2}^2 - b{\beta_2}^2\right)}{{\beta_2}^3}\sqrt{\frac{t}{\pi}},$$

and

$$u^{2l_2}(t) = \mathcal{L}^{-1}\{U^{2l_2}(p)\} = -2\,\frac{\alpha_2 a}{\sqrt{\pi\,t}\beta_2} + 2\,\frac{a^2{\alpha_2}^2}{{\beta_2}^2} + 2\,\frac{\left(-2\,a^2{\alpha_2}^2 + b{\beta_2}^2\right)\alpha_2 a}{{\beta_2}^3}\sqrt{\frac{t}{\pi}}.$$

(3) Likewise, for the analytical approximations of $U_x(l_1,p)$ and $u_x(l_1,t)$, the results are:

$$U_x(l_1,p) = \frac{\sqrt{b+p}\,G_1\left(p\right)}{\beta_1\sqrt{b+p} - a\alpha_1} - 2\,\frac{\alpha_1\sqrt{b+p}\,R\left(l_1,p\right)}{\beta_1\sqrt{b+p} - a\alpha_1},$$

and

$$U_x^a(l_1,p) = {\beta_1}^{-1}\,G_1(p) + U_x^{1l_1}(p)G_1(p) + U_x^{2l_1}(p)R(l_1,p) - 2\,\frac{\alpha_1}{\beta_1}\,R(l_1,p) + O\left(\frac{1}{p^2}\right),$$

where

$$U_x^{1l_1}(p) = \frac{a\alpha_1}{{\beta_1}^2}\sqrt{\frac{1}{p}} + \frac{a^2{\alpha_1}^2}{{\beta_1}^3 p} + \frac{1}{\beta_1}\left(-\frac{1}{2}\,\frac{a\alpha_1 b}{\beta_1} + \frac{a^3{\alpha_1}^3}{{\beta_1}^3}\right)\left(\frac{1}{p}\right)^{3/2},$$

and

$$U_x^{2l_1}(p) = -2\,\frac{{\alpha_1}^2 a}{{\beta_1}^2}\sqrt{\frac{1}{p}} - 2\,\frac{{\alpha_1}^3 a^2}{{\beta_1}^3 p} - \frac{2}{\beta_1}\left(-\frac{1}{2}\,\frac{{\alpha_1}^2 ab}{\beta_1} + \frac{{\alpha_1}^4 a^3}{{\beta_1}^3}\right)\left(\frac{1}{p}\right)^{3/2}.$$

The related approximation in the time domain during a short time step $t \in [0, \Delta t]$ is:

$$u_x^a(l_1,t) = {\beta_1}^{-1}\,g_1(t) + u_x^{1l_1}(t) * g_1(t) + u_x^{2l_1}(t) * r(l_1,t) - 2\,\frac{\alpha_1}{\beta_1}\,r(l_1,t) + O\left(\Delta t\right),$$

where

$$u_x^{1l_1}(t) = \mathcal{L}^{-1}\{U_x^{1l_1}(p)\} = \frac{a\alpha_1}{\sqrt{\pi\,t}{\beta_1}^2} + \frac{a^2{\alpha_1}^2}{{\beta_1}^3} + \frac{a\alpha_1\left(2\,a^2{\alpha_1}^2 - b{\beta_1}^2\right)}{{\beta_1}^4}\sqrt{\frac{t}{\pi}},$$

and

$$u_x^{2l_1}(t) = \mathcal{L}^{-1}\{U_x^{2l_1}(p)\} = -2\,\frac{{\alpha_1}^2 a}{\sqrt{\pi\,t}{\beta_1}^2} - 2\,\frac{{\alpha_1}^3 a^2}{{\beta_1}^3} + 2\,\frac{\left(-2\,a^2{\alpha_1}^2 + b{\beta_1}^2\right){\alpha_1}^2 a}{{\beta_1}^4}\sqrt{\frac{t}{\pi}}.$$

(4) Finally for approximating $U_x(l_2,p)$ and $u_x(l_2,t)$, calculations give:

$$U_x(l_2,p) = \frac{\sqrt{b+p}\,G_2\left(p\right)}{\beta_2\sqrt{b+p} + \alpha_2 a} - 2\,\frac{\alpha_2\sqrt{b+p}\,R\left(l_2,p\right)}{\beta_2\sqrt{b+p} + \alpha_2 a},$$

and

$$U_x^a(l_2,p) = {\beta_2}^{-1}\, G_2(p) + U_x^{1l_2}(p)G_2(p) + U_x^{2l_2}(p)R(l_2,p) - 2\,\frac{\alpha_2}{\beta_2}\,R(l_2,p) + O\left(\frac{1}{p^2}\right),$$

where

$$U_x^{1l_2}(p) = -\frac{\alpha_2 a}{{\beta_2}^2}\sqrt{\frac{1}{p}} + \frac{a^2{\alpha_2}^2}{{\beta_2}^3 p} + \frac{1}{\beta_2}\left(\frac{1}{2}\,\frac{\alpha_2 ab}{\beta_2} - \frac{{\alpha_2}^3 a^3}{{\beta_2}^3}\right)\left(\frac{1}{p}\right)^{3/2},$$

and

$$U_x^{2l_2}(p) = 2\,\frac{{\alpha_2}^2 a}{{\beta_2}^2}\sqrt{\frac{1}{p}} - 2\,\frac{{\alpha_2}^3 a^2}{{\beta_2}^3 p} - \frac{2}{\beta_2}\left(\frac{1}{2}\,\frac{{\alpha_2}^2 ab}{\beta_2} - \frac{{\alpha_2}^4 a^3}{{\beta_2}^3}\right)\left(\frac{1}{p}\right)^{3/2}.$$

The corresponding truncation in the time domain during a short time step $t \in [0, \Delta t]$ is written as:

$$u_x^a(l_2,t) = {\beta_2}^{-1}\, g_2(t) + u_x^{1l_2}(t) * g_2(t) + u_x^{2l_2}(t) * r(l_2,t) - 2\,\frac{\alpha_2}{\beta_2}\,r(l_2,t) + O\left(\Delta t\right),$$

where

$$u_x^{1l_2}(t) = \mathcal{L}^{-1}\{U_x^{1l_2}(p)\} = -\frac{\alpha_2 a}{\sqrt{\pi\, t}{\beta_2}^2} + \frac{a^2{\alpha_2}^2}{{\beta_2}^3} + \frac{\left(-2\,a^2{\alpha_2}^2 + b{\beta_2}^2\right)\alpha_2 a}{{\beta_2}^4}\sqrt{\frac{t}{\pi}},$$

and

$$u_x^{2l_2}(t) = \mathcal{L}^{-1}\{U_x^{2l_2}(p)\} = 2\,\frac{{\alpha_2}^2 a}{\sqrt{\pi\, t}{\beta_2}^2} - 2\,\frac{{\alpha_2}^3 a^2}{{\beta_2}^3} + 2\,\frac{{\alpha_2}^2 a\left(2\,a^2{\alpha_2}^2 - b{\beta_2}^2\right)}{{\beta_2}^4}\sqrt{\frac{t}{\pi}}.$$

- Case $\beta_1 \neq 0$ and $\beta_2 = 0$

  (1) As in the first case, the system is solved using the reduced determinant $\det(\mathcal{S}^a)$. This leads to:

  $$U(l_1,p) = -\frac{G_1\left(p\right)a}{\beta_1\sqrt{b+p} - a\alpha_1} + 2\,\frac{\sqrt{b+p}\,R\left(l_1,p\right)\beta_1}{\beta_1\sqrt{b+p} - a\alpha_1},$$

  and $U(l_1,p)$ and its asymptotic expansion $U^a(l_1,p)$, as well as $u(l_1,t)$ and its truncation expansion $u^a(l_1,t)$ are the same as in the case $\beta_1\beta_2 \neq 0$, point (1).

  (2) Concerning $U(l_2,p)$ and $u(l_2,t)$, the results are reduced to:

  $$U(l_2,p) = \frac{G_2\left(p\right)}{\alpha_2},$$

  and the exact inverse in the time domain during a short time step $t \in [0, \Delta t]$ is:

  $$u(l_2,t) = \frac{g_2(t)}{\alpha_2}.$$

  (3) The functions $U_x(l_1,p)$, $u_x(l_1,t)$ and their asymptotic and truncation expansions $U_x^a(l_1,p)$, $u_x^a(l_1,t)$ are the same as in the case $\beta_1\beta_2 \neq 0$, point (3), since

  $$U_x(l_1,p) = \frac{\sqrt{b+p}\,G_1\left(p\right)}{\beta_1\sqrt{b+p} - a\alpha_1} - 2\,\frac{\alpha_1\sqrt{b+p}\,R\left(l_1,p\right)}{\beta_1\sqrt{b+p} - a\alpha_1}.$$

(4) About $U_x(l_2,p)$ and $u_x(l_2,t)$, one has:

$$U_x(l_2,p) = \frac{\sqrt{b+p}\, G_2\left(p\right)}{\alpha_2 a} - 2\,\frac{\sqrt{b+p}\, R\left(l_2,p\right)}{a},$$

and the calculations will need here the assumptions that $r(l_2,t)$ and $g_2(t)$ are once piecewise differentiable on $t \in [0,\Delta t]$, and their respective derivatives $r'(l_2,t)$ and $g_2'(t)$ verify the relations: $r(l_2,t) - r(l_2,0) = \int_0^t r'(l_2,\tau)d\tau$, and $g_2(t) - g_2(0) = \int_0^t g_2'(\tau)d\tau$. Their derivatives are assumed to be of exponential order, so that if $R^q(l_2,p)$ and $G_2^q(p)$ are the Laplace transforms of the derivatives, we have according to the transform properties:

$$R(l_2,p) = \mathcal{L}\left\{\int_0^t r'(l_2,\tau)d\tau\right\} + \frac{r(l_2,0)}{p} = \frac{R^q(l_2,p)}{p} + \frac{r(l_2,0)}{p},$$

and

$$G_2(p) = \mathcal{L}\left\{\int_0^t g_2'(\tau)d\tau\right\} + \frac{g_2(0)}{p} = \frac{G_2^q(p)}{p} + \frac{g_2(0)}{p}.$$

Then, $U_x(l_2,p)$ can be rewritten as

$$U_x(l_2,p) = \frac{\sqrt{b+p}\ \left(G_2^q(p) + g_2(0)\right)}{\alpha_2 a\, p} - 2\,\frac{\sqrt{b+p}\ \left(R^q(l_2,p) + r(l_2,0)\right)}{a\, p},$$

and

$$U_x^a(l_2,p) = U_x^{l_2}(p)\left(\frac{G_2^q(p) + g_2(0)}{a\alpha_2} - 2\frac{R^q(l_2,p) + r(l_2,0)}{a}\right) + O\left(\frac{1}{p^{5/2}}\right),$$

where

$$U_x^{l_2}(p) = \sqrt{\frac{1}{p}} + \frac{b}{2}\left(\frac{1}{p}\right)^{3/2}.$$

In this sub-case, a truncation expansion in the time domain during a short time step $t \in [0,\Delta t]$ is:

$$u_x^a(l_2,t) = \frac{1}{a}\left(\frac{g_2(0)}{\alpha_2} - 2r(l_2,0)\right)u_x^{l_2}(t) + \frac{1}{a}u_x^{l_2}(t) * \left(\frac{g_2'(t)}{\alpha_2} - 2r'(l_2,t)\right) + O\left(\Delta t\right),$$

where

$$u_x^{l_2}(t) = \mathcal{L}^{-1}\{U_x^{l_2}(p)\} = \frac{bt+1}{\sqrt{\pi\, t}}.$$

- Case $\beta_1 = 0$ and $\beta_2 \neq 0$

(1) The function $U(l_1,p)$ and its inverse $u(l_1,t)$ in the time domain ($t \in [0,\Delta t]$) are respectively:

$$U(l_1,p) = \frac{G_1\left(p\right)}{\alpha_1},$$

and

$$u(l_1,t) = \frac{g_1(t)}{\alpha_1}.$$

(2) For $U(l_2,p)$ and $u(l_2,t)$, the results are identical to those obtained in the case $\beta_1\beta_2 \neq 0$, point (2) since:

$$U(l_2,p) = \frac{G_2\,(p)\,a}{\beta_2\sqrt{b+p}+\alpha_2 a} + 2\,\frac{R\,(l_2,p)\,\sqrt{b+p}\beta_2}{\beta_2\sqrt{b+p}+\alpha_2 a},$$

(3) For $U_x(l_1,p)$ and $u_x(l_1,t)$, calculations give:

$$U_x(l_1,p) = -\frac{\sqrt{b+p}\,G_1\,(p)}{\alpha_1 a} + 2\,\frac{\sqrt{b+p}\,R\,(l_1,p)}{a},$$

and the assumptions are that $r(l_1,t)$ and $g_1(t)$ are once piece-wise differentiable for $t \in [0,\Delta t]$. The respective derivatives $r'(l_1,t)$ and $g_1'(t)$ can be related to the original functions by the relations: $r(l_1,t)-r(l_1,0) = \int_0^t r'(l_1,\tau)d\tau$, and $g_1(t)-g_1(0) = \int_0^t g_1'(\tau)d\tau$. Since the derivatives are assumed to be of exponential order, their Laplace transforms $R^q(l_1,p)$ and $G_1^q(p)$ verify the properties:

$$R(l_1,p) = \mathcal{L}\left\{\int_0^t r'(l_1,\tau)d\tau\right\} + \frac{r(l_1,0)}{p} = \frac{R^q(l_1,p)}{p} + \frac{r(l_1,0)}{p},$$

and

$$G_1(p) = \mathcal{L}\left\{\int_0^t g_1'(\tau)d\tau\right\} + \frac{g_1(0)}{p} = \frac{G_1^q(p)}{p} + \frac{g_1(0)}{p}.$$

Then,

$$U_x(l_1,p) = -\frac{\sqrt{b+p}\;(G_1^q(p)+g_1(0))}{\alpha_1 a\,p} + 2\,\frac{\sqrt{b+p}\;(R^q(l_1,p)+r(l_1,0))}{a\,p},$$

and

$$U_x^a(l_1,p) = U_x^{l_1}(p)\left(-\frac{G_1^q(p)+g_1(0)}{a\alpha_1} + 2\frac{R^q(l_1,p)+r(l_1,0)}{a}\right) + O\left(\frac{1}{p^{5/2}}\right),$$

where

$$U_x^{l_1}(p) = \sqrt{\frac{1}{p}} + \frac{b}{2}\left(\frac{1}{p}\right)^{3/2}.$$

A truncation expansion in the time domain during a short time step $t \in [0,\Delta t]$ corresponds to:

$$u_x^a(l_1,t) = \frac{1}{a}\left(-\frac{g_1(0)}{\alpha_1} + 2r(l_1,0)\right)u_x^{l_1}(t) + \frac{1}{a}u_x^{l_1}(t) * \left(-\frac{g_1'(t)}{\alpha_1} + 2r'(l_1,t)\right) + O\left(\Delta t\right),$$

where

$$u_x^{l_1}(t) = \frac{bt+1}{\sqrt{\pi\,t}}.$$

(4) Since:

$$U_x(l_2,p) = \frac{\sqrt{b+p}\,G_2\,(p)}{\beta_2\sqrt{b+p}+\alpha_2 a} - 2\,\frac{\alpha_2\sqrt{b+p}\,R\,(l_2,p)}{\beta_2\sqrt{b+p}+\alpha_2 a},$$

the results for $U_x(l_2,p)$ and $u_x(l_2,t)$ in this sub-case are identical to those obtained in the case $\beta_1\beta_2 \neq 0$, point (4).

- <u>Case $\beta_1 = 0$ and $\beta_2 = 0$</u>
  (1) The expression of $U(l_1, p)$ and its inverse $u(l_1, t)$ are identical to those obtained in the case $\beta_1 = 0$ and $\beta_2 \neq 0$, point (1).
  (2) The expression of $U(l_2, p)$ and of its inverse $u(l_2, t)$ are the same as those obtained in the case $\beta_1 \neq 0$ and $\beta_2 = 0$, point (2).
  (3) The expansions of $U_x(l_1, p)$ and of its inverse $u_x(l_1, t)$ are equal to those obtained in the case $\beta_1 = 0$ and $\beta_2 \neq 0$, point (3).
  (4) The expansions of $U_x(l_2, p)$ and of its inverse $u_x(l_2, t)$ are identical to those obtained in the case $\beta_1 \neq 0$ and $\beta_2 = 0$, point (4).

6. Discussion of the results and applications

First, the results mentioned above can be compared and discussed on a specific example. Some powerful PDE toolbox functions exist in MATLAB software, for example, and scripts based on Gauss-Seidel and finite difference methods are available online. The curves obtained by the Fourier decomposition method, when using partial sums of the infinite series solution, and those obtained from the present method of approximate analytical solutions in short time limits, can both be compared to Matlab solution curves, when using toolbox functions. For this purpose, the example titled "Example 6.1." is considered from Henner et al. (2019). It consists of solving a problem where the initial condition matches with the boundary conditions. The equation is specified as:

$$\frac{\partial u}{\partial t} - a^2 \frac{\partial^2 u}{\partial x^2} = 0, \quad 0 < x < l, \quad t > 0, \tag{44}$$

subject to consistent initial and boundary conditions

$$u(x,0) = \varphi(x), \qquad u(0,t) = u(l,t) = 0, \tag{45}$$

where

$$\varphi(x) = \begin{cases} \dfrac{x}{l} u_0 & \text{for} \quad 0 \le x \le \dfrac{l}{2}, \\[2ex] \dfrac{l-x}{l} u_0 & \text{for} \quad \dfrac{l}{2} < x \le l, \end{cases} \tag{46}$$

with u_0 being a constant. The exact solution obtained by the Fourier decomposition method can be reported as the following infinite series:

$$u(x,t) = \frac{4u_0}{\pi^2} \sum_{k=1}^{+\infty} \frac{(-1)^{k+1}}{(2k-1)^2} \exp\left(-\frac{a^2(2k-1)^2\pi^2}{l^2} t\right) \sin\frac{(2k-1)\pi x}{l}. \tag{47}$$

Assigning the corresponding values to parameters in the general problem (1)-(4), that is, $l_1 = 0,\ l_2 = l,\ b = 0,\ f(x,t) = 0,\ \beta_1 = \beta_2 = 0,\ \alpha_1 = \alpha_2 = 1,\ g_1(t) = g_2(t) = 0$, the p-domain solution (27) is reduced to:

$$\begin{aligned} U(x,p) =& \frac{a}{2\sqrt{p}} \left[U_x(l,p) \exp\left(\frac{-(l-x)\sqrt{p}}{a}\right) - U_x(0,p) \exp\left(\frac{-x\sqrt{p}}{a}\right) \right] \\ &+ \mathcal{L}\left\{ \frac{1}{2a\sqrt{\pi t}} \int_0^l \varphi(\xi) \left[\exp\left(-\frac{(\xi - x)^2}{4a^2 t}\right) \right] d\xi \right\}. \end{aligned} \tag{48}$$

Then, for the case where $l = 10$, $u_0 = 5$ and $a^2 = 0.25$ as in the Example 6.1. from Henner et al. (2019), the function φ is rewritten as:

$$\varphi(x) = \begin{cases} \dfrac{1}{2}x \quad \text{for} \quad 0 \leq x \leq 5, \\ \\ 5 - \dfrac{1}{2}x \quad \text{for} \quad 5 < x \leq 10, \end{cases}$$

while the series (47) becomes:

$$u(x,t) = \frac{20}{\pi^2} \sum_{k=1}^{+\infty} \frac{(-1)^{k+1}}{(2k-1)^2} \exp\left(-\frac{(2k-1)^2\pi^2}{400}t\right) \sin\frac{(2k-1)\pi x}{10}. \tag{49}$$

Using Equations (37) and (39) and denoting $\exp(x)$ by e^x, the exact operational solution (48) can be written as:

$$U(x,p) = \frac{1}{4\sqrt{p}}\left[U_x(10,p)\, \exp\left(-2(10-x)\sqrt{p}\right) - U_x(0,p)\, \exp\left(-2x\sqrt{p}\right)\right] + R(x,p). \tag{50}$$

In the above expression,

$$U_x(0,p) = -U_x(10,p) = -\frac{1}{2}\,\frac{-\mathrm{e}^{-20\sqrt{p}} + 2\,\mathrm{e}^{-10\sqrt{p}} - 1}{(\mathrm{e}^{-20\sqrt{p}} + 1)\, p}; \tag{51}$$

and

$$R(x,p) = \begin{cases} -\dfrac{1}{8}\,\dfrac{-4\,x\sqrt{p} + 2\,\mathrm{e}^{2\,(-5+x)\sqrt{p}} - \mathrm{e}^{-2\,x\sqrt{p}} - \mathrm{e}^{2\,(-10+x)\sqrt{p}}}{p^{\frac{3}{2}}} \quad \text{for} \quad 0 \leq x \leq 5, \\ \\ -\dfrac{1}{8}\,\dfrac{4\,x\sqrt{p} - 40\,\sqrt{p} + 2\,\mathrm{e}^{-2\,(-5+x)\sqrt{p}} - \mathrm{e}^{-2\,x\sqrt{p}} - \mathrm{e}^{2\,(-10+x)\sqrt{p}}}{p^{\frac{3}{2}}} \quad \text{for} \quad 5 < x \leq 10. \end{cases}$$

The formula (23) can then be checked, and the exactness of the operational solution (50) is proved, that is:

$$-a^2\frac{d^2U}{dx^2}(x,p) + pU(x,p) = \varphi(x).$$

Now, according to the results obtained in Section 5 relatively to the present case ($\beta_1 = \beta_2 = 0$), the corresponding truncation expansion in the time domain during the short time step $t \in [0, \Delta t]$, are respectively recalled as:

$$u^a(x,t) = r(x,t) + o\left((\Delta t)^n\right),$$

and

$$u_x^a(l_1,t) = \frac{1}{a}\left(-\frac{g_1(0)}{\alpha_1} + 2r(l_1,0)\right)u_x^{l_1}(t) + \frac{1}{a}u_x^{l_1}(t) * \left(-\frac{g_1'(t)}{\alpha_1} + 2r'(l_1,t)\right) + O\left(\Delta t\right),$$

where

$$u_x^{l_1}(t) = \frac{bt+1}{\sqrt{\pi\, t}},$$

and again

$$u_x^a(l_2,t) = \frac{1}{a}\left(\frac{g_2(0)}{\alpha_2} - 2r(l_2,0)\right)u_x^{l_2}(t) + \frac{1}{a}u_x^{l_2}(t) * \left(\frac{g_2'(t)}{\alpha_2} - 2r'(l_2,t)\right) + O\left(\Delta t\right),$$

where

$$u_x^{l_2}(t) = \frac{bt+1}{\sqrt{\pi\, t}}.$$

By using Maple software, for example, the analytical approximation $u^a(x,t) \simeq r(x,t)$ in short time limits ($t \in [0, \Delta t]$), can be explicitly computed from Equation (20) as:

$$\begin{aligned} u^a(x,t) &= \frac{1}{\sqrt{\pi t}} \int_0^{10} \varphi(\xi) \left[\exp\left(-\frac{(\xi-x)^2}{t}\right)\right] d\xi \\ &= \frac{1}{4}\left((x-10)\mathrm{erf}\left(\frac{-10+x}{\sqrt{t}}\right) + (-2x+10)\mathrm{erf}\left(\frac{-5+x}{\sqrt{t}}\right) + x\,\mathrm{erf}\left(\frac{x}{\sqrt{t}}\right)\right) \\ &\quad + \frac{\sqrt{t}}{4\sqrt{\pi}}\left(-2\,\mathrm{e}^{-\frac{(x-5)^2}{t}} + \mathrm{e}^{-\frac{(x-10)^2}{t}} + \mathrm{e}^{-\frac{x^2}{t}}\right). \end{aligned} \tag{52}$$

Note that in cases where it is not possible to compute the integral explicitly, $r(x,t)$ remains an analytic expression that can be represented in a graphical way. From the above expressions of $u_x^a(l_1,t)$ and $u_x^a(l_2,t)$, one has:

$$\begin{aligned} u_x^a(0,t) &= -u_x^a(10,t) \\ &= -\frac{1}{2}\mathrm{erf}\left(\frac{10}{\sqrt{t}}\right) + \mathrm{erf}\left(\frac{5}{\sqrt{t}}\right) + O\left(\Delta t\right), \end{aligned} \tag{53}$$

where $\mathrm{erf}(x)$ is the Error Function defined by:

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \mathrm{e}^{y^2} dy.$$

Compared to the function $\varphi(x)$ characterizing the initial condition, it's remarkable that the derivatives coincide at the boundaries $l_1 = 0$ and $l_2 = 10$, that is:

$$\lim_{t\to 0} u_x^a(0,t) = \varphi'(0) = \frac{1}{2}, \tag{54}$$

and

$$\lim_{t\to 10} u_x^a(10,t) = \varphi'(10) = -\frac{1}{2}. \tag{55}$$

In the limits of our knowledge, the truncation expansion (53) is not available directly from the infinite series solution (49).

Figure 1 shows curves of the solution of the problem (44)-(46) and of its derivative during the first time step $t \in [0, \Delta t]$, with $\Delta t = 10^{-2}$. The curves are obtained by using the three different methods mentioned above. The numerical solution is computed by using toolbox functions of MATLAB, while the series solution provided by the Fourier decomposition method is truncated at its first 20 terms. The approximate analytical solution is represented by the function u^a given by the above formula (52). The three resulting curves of the solution u are almost identical as shown on figure 1(a). Except around the peak point of abscissa $x = 5$, their shapes seem to be very similar to that of the function (46) characterizing the initial condition, since the time passed from $t = 0$ to $t = \Delta t = 10^{-2}$ is still relatively small. On Figure 1(b), the derivative curves almost coincide for the approximate analytical and the numerical solutions, except at the domain boundaries $x = 0$

and $x = 10$. However, the curve for the series solution exhibits deviations in the form of small oscillations along the two former curves, especially around the peak point ($x = 5$). On the one hand, this suggests the potential consistency of both numerical and approximate analytical methods in correctly accounting for non-physical phenomena such as derivative jumps and the so-called Gibbs phenomenon, which appears in generalized solutions of infinite series. On the other hand, it illustrates the relevance of refined truncation expansions for the approximate solution at the domain boundaries. Note that $u^a(0,t)$ and $u^a(10,t)$ are kept zero in the present case, and only the derivatives $u^a_x(0,t)$ and $u^a_x(10,t)$ are concerned.

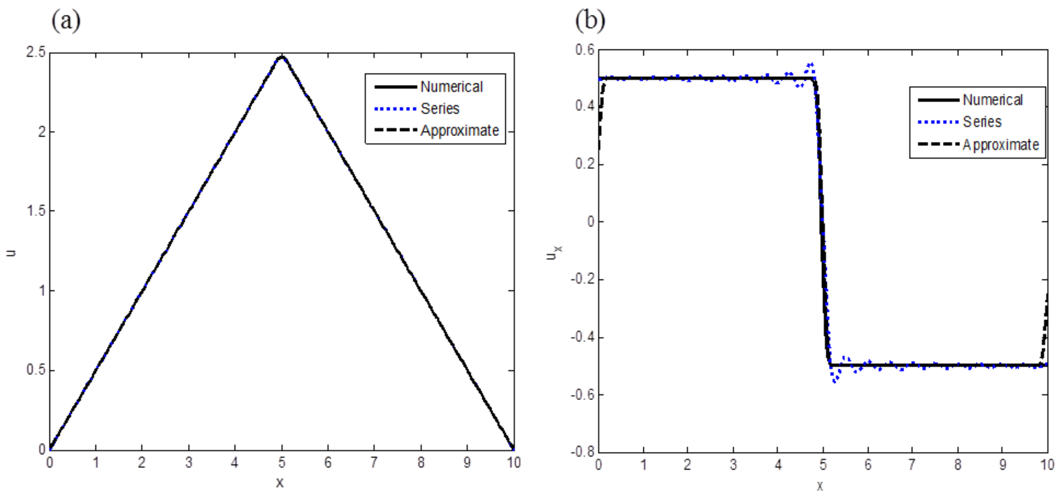

Figure 1. (a) Solution $u(x,t)$ at $t = \Delta t = 10^{-2}$. (b) Partial derivative $u_x(x,t)$ at $t = \Delta t = 10^{-2}$

Figures 2(a) and 2(b) respectively show evolution of the derivatives $u^a_x(0,t)$ and $u^a_x(10,t)$ expressed in formulae (53), versus that of the series solution (47) truncated at the first 20 terms. During the short time step $[0, \Delta t]$, the derivative curves obtained by the approximate and the series solutions can be compared. While the convergence of the series solution is weak, especially at points close to the ends of the domain, the shapes shown by the curves of $u^a_x(0,t)$ and $u^a_x(10,t)$ correspond well to the similarity highlighted above with the function characterizing the initial condition, and confirmed by the calculation of limits (54) and (55). Beyond its precision or consistency, another advantage of performing the approximate analytical solutions $u^a_x(0,t)$ and $u^a_x(10,t)$ during the short time limit $[0, \Delta t]$, consists of the computational efficiency of these solutions. Indeed, at the first execution of the code source, the computer runtime of their procedure is far reduced (about 20 times) compared to that of the series solution. More generally, the procedure for calculating analytical approximations during the first time step $t \in [0, \Delta t]$ can be repeated for the next time step. All that needs to be done is to update, from the previous step, the source term $f(x,t)$, the functions $\varphi(x)$, $g_1(t)$ and $g_2(t)$, which are linked to the initial and boundary conditions. Thus, computational models for one-dimensional Stefan problems, as discussed in Javierre et al. (2006), can be handled efficiently when using the truncation expansions $u^a(l_1,t)$, $u^a_x(l_1,t)$, $u^a(l_2,t)$ and $u^a_x(l_2,t)$ of the solution at the boundaries of the domain, as discussed in section 5. An example of the use of similar formulas for a specific problem of spherically symmetric droplet evaporation can be seen in Anani (2021).

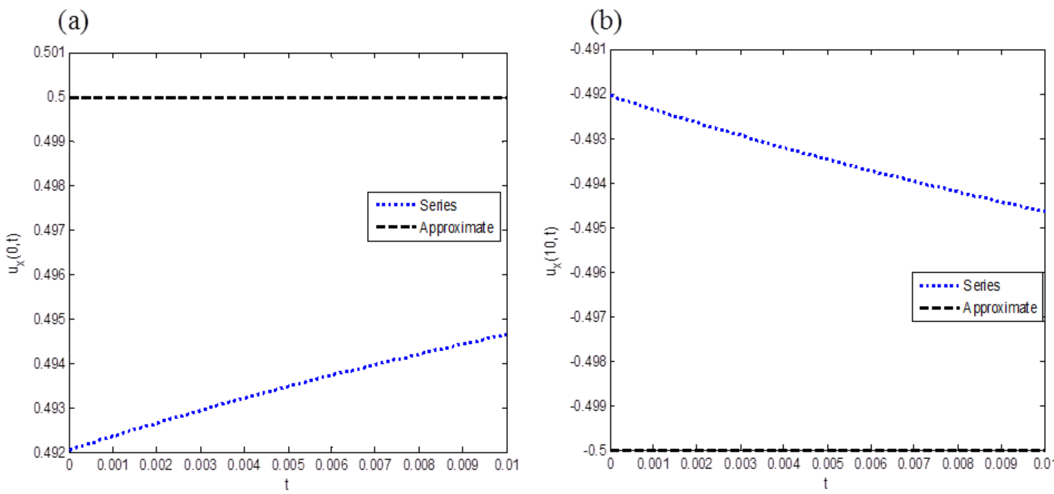

Figure 2. (a) Evolution of the derivative $u_x(0,t)$ for $t \in [0, \Delta t]$, $\Delta t = 10^{-2}$. (b) Evolution of the derivative $u_x(10,t)$ for $t \in [0, \Delta t]$, $\Delta t = 10^{-2}$

Next, the exact solution in the Laplace domain can be extended to unbounded domains as infinite or semi-infinite intervals for the space variable x. Indeed when $l_1 = -\infty$ and $l_2 = +\infty$, one has $U_x(l_1, p) = U_x(l_2, p) = 0$, and by taking the limits in Equation (27), the exact p-domain solution is reduced to $U(x,p) = R(x,p)$, which corresponds to the following solution in the time domain:

$$\begin{aligned} u(x,t) = r(x,t) =& \frac{\exp(-bt)}{2a\sqrt{\pi t}} \int_{-\infty}^{+\infty} \varphi(\xi) \left[\exp\left(-\frac{(\xi - x)^2}{4a^2 t} \right) \right] d\xi \\ &+ \frac{1}{2a\sqrt{\pi}} \int_0^t d\theta \int_{-\infty}^{+\infty} \frac{\exp(-b(t-\theta))}{\sqrt{(t-\theta)}} \exp\left(-\frac{(\xi - x)^2}{4a^2(t-\theta)} \right) f(\xi,\theta) d\xi. \end{aligned}$$

In the case where $b = 0$, the above solution is identical to the one reported by Henner et al. (2019) in their book at Section 6.8 titled "The Heat Equation in an Infinite Region," where the Fourier decomposition method was used. About semi-infinite domains, let us chose $l_1 = 0$ and $l_2 = +\infty$ for example, then $U_x(l_2, p) = 0$, and $u(l_2, t)$ is to be considered as constant. The p-domain solution (27) reduces to:

$$U(x,p) = \frac{1}{2} U(0,p) \exp\left(\frac{-x\sqrt{b+p}}{a} \right) - \frac{a U_x(0,p)}{2\sqrt{b+p}} \exp\left(\frac{-x\sqrt{b+p}}{a} \right) + R(x,p), \tag{56}$$

where $R(x,p) = \mathcal{L}\{r(x,t)\}$, and

$$\begin{aligned} r(x,t) =& \frac{\exp(-bt)}{2a\sqrt{\pi t}} \int_{0}^{+\infty} \varphi(\xi) \left[\exp\left(-\frac{(\xi - x)^2}{4a^2 t} \right) \right] d\xi \\ &+ \frac{1}{2a\sqrt{\pi}} \int_0^t d\theta \int_{0}^{+\infty} \frac{\exp(-b(t-\theta))}{\sqrt{(t-\theta)}} \exp\left(-\frac{(\xi - x)^2}{4a^2(t-\theta)} \right) f(\xi,\theta) d\xi. \end{aligned} \tag{57}$$

Setting as in the statement of Problem a. of Chapter 8.1 in Luikov (2012), $b = 0$, $\varphi(x) = t_0 =$ (constant), $u(0,t) = g_1(t) = t_a =$ (constant) implying $U(0,p) = t_a/p$, it can be verified for $f(x,t) = w/c\gamma =$ (constant) that:

$$Ux(0,p) = \frac{2\sqrt{p}}{a} R(0,p) - \frac{t_a}{a\sqrt{p}},$$

and

$$R(x,p) = \frac{t_0}{p} + \frac{w}{p^2 c\gamma} - \frac{t_0}{2p}\exp\left(-\frac{x\sqrt{p}}{a}\right) - \frac{w}{2p^2 c\gamma}\exp\left(-\frac{x\sqrt{p}}{a}\right).$$

This leads to reduce the exact p-domain solution (56) into the form of:

$$U(x,p) = \frac{t_0}{p} + \frac{w}{p^2 c\gamma} + \frac{(t_a - t_0)}{p}\exp\left(-\frac{x\sqrt{p}}{a}\right) - \frac{w}{p^2 c\gamma}\exp\left(-\frac{x\sqrt{p}}{a}\right),$$

which is identical to the solution (8.1.11) reported in Luikov (2012), provided that a is replaced by $\sqrt{a}$ as specified in the statement of the problem. Note in the latter reference that the Laplace transform method was directly used to solve the problem, since the initial condition is specified as a constant function.

7. Conclusion

This study has made it possible to compute the explicit solution in the Laplace domain and accurate approximations in the earlier time step for initial-boundary value problems of the one-dimensional parabolic equation with constant coefficients. The problem is solved in its most general form with boundary conditions specified in a unified manner on an arbitrary bounded interval of the real line. Analytical approximations in short time limits are proven to be more consistent and sufficiently simple to improve computational efficiency in numerical schemes and simulations, compared to classical or generalized series solutions. Early time behaviors of heat or mass reaction-diffusion processes are of great interest in engineering, and have a wide range of applications in fields like Computational Fluid Dynamics (CFD) and Nuclear Energy. Additionally, the operational solution obtained for the problem can be extended to unbounded domains. This solution represents a significant advancement in the study of linear parabolic equations. While Laplace inversion theorems can be used to search for solutions, it is important to note that most tables and mathematical software have a limited number of analytic inverses of the Laplace domain in the time domain. However, regardless of the complexity of operational solutions, numerical inverse Laplace transforms can still be performed. Therefore, the exact operational solution can be numerically transformed to derive precise curves of the solution in the entire time domain. Finally, to obtain the exact solution in the Laplace domain, it is necessary to study more precisely the largest class to which the functions involved in the problem must belong.

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