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Dual Quaternion Matrices and MATLAB Applications

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Abstract

There are many studies in the literature on real quaternions and real quaternion matrices. There are few studies in the literature on dual quaternions. Definitions of the matrices of dual quaternions used in this study will be given. The originality of our research, the set of dual quaternion matrix we studied, will be defined for the first time in this study, and its properties will be given. Moreover, this study is critical because it is an applied study related to dual quaternion matrices. It will be easier to solve examples with large matrix sizes with MATLAB. People who use different programs can also write applications inspired by them. In this work, the set of dual quaternion matrices is examined. Among the dual quaternion matrices, additional features such as addition, multiplication, inverse, transpose, conjugate, power, and trace are explored. In addition, real matrix representations of dual quaternion matrices and their characteristics were developed. These were utilized to determine the dual quaternion matrices' determinants and inverses. In addition, many types of determinants and inverses of dual quaternion matrices were created, and MATLAB applications were developed to facilitate the solution of cases utilizing these techniques. Finally, the results and methodologies were compared to the actual quaternion matrix characteristics.

Keywords: Quaternions; Dual quaternions; Dual quaternion matrices; Real matrix representation; MATLAB applications; Inverse; Power; Determinant

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1. Introduction

Quaternions are a crucial component of the rotational representation in computer graphics, particularly when it comes to animation and user interfaces. Due to the challenges involved in accurately depicting the four-dimensional space occupied by quaternions, quaternion rotation is typically reserved for study at a more advanced level in computer graphics education. This is unfortunate. The quaternion demonstration is a tangible visual aid consisting primarily of a belt. It is one of the tools that may be used to overcome these challenges. One end of the belt can be held still while the other is rotated to symbolize any quaternion used to define a rotation. The composition of rotations serves as a demonstration of the multiplication of quaternions, and the twists that arise in the belt serve as a visual representation of how quaternions interpolate rotation (Hart et al. (1994)). Rotating vectors in three dimensions may be accomplished through quaternion multiplication. As a result, quaternions are frequently utilized in place of matrices in computer graphics regarding the representation of rotations in three dimensions. However, although the formal algebra of quaternions is well-known in the graphics community, the derivations of the formulae for this algebra and the geometric concepts that underlie this algebra are not generally understood (Goldman (2011)).

Multiple writers researched the algebraic characteristics of quaternion matrices (Bitim (2019); Brenner (1951); Erdoğdu and Özdemir (2013a); Erdoğdu and Özdemir (2013b); Erdoğdu and Özdemir (2015a); Erdoğdu and Özdemir (2015b); Flaut and Shpakivskyi (2013); Halici and Deveci (2021); He (2019); Kösal and Tosun (2014); Kösal and Tosun (2017); Nalbant and Yüce (2019); Özyurt and Alagöz (2018); Qi et al. (2022); Wiegmann (1955); Wolf (1936); Zhang (1997)). Zhang (1997) provided a quick overview of quaternions and their matrices. Alagöz et al. (2012) defined split quaternions and matrices for them using complex matrices. Erdoğdu and Özdemir (2013b) defined the representation of split quaternion matrices in real matrices and provided several key algebraic features of split quaternion matrices by utilizing real matrices as the basis for their work. Alagöz and Özyurt (2019) looked at the characteristics of complex split quaternion and complex quaternion matrix.

The academic literature has a significant amount of study on real quaternion and real quaternion matrices. In the body of work devoted to research, one does not come across a substantial number of papers that focus on dual quaternions. There will be a discussion on the definitions of the matrices of dual quaternions used in this study. These matrices have been employed in the current investigation. The first time that the set of dual quaternion matrix that we examined will be defined, and the very first time that their properties will be detailed, will both take place within the scope of this study. This is an original and essential addition that comes from our research. In addition, the significance of this work cannot be understated because it is an applied research addressing dual quaternion matrices, and that feature alone makes it extremely important. Users who do not work with MATLAB can take ideas from MATLAB applications and implement them in the software they work with. When dealing with issues involving situations that have large matrix sizes, using MATLAB will make the process much simpler.

The present study introduces a novel contribution by providing the initial definition of dual quaternion matrices and their associated properties. The definitions, theorems, methods, and applications in the study are unique. The calculation of the determinant can be facilitated by utilizing Theorem 6.1, Theorem 6.3, and Theorem 6.4. The studied samples were selected for comparison, and the same results were obtained with the methods used. This shows the accuracy of the methods. Furthermore, MATLAB applications facilitate the facile computation of the inverse, power, real matrix representation, and determinant of matrices with exceedingly high orders. A comparison of real quaternion matrices and dual quaternion matrices that we defined is given in Table 2. When the literature is examined, a comparison table for other quaternion matrices is not encountered. The objective of this investigation is to enhance the existing body of knowledge by introducing novel dual quaternion matrices, MATLAB implementations, and comparative tables. As can be seen in Table 2, when real and dual quaternion matrices are compared, there are many features that cannot be achieved with real quaternion matrices. However, with dual quaternion matrices, many generalizations can be obtained for power, inverse, and determinant, etc.

In Section 1, the literature of the real set of quaternions is examined. The characteristics of dual quaternion matrices will be examined in this research. A brief discussion of the fundamental features of dual quaternions will be mentioned in Section 3. After that, in Section 4, we will define dual quaternion matrices and examine their properties. MATLAB will then be used to obtain the inverse of dual quaternion matrix. Then, we will define the dual quaternion matrix's real matrix representation and give its characteristics in Section 5. Besides, we will find this representation matrix using MATLAB. Section 6 will cover the determinant of dual quaternion matrices and several methods for determining the inverse of dual quaternion matrices. Furthermore, examples are given for each topic. To show the accuracy of the given theorems and the results, we always paid attention to taking the same matrix in the examples. Finally, the study is concluded with a summary of the results and a discussion of future research prospects in Section 7.

2. Quaternions

Hamilton (1866) presented the quaternion set in 1843. People may deduce that the quaternion product is non-commutative based on these rules. Following the publication of Hamilton's paper in 1849, split quaternions were first defined by Cockle (Clifford (1873)) and H_S is non-commutative, too. Akyiğit et al. (2013) established the split Lucas and Fibonacci quaternions in 2013. Kula and Yaylı (2007) defined complex split quaternions. Majernik (2006) defined dual quaternions. See its definition at the beginning of Section 3. Yüce and Ercan (2011) worked on dual quaternions. In their study, the equations of De Moivre and Euler are generalized for dual quaternions. Ata and Yaylı (2009) introduced dual quaternions with dual number coefficients and denoted by:

$$H(\mathbb{D}) = \left\{ \begin{array}{c} Q = A + B\, i\, + C\, j\, + D\, k \;\mid\; A,\, B,\, C,\, D \in \mathbb{D}, \\ i^2 = j^2 = k^2 = -1 = i\, j\, k \end{array} \right\}. \tag{1}$$

Here $\mathbb{D}$ is used for a set of dual numbers. $H(\mathbb{D})$ and $H_{\mathbb{D}}$ are different sets. Essentially, these quaternions in Equation (1) must be called dual coefficient quaternions.

3. Dual Quaternions

The dual quaternion set is expressed as

$$H_{\mathbb{D}} = \{\, q = a + b\,i + c\,j + d\,k \mid a,\, b,\, c,\, d \in \mathbb{R},\, i,\, j,\, k \notin \mathbb{R}\,\},$$

where i, j, k follow these multiplication rules:

$$i^2 = j^2 = k^2 = 0, \quad i\,j = j\,i = j\,k = k\,j = k\,i = i\,k = 0.$$

If $a = 0$, q is called non-pure dual quaternion. The addition rule maintains the addition's associativity and commutativity features. The algebra $H_{\mathbb{D}}$ is not a division algebra because it is not a division ring (Rotman (2002)). However, the nilpotent basis elements i, j, k in the definition of $H_{\mathbb{D}}$, do not have multiplicative inverses. If $u = a + bi + cj + dk$ were an inverse of i, then $1 = iu = i(a + bi + cj + dk) = ia$. Thus, a would be a non-zero real number, and so, $i = a^{-1}$ would be a real number, yet $i \notin \mathbb{R}$. $H_{\mathbb{D}}$ is isomorphic to the Galilean space G^4 (Majernik (2006)).

4. Dual Quaternion Matrices

An $m \times n$ matrix whose elements may contain dual quaternions is called an $m \times n$ dual quaternion matrix. Also, $m \times n$ matrices with dual quaternion coefficients is expressed as $M_{m\times n}(H_{\mathbb{D}})$ and defined as

$$M_{m\times n}(H_{\mathbb{D}}) = \{\hat{A} = [\hat{a}_{rs}] = [a_{rs}] + [b_{rs}]\,i + [c_{rs}]\,j + [d_{rs}]\,k \mid \hat{a}_{rs} = a_{rs} + b_{rs}\,i + c_{rs}\,j + d_{rs}\,k \in H_{\mathbb{D}}\} \tag{2}$$

where $a_{rs},\, b_{rs},\, c_{rs},\, d_{rs} \in \mathbb{R}$ and

$$i^2 = j^2 = k^2 = i\;j\;k = 0\,, \;\; i\;j = j\;i = j\;k = k\;j = k\;i = i\;k = 0.$$

Moreover, dual quaternion matrices are expressed as

$$M_{m\times n}(H_{\mathbb{D}}) = \{\hat{A} = A_1 + B_1\,i + C_1\,j + D_1\,k \mid A_1,\, B_1,\, C_1,\, D_1 \in M_{m\times n}(\mathbb{R})\},$$

where $A_1 = [a_{rs}],\, B_1 = [b_{rs}],\, C_1 = [c_{rs}],\, D_1 = [d_{rs}]$.

If $m = n$, then these matrices are indicated by $M_n(H_{\mathbb{D}})$.

For $\hat{A} = [\hat{a}_{rs}] = A_1 + B_1\,i + C_1\,j + D_1\,k$, $\hat{B} = [\hat{b}_{rs}] = A_2 + B_2\,i + C_2\,j + D_2\,k \in M_{m\times n}(H_{\mathbb{D}})$, the matrix addition is expressed as

$$\hat{A} + \hat{B} = [\hat{a}_{rs} + \hat{b}_{rs}] \in M_{m\times n}(H_{\mathbb{D}}),$$

or

$$\hat{A} + \hat{B} = (A_1 + A_2) + (B_1 + B_2)i + (C_1 + C_2)j + (D_1 + D_2)k.$$

For $\hat{A} = [\hat{a}_{rs}] = A_1 + B_1\,i + C_1\,j + D_1\,k \in M_{m\times n}(H_{\mathbb{D}})$, $\hat{B} = [\hat{b}_{so}] = A_2 + B_2\,i + C_2\,j + D_2\,k \in M_{n\times p}(H_{\mathbb{D}})$, the matrix multiplication is expressed as

$$\hat{A}\,\hat{B} = \left[\sum_{s=1}^{n} \hat{a}_{rs}\,\hat{b}_{so}\right] \in M_{m\times p}(H_{\mathbb{D}}),$$

or

$$\hat{A}\,\hat{B} = A_1\,A_2 + (A_1\,B_2 + B_1\,A_2)i + (A_1\,C_2 + C_1\,A_2)j + (A_1\,D_2 + D_1\,A_2)k, \tag{3}$$

or

$$\hat{A}\,\hat{B} = A_1\,A_2 + A_1(B_2\,i + C_2\,j + D_2\,k) + (B_1\,i + C_1\,j + D_1\,k)A_2.$$

The multiplication of dual quaternion matrices is non-commutative.

For $q \in H_{\mathbb{D}}$ and $\hat{A} = [\hat{a}_{rs}] \in M_{m\times n}(H_{\mathbb{D}})$, the scalar multiplication is defined by

$$q\hat{A} = \hat{A}q = [q\hat{a}_{rs}].$$

The product of dual quaternion matrices is associative. The proof is a simple routine.

Lemma 4.1.

For $\hat{A}, \hat{B} \in M_n(H_{\mathbb{D}})$ and $p, q \in H_{\mathbb{D}}$, we achieve

(i) $q(\hat{A} + \hat{B}) = q\,\hat{A} + q\,\hat{B}$,
(ii) $(p + q)\hat{A} = p\,\hat{A} + q\,\hat{A}$,
(iii) $(p\,q)\,\hat{A} = p\,(q\,\hat{A})$,
(iv) $1\,\hat{A} = \hat{A}$,
(v) $(q\,\hat{A})\,\hat{B} = q\,(\hat{A}\,\hat{B})$,
(vi) $(\hat{A}\,q)\,\hat{B} = \hat{A}\,(q\,\hat{B})$.

Proof:

(i), (ii), (iv), (v) and (vi) are easily demonstrable. We shall demonstrate (iii):
Let $\hat{A} = A + B\,i + C\,j + D\,k \in M_{m\times n}(H_{\mathbb{D}})$ where $A,\ B,\ C,\ D \in M_{m\times n}(\mathbb{R})$ and $p, q \in H_{\mathbb{D}}$,

$$\begin{aligned}(p\,q)\hat{A} &= (p\,q)A + (p\,q)B\,i + (p\,q)C\,j + (p\,q)D\,k\\ &= (p(q\,A)) + (p(q\,B))\,i + (p(q\,C))\,j + (p(q\,D))\,k\\ &= p((q\,A) + (q\,B)\,i + (q\,C)\,j + (q\,D)\,k)\\ &= p(q\hat{A}).\end{aligned}$$

■

4.1. Module $M_n(H_{\mathbb{D}})$ Structure Over the Ring $M_n(\mathbb{R})$

$M_n(H_{\mathbb{D}})$ is a module over the ring $H_{\mathbb{D}}$. Also, it is spanned by

$$\left\{ \begin{bmatrix} 1 & 0 & \dots 0 \\ 0 & 0 & \dots 0 \\ \dots\dots & \ddots & \vdots \\ 0 & 0 & \dots 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & \dots 0 \\ 0 & 0 & \dots 0 \\ \dots\dots & \ddots & \vdots \\ 0 & 0 & \dots 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & 0 & \dots 0 \\ 0 & 0 & \dots 0 \\ \dots\dots & \ddots & \vdots \\ 0 & 0 & \dots 1 \end{bmatrix} \right\}.$$

Then, $dim(M_n(H_{\mathbb{D}})) = n^2$.

For $\hat{A} = [\hat{a}_{rs}] = A + B\,i + C\,j + D\,k \in M_n(H_{\mathbb{D}})$, $Q = [q_{tr}] \in M_n(\mathbb{R})$, the product of $\hat{A}$ with Q is expressed as

$$Q\,\hat{A} = \left[\sum_{r=1}^{n} q_{tr}\,\hat{a}_{rs}\right] \in M_n(H_{\mathbb{D}}),$$

or

$$Q\,\hat{A} = Q\,A + Q\,B\,i + Q\,C\,j + Q\,D\,k. \tag{4}$$

Similarly, the right multiplication may be defined.

Lemma 4.2.

For $\hat{A}, \hat{B} \in M_n(H_{\mathbb{D}})$, $Q_1,\ Q_2 \in M_n(\mathbb{R})$, the left multiplication has the properties below:

(i) $(Q_1 + Q_2)\,\hat{A} = Q_1\,\hat{A} + Q_2\,\hat{A}$,
(ii) $(Q_1\,Q_2)\,\hat{A} = Q_1\,(Q_2\,\hat{A})$,
(iii) $Q_1(\hat{A} + \hat{B}) = Q_1\,\hat{A} + Q_1\,\hat{B}$,
(iv) $I_n\,(\hat{A}) = \hat{A}$,
(v) $(Q_1\,\hat{A})\,\hat{B} = Q_1\,(\hat{A}\,\hat{B})$,
(vi) $(\hat{A}\,Q_1)\,\hat{B} = \hat{A}\,(Q_1\,\hat{B})$.

In the same way, the right multiplication characteristics may be illustrated.

Proof:

(i), (ii), (iv), (v) and (vi) are easily demonstrable. We shall demonstrate (iii):
Let $\hat{A} = A + B\,i + C\,j + D\,k \in M_n(H_{\mathbb{D}})$, $Q_1, Q_2 \in M_n(\mathbb{R})$ and

$$\begin{aligned}(Q_1\,Q_2)\hat{A} &= (Q_1\,Q_2)A + (Q_1\,Q_2)B\,i + (Q_1\,Q_2)C\,j + (Q_1\,Q_2)D\,k\\ &= (Q_1(Q_2\,A)) + (Q_1(Q_2\,B))\,i + (Q_1(Q_2\,C))\,j + (Q_1(Q_2\,D))\,k\\ &= Q_1((Q_2\,A) + (Q_2\,B)\,i + (Q_2\,C)\,j + (Q_2\,D)\,k)\\ &= Q_1(Q_2\hat{A}).\end{aligned}$$

■

Remark 4.1.

$M_n(H_{\mathbb{D}})$ is a left (right) module over $M_n(\mathbb{R})$.

Also, for all $\hat{A} = A + B\,i + C\,j + D\,k \in M_n(H_{\mathbb{D}})$ we obtain

$$\hat{A} = A\,\hat{1} + B\,\hat{I} + C\,\hat{J} + D\,\hat{K},$$

such that

$$\hat{1} = \hat{I}_n,\ \hat{I} = i\,\hat{I}_n,\ \hat{J} = j\,\hat{I}_n,\ \hat{K} = k\,\hat{I}_n,\ \hat{1}^2 = \hat{I}_n,$$
$$\hat{I}^2 = \hat{J}^2 = \hat{K}^2 = \hat{0}_n,\ \hat{I}\,\hat{J} = \hat{J}\,\hat{I} = \hat{J}\,\hat{K} = \hat{K}\,\hat{J} = \hat{K}\,\hat{I} = \hat{I}\,\hat{K} = \hat{0}_n.$$

So, we obtain

$$M_n(H_{\mathbb{D}}) = sp\{\hat{1}, \hat{I}, \hat{J}, \hat{K}\}.$$

Then, $dim(M_n(H_{\mathbb{D}})) = 4$.

4.2. Conjugate, Transpose and Conjugate Transpose of Dual Quaternion Matrix

The dual quaternion matrix's conjugate $\hat{A} = [\hat{a}_{rs}] = A + B\,i + C\,j + D\,k \in M_{m\times n}(H_{\mathbb{D}})$ is defined as

$$\overline{\hat{A}} = [\overline{\hat{a}_{rs}}] = [a_{rs} - b_{rs}\,i - c_{rs}\,j - d_{rs}\,k] \in M_{m\times n}(H_{\mathbb{D}}), \tag{5}$$

or

$$\overline{\hat{A}} = A - B\,i - C\,j - D\,k \in M_{m\times n}(H_{\mathbb{D}}).$$

Theorem 4.1.

Let $\hat{A}$, $\hat{B}$ be dual quaternion matrices, $q \in H_{\mathbb{D}}$ and $Q \in M_n(\mathbb{R})$. The conjugate of dual quaternion matrices meets the properties below:

(i) $\overline{(\overline{\hat{A}})} = \hat{A}$, for $\hat{A} \in M_{m\times n}(H_{\mathbb{D}})$,
(ii) $\overline{(q\,\hat{A})} = \overline{q}\,\overline{\hat{A}} = \overline{\hat{A}}\,\overline{q}$, for $\hat{A} \in M_{m\times n}(H_{\mathbb{D}})$,
(iii) $\overline{\hat{A} + \hat{B}} = \overline{\hat{A}} + \overline{\hat{B}}$, for $\hat{A},\ \hat{B} \in M_{m\times n}(H_{\mathbb{D}})$,
(iv) $\overline{\hat{A}\,\hat{B}} = \overline{\hat{A}}\,\overline{\hat{B}}$, for $\hat{A} \in M_{m\times n}(H_{\mathbb{D}}),\ \hat{B} \in M_{n\times p}(H_{\mathbb{D}})$,
(v) $\overline{Q\,\hat{A}} = Q\,\overline{\hat{A}}$, for $Q \in M_n(\mathbb{R}),\ \hat{A} \in M_n(H_{\mathbb{D}})$,
(vi) $\hat{A}\,\overline{\hat{A}} = \overline{(\overline{\hat{A}}\,\hat{A})}$, for $\hat{A} \in M_n(H_{\mathbb{D}})$.

Proof:

(i), (ii), (iii), (v) and (vi) are easily demonstrable. We shall demonstrate (iv):
Let $\hat{A} = A + B\,i + C\,j + D\,k \in M_{m\times n}(H_{\mathbb{D}})$ and $\hat{B} = A_2 + B_2\,i + C_2\,j + D_2\,k \in M_{n\times p}(H_{\mathbb{D}})$.
Then, from Equation (3),

$$\overline{\hat{A}\,\hat{B}} = A\,A_2 - (A\,B_2 + B\,A_2)\,i - (A\,C_2 + C\,A_2)\,j - (A\,D_2 + D\,A_2)\,k.$$

On the other hand, we obtain

$$\overline{\hat{A}}\,\overline{\hat{B}} = A\,A_2 - (A\,B_2 + B\,A_2)\,i - (A\,C_2 + C\,A_2)\,j - (A\,D_2 + D\,A_2)\,k.$$

Thus, we get $\overline{\hat{A}\,\hat{B}} = \overline{\hat{A}}\,\overline{\hat{B}}$. ■

The transpose of dual quaternion matrix $\hat{A} = [\hat{a}_{rs}] = A + B\,i + C\,j + D\,k \in M_{m\times n}(H_{\mathbb{D}})$ is defined as

$$\hat{A}^T = [\hat{a}_{rs}]^T = [a_{sr}] \in M_{n\times m}(H_{\mathbb{D}}),$$

or

$$\hat{A}^T = A^T + B^T\, i + C^T\, j + D^T\, k \in M_{n\times m}(H_{\mathbb{D}}).$$

Theorem 4.2.

Let $\hat{A}$, $\hat{B}$ be dual quaternion matrices, $q \in H_{\mathbb{D}}$ and $Q \in M_n(\mathbb{R})$. The transpose of dual quaternion matrices satisfies the following properties:

(i) $(\hat{A}^T)^T = \hat{A},\ for\ \hat{A} \in M_{m\times n}(H_{\mathbb{D}}),$
(ii) $(q\,\hat{A})^T = q\,\hat{A}^T = \hat{A}^T\, q,\ for\ \hat{A} \in M_{m\times n}(H_{\mathbb{D}}),$
(iii) $(\hat{A}+\hat{B})^T = \hat{A}^T + \hat{B}^T,\ for\ \hat{A},\ \hat{B} \in M_{m\times n}(H_{\mathbb{D}}),$
(iv) $(\hat{A}\,\hat{B})^T = \hat{B}^T\,\hat{A}^T,\ for\ \hat{A},\ \hat{B} \in M_n(H_{\mathbb{D}}).$

Proof:

(i), (ii) and (iii) are easily demonstrable. We shall demonstrate (iv).
(iv) Let $\hat{A} = A + B\, i + C\, j + D\, k$, $\hat{B} = A_2 + B_2\, i + C_2\, j + D_2\, k\ \in M_n(H_{\mathbb{D}})$.
Then from Equation (3)

$$(\hat{A}\,\hat{B})^T = A_2^T\, A^T + (B_2^T\, A^T + A_2^T\, B^T)\, i + (C_2^T\, A^T + A_2^T\, C^T)\, j + (D_2^T\, A^T + A_2^T\, D^T)\, k.$$

On the other hand, for $\hat{A}^T = A^T + B^T\, i + C^T\, j + D^T\, k$ and $\hat{B}^T = {A_2}^T + {B_2}^T\, i + {C_2}^T\, j + {D_2}^T\, k$ we can obtain

$$\hat{B}^T\,\hat{A}^T = A_2^T\, A^T + (B_2^T\, A^T + A_2^T\, B^T)\, i + (C_2^T\, A^T + A_2^T\, C^T)\, j + (D_2^T\, A^T + A_2^T\, D^T)\, k.$$

Thus, we get $(\hat{A}\,\hat{B})^T = \hat{B}^T\hat{A}^T$. ■

The conjugate transpose of dual quaternion matrix $\hat{A} = [\hat{a}_{rs}] = A + B\, i + C\, j + D\, k \in M_{m\times n}(H_{\mathbb{D}})$ is defined as

$$\hat{A}^* = [\hat{a}_{rs}]^* = [\overline{\hat{a}_{sr}}] \in M_{n\times m}(H_{\mathbb{D}}) \quad \text{or} \quad \hat{A}^* = A^T - B^T\, i - C^T\, j - D^T\, k \in M_{n\times m}(H_{\mathbb{D}}).$$

Theorem 4.3.

Let $\hat{A}, \hat{B}$ be dual quaternion matrices, $q \in H_{\mathbb{D}}$ and $Q \in M_n(\mathbb{R})$. The conjugate transpose of dual quaternion matrices satisfies the next properties:

(i) $(\overline{\hat{A}})^T = (\overline{\hat{A}^T}),$ for $\hat{A} \in M_{m\times n}(H_{\mathbb{D}}),$
(ii) $(q\,\hat{A})^* = \overline{q}\,\hat{A}^* = \hat{A}^*\,\overline{q},$ for $\hat{A} \in M_{m\times n}(H_{\mathbb{D}}),$
(iii) $(\hat{A}+\hat{B})^* = \hat{A}^* + \hat{B}^*,$ for $\hat{A},\ \hat{B} \in M_{m\times n}(H_{\mathbb{D}}),$
(iv) $(\hat{A}\,\hat{B})^* = \hat{B}^*\,\hat{A}^*,$ for $\hat{A},\ \hat{B} \in M_n(H_{\mathbb{D}}),$
(v) $(Q\,\hat{A})^* = \hat{A}^*\, Q^T,$ for $Q \in M_n(\mathbb{R}),\ \hat{A} \in M_n(H_{\mathbb{D}}).$

Proof:

(i), (ii), (iii) and (v) are easily demonstrable. Now we will demonstrate (iv):

Let $\hat{A} = A + B\,i + C\,j + D\,k$, $\hat{B} = A_2 + B_2\,i + C_2\,j + D_2\,k \in M_n(H_{\mathbb{D}})$. Then, from Equation (3)

$$\begin{aligned}(\hat{A}\,\hat{B})^* &= [\overline{A}\,\overline{A_2} - (A\,B_2 + B\,A_2)\,i - (A\,C_2 + C\,A_2)\,j - (A\,D_2 + D\,A_2)\,k]^T \\ &= A_2^T\,A^T - (B_2^T\,A^T + A_2^T\,B^T)\,i - (C_2^T\,A^T + A_2^T\,C^T)\,j - (D_2^T\,A^T + A_2^T\,D^T)\,k.\end{aligned}$$

On the other hand, for $\hat{A}^* = A^T - B^T\,i - C^T\,j - D^T\,k$ and $\hat{B}^* = {A_2}^T - {B_2}^T\,i - {C_2}^T\,j - {D_2}^T\,k$ we can obtain

$$\hat{B}^*\,\hat{A}^* = A_2^T\,A^T - (B_2^T\,A^T + A_2^T\,B^T)\,i - (C_2^T\,A^T + A_2^T\,C^T)\,j - (D_2^T\,A^T + A_2^T\,D^T)\,k.$$

Thus, we find $(\hat{A}\,\hat{B})^* = \hat{B}^*\,\hat{A}^*$. ■

4.3. The Inverse of Dual Quaternion Matrix

We shall examine the inverse of dual quaternion matrix using three distinct methods. Here, we will define the inverse of this matrix with the inverse of the real part. However, we will show that the dual quaternion matrix inverse can also be found using adjoint matrix, real matrix representation of this matrix by Theorem 5.1 / (v) and Corollary 6.1.

Theorem 4.4.

Let $\hat{A}, \hat{B} \in M_n(H_{\mathbb{D}})$, if $\hat{A}\,\hat{B} = \hat{I}_n$ then $\hat{B}\,\hat{A} = \hat{I}_n$.

Proof:

Observe first that the theorem is true for real matrices. Now, let $\hat{A} = A + B\,i + C\,j + D\,k$ and $\hat{B} = A_2 + B_2\,i + C_2\,j + D_2\,k$ be dual quaternions.

$$\begin{aligned}\hat{A}\,\hat{B} &= A\,A_2 + (A\,B_2 + B\,A_2)\,i + (A\,C_2 + C\,A_2)\,j + (A\,D_2 + D\,A_2)\,k \\ &= \hat{I}_n = I_n + 0_n\,i + 0_n\,j + 0_n\,k. \qquad (6)\end{aligned}$$

From Equation (6) we get

$$\begin{aligned}A\,A_2 &= I_n, \\ A\,B_2 + B\,A_2 &= 0_n, \\ A\,C_2 + C\,A_2 &= 0_n, \\ A\,D_2 + D\,A_2 &= 0_n.\end{aligned} \qquad (7)$$

When we solve Equation (7), we obtain

$$\begin{aligned}A_2\,A &= I_n, \\ B &= -A\,B_2\,A, \\ C &= -A\,C_2\,A, \\ D &= -A\,D_2\,A.\end{aligned} \qquad (8)$$

By using Equation (8), we get

$$\begin{aligned}A_2\,A &= I_n, \\ B_2\,A + A_2\,B &= 0_n, \\ C_2\,A + A_2\,C &= 0_n, \\ D_2\,A + A_2\,D &= 0_n.\end{aligned} \qquad (9)$$

Finally, by using Equation (9), we find

$$\begin{aligned} A_2\, A + & (B_2\, A + A_2\, B)\, i + (C_2\, A + A_2\, C)\, j + (D_2\, A + A_2\, D)\, k \\ & = (A_2 + B_2\, i + C_2\, j + D_2\, k)\,(A + B\, i + C\, j + D\, k) \\ & = \hat{B}\,\hat{A} = I_n + 0\, i + 0\, j + 0\, k \\ & = \hat{I}_n. \end{aligned}$$

■

Theorem 4.5.

Let $\hat{A} = A + B\,i + C\,j + D\,k \in M_n(H_{\mathbb{D}})$ and $A,\ B,\ C,\ D \in M_n(\mathbb{R})$. If $det(A) \neq 0$ and $\hat{A}$ is invertible, and the inverse of this matrix can be found as follows:

$$\hat{A}^{-1} = A^{-1}\,\overline{\hat{A}}\,A^{-1}. \tag{10}$$

Proof:

Let $\hat{A} = A + B\,i + C\,j + D\,k$ and $\hat{B} = A_2 + B_2\,i + C_2\,j + D_2\,k$ be dual quaternions. When we solve Equation (7), we can also obtain

$$\begin{aligned} A_2 &= A^{-1}, \\ B_2 &= -A^{-1}\,B\,A^{-1}, \\ C_2 &= -A^{-1}\,C\,A^{-1}, \\ D_2 &= -A^{-1}\,D\,A^{-1}. \end{aligned} \tag{11}$$

By using Equation (11) in $\hat{B}$, we get

$$\begin{aligned} \hat{B} = \hat{A}^{-1} &= A^{-1} - (A^{-1}\,B\,A^{-1})\,i - (A^{-1}\,C\,A^{-1})\,j - (A^{-1}\,D\,A^{-1})\,k \\ &= A^{-1}\,(A - B\,i - C\,j - D\,k)\,A^{-1} \\ &= A^{-1}\,\overline{\hat{A}}\,A^{-1}. \end{aligned}$$

■

Remark 4.2.

$\hat{A} \in M_n(H_{\mathbb{D}})$ is invertible if and only if $A \in M_n(\mathbb{R})$ is invertible, namely $det(A) \neq 0$.

Example 4.1.

Let $\hat{A} = \begin{bmatrix} 1-i & 2-j & 3-k \\ -i+j & 1-k & 2+2\,j \\ 3-i & 2+i & -i-j-k \end{bmatrix} \in M_3(H_{\mathbb{D}})$. We can show $\hat{A}$ as

$$\hat{A} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 3 & 2 & 0 \end{bmatrix} + \begin{bmatrix} -1 & 0 & 0 \\ -1 & 0 & 0 \\ -1 & 1 & -1 \end{bmatrix} i + \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 2 \\ 0 & 0 & -1 \end{bmatrix} j + \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} k.$$

Since $det(A) \neq 0$, $A^{-1} = \begin{bmatrix} 4 & -6 & -1 \\ -6 & 9 & 2 \\ 3 & -4 & -1 \end{bmatrix}$ and by using Equation (10), we get

$$\begin{aligned} \hat{A}^{-1} &= \begin{bmatrix} 4 & -6 & -1 \\ -6 & 9 & 2 \\ 3 & -4 & -1 \end{bmatrix} \begin{bmatrix} 1+i & 2+j & 3+k \\ i-j & 1+k & 2-2j \\ 3+i & 2-i & i+j+k \end{bmatrix} \begin{bmatrix} 4 & -6 & -1 \\ -6 & 9 & 2 \\ 3 & -4 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 4-21i+33j+45k & -6+31i-44j-66k & -1+6i-9j-15k \\ -6+38i-48j-66k & 9-56i+64j+97k & 2-11i+13j+22k \\ 3-17i+19j+30k & -4+25i-25j-44k & -1+5i-5j-10k \end{bmatrix} \\ &= \begin{bmatrix} 4 & -6 & -1 \\ -6 & 9 & 2 \\ 3 & -4 & -1 \end{bmatrix} + \begin{bmatrix} -21 & 31 & 6 \\ 38 & -56 & -11 \\ -17 & 25 & 5 \end{bmatrix} i + \begin{bmatrix} 33 & -44 & -9 \\ -48 & 64 & 13 \\ 19 & -25 & -5 \end{bmatrix} j + \begin{bmatrix} 45 & -66 & -15 \\ -66 & 97 & 22 \\ 30 & -44 & -10 \end{bmatrix} k. \end{aligned}$$

MATLAB allows us to obtain the dual quaternion matrix's inverse by Equations (11) in Example 4.1. Equations (11) are the real matrices of the inverse matrix. In the box below is the MATLAB command followed by the output. With this command, we can find the real matrices of the inverse of the dual quaternion matrix.

```
≫ A=[1 2 3;0 1 2;3 2 0];
B=[-1 0 0;-1 0 0;-1 1 -1];
C=[0 -1 0;1 0 2;0 0 -1];
D=[0 0 -1;0 -1 0;0 0 -1];
X1=inv(A)
X2= -inv(A)*B*inv(A)
X3= -inv(A)*C*inv(A)
X4= -inv(A)*D*inv(A)
[transpose(X1′) transpose(X2′) transpose(X3′) transpose(X4′)]
ans =
4 -6 -1   -21  31  6     33 -44 -9    45 -66 -15
-6 9  2    38 -56 -11    -48 64 13    -66  97  22
3 -4 -1   -17  25  5     19 -25 -5    30 -44 -10
```

Theorem 4.6.

Let $\hat{A}, \hat{B} \in M_n(H_{\mathbb{D}})$, $q \in H_{\mathbb{D}}$ (non-pure dual quaternion) and $Q \in M_n(\mathbb{R})$. After that, conditions are fulfilled below:

(i) $(q\,\hat{A})^{-1} = q^{-1}\,\hat{A}^{-1}$, if $\hat{A}$ is invertible
(ii) $(\hat{A}\,\hat{B})^{-1} = \hat{B}^{-1}\hat{A}^{-1}$, if $\hat{A}$, $\hat{B}$ are invertible,
(iii) $(\hat{A}^{*})^{-1} = (\hat{A}^{-1})^{*}$, if $\hat{A}$ is invertible,
(iv) $(\overline{\hat{A}})^{-1} = (\overline{\hat{A}^{-1}})$, if $\hat{A}$ is invertible,
(v) $(\hat{A}^{T})^{-1} = (\hat{A}^{-1})^{T}$, if $\hat{A}$ is invertible.

Proof:

(i), (ii), (iv) and (v) may be simply demonstrated. Now, we will demonstrate (iii).

If $\hat{A}$ is an invertible dual quaternion matrix, we can write

$$\hat{A}\,\hat{A}^{-1} = \hat{A}^{-1}\,\hat{A} = \hat{I}_n \tag{12}$$

Then, we take the conjugate transpose of Equation (12), $(\hat{A}\,\hat{A}^{-1})^* = (\hat{A}^{-1}\,\hat{A})^* = (\hat{I}_n)^*$, and by using Theorem 4.3 property (iv), we get

$$(\hat{A}^{-1})^*\,(\hat{A})^* = (\hat{A})^*\,(\hat{A}^{-1})^* = \hat{I}_n.$$

So, $(\hat{A})^*$ is an invertible, and we get $(\hat{A}^*)^{-1} = (\hat{A}^{-1})^*$. ■

Example 4.2.

Let $\hat{A} = \begin{bmatrix} i+j-k & 1-k \\ 1-i+k & i+j-k \end{bmatrix}$, $\hat{B} = \begin{bmatrix} i+j & 1+i+j+k \\ -1+j-k & 1-j \end{bmatrix} \in M_2(H_{\mathbb{D}})$. Then,

(i) $\overline{\hat{A}\,\hat{B}} = \begin{bmatrix} -1-j & 1-i+2k \\ -k & 1-i-2j-k \end{bmatrix} = \overline{\hat{A}}\,\overline{\hat{B}}$,

(ii) $(\hat{A}\,\hat{B})^T = \begin{bmatrix} -1+j & k \\ 1+i-2k & 1+i+2j+k \end{bmatrix} = \hat{B}^T\,\hat{A}^T$.

The feature (ii) is obtained on Example 4.2 with MATLAB. In the box below are the MATLAB command and output. We can find the real matrices of property (ii) on Example 4.2 with this command.

```
>> A=[0 1;1 0];
B=[1 0;-1 1];
C=[1 0;0 1];
D=[-1 -1;1 -1];
A2=[0 1;-1 1];
B2=[1 1;0 0];
C2=[1 1;1 -1];
D2=[0 1;-1 0];
X1=transpose(A*A2)
X2=transpose(A*B2+B*A2)
X3=transpose(A*C2+C*A2)
X4=transpose(A*D2+D*A2)
[transpose(X1′) transpose(X2′) transpose(X3′) transpose(X4′)]
ans =
-1 0   0 0   1 0    0 1
1  1   1 1   0 2   -2 1
```

4.4. The Power of the Dual Quaternion Matrix

Dual quaternion matrices' power can be calculated with the theorem below.

Theorem 4.7.

Let $\hat{A} = A + B\,i + C\,j + D\,k \in M_n(H_{\mathbb{D}})$. Then, we get for $m > 0$,

$$\hat{A}^m = A^m + \sum_{r=0}^{m-1} A^{m-r-1}(B\,i + C\,j + D\,k)\,A^r. \tag{13}$$

Proof:

We will prove Theorem 4.7 by the inductive method. When $m = 1$, both sides are the same, so Equation (13) is correct for $m = 1$. Let $t \in \mathbb{Z}^+$ and consider Equation (13) is correct for $m = t$. Later

$$\begin{aligned}\hat{A}^{t+1} &= (A + B\,i + C\,j + D\,k)\,(A^t + \sum_{r=0}^{t-1} A^{t-r-1}(B\,i + C\,j + D\,k)\,A^r)\\ &= A^{t+1} + (B\,i + C\,j + D\,k)\,A^t + A\,(\sum_{r=0}^{t-1} A^{t-r-1}(B\,i + C\,j + D\,k)\,A^r).\end{aligned}$$

It follows that:

$$\begin{aligned}\hat{A}^{t+1} &= A^{t+1} + (B\,i + C\,j + D\,k)\,A^t + (\sum_{r=0}^{t-1} A^{t-r}(B\,i + C\,j + D\,k)\,A^r)\\ &= A^{t+1} + \sum_{r=0}^{t} A^{t-r}(B\,i + C\,j + D\,k)\,A^r.\end{aligned}$$

Thus, Equation (13) holds for $m = t + 1$ and using induction method, Equation (13) is correct for all $m \in \mathbb{Z}^+$. ■

Example 4.3.

Let $\hat{A} = \begin{bmatrix} 4 - i + 2j + k & -1 + i + 2k \\ 2 + 3j - k & 2 + i - j \end{bmatrix} \in M_2(H_{\mathbb{D}})$. We can show $\hat{A}$ as

$$\hat{A} = \begin{bmatrix} 4 & -1 \\ 2 & 2 \end{bmatrix} + \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} i + \begin{bmatrix} 2 & 0 \\ 3 & -1 \end{bmatrix} j + \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} k.$$

Then, we get the 5-th power of dual quaternion matrix $\hat{A}$ by using Equation (13),

$$\begin{aligned}\hat{A}^5 &= \begin{bmatrix} 304 - 316\,i + 784\,j + 2436\,k & -316 + 316\,i - 240\,j - 136\,k \\ 632 + 1428\,j + 1220\,k & -328 + 316\,i - 644\,j + 584\,k \end{bmatrix}\\ &= \begin{bmatrix} 304 & -316 \\ 632 & -328 \end{bmatrix} + \begin{bmatrix} -316 & 316 \\ 0 & 316 \end{bmatrix} i + \begin{bmatrix} 784 & -240 \\ 1428 & -644 \end{bmatrix} j + \begin{bmatrix} 2436 & -136 \\ 1220 & 584 \end{bmatrix} k.\end{aligned}$$

We can obtain the power of the dual quaternion matrix in Example 4.3 with MATLAB by Equation (13). In the box below are the MATLAB command and output. We can find the real matrices of the power of dual quaternion matrix in Example 4.3 with this command. Moreover, the higher power of the dual quaternion matrix can be calculated.

```
≫ n=2;
m=5;
A=[4 -1;2 2];
B=[-1 1;0 1];
C=[2 0; 3 -1];
D=[1 2;-1 0];
X1=zeros(n,n);
X2=zeros(n,n);
X3=zeros(n,n);
X4=zeros(n,n);
```

```
≫ X1 = A^m;
for    r = 0 : m − 1
X2 = X2 + (A^(m − r − 1)) ∗ B ∗ A^r;
X3 = X3 + (A^(m − r − 1)) ∗ C ∗ A^r;
X4 = X4 + (A^(m − r − 1)) ∗ D ∗ A^r;
end
[transpose(X1′) transpose(X2′) transpose(X3′) transpose(X4′)]
ans =
304 -316   -316 316     784 -240   2436 -136
632 -328       0 316   1428  -644   1220  584
```

4.5. The Trace of the Dual Quaternion Matrix

For $\hat{A} = A + B\,i + C\,j + D\,k \in M_n(H_{\mathbb{D}})$, the trace of dual quaternion matrix $\hat{A}$ is described as

$$\begin{aligned} tr(\hat{A}) &= \sum_{r=1}^{n} \hat{a}_{rr} = \sum_{r=1}^{n} a_{rr} + \sum_{r=1}^{n} b_{rr} i + \sum_{r=1}^{n} c_{rr} j + \sum_{r=1}^{n} d_{rr} k \\ &= tr(A) + tr(B)i + tr(C)j + tr(D)k, \end{aligned}$$

and indicated by $tr(\hat{A})$.

Theorem 4.8.

Let $q \in H_{\mathbb{D}}$, $\hat{A}$ and $\hat{B} \in M_n(H_{\mathbb{D}})$. In that case, the following conditions are met:

(i) $tr(\hat{A}\hat{B}) = tr(\hat{B}\hat{A})$,
(ii) $tr(\hat{A} + \hat{B}) = tr(\hat{A}) + tr(\hat{B})$,
(iii) $tr(\hat{A}q) = tr(\hat{A})q$ and $tr(q\hat{A}) = q\,tr(\hat{A})$.

Proof:

(ii) and (iii) may be simply shown. Now we will demonstrate (i).

Let $\hat{A} = [\hat{a}_{rs}], \hat{B} = [\hat{b}_{st}] \in M_n(H_{\mathbb{D}})$.

$$\hat{A}\hat{B} = [\hat{a}_{rs}][\hat{b}_{st}] = \left[\sum_{s=1}^{n} \hat{a}_{rs}\hat{b}_{st}\right] = [\hat{c}_{rt}],$$

and

$$\hat{B}\hat{A} = [\hat{b}_{st}][\hat{a}_{tr}] = \left[\sum_{t=1}^{n} \hat{b}_{st}\hat{a}_{tr}\right] = [\hat{d}_{sr}].$$

Then,

$$tr(\hat{A}\hat{B}) = \sum_{r=1}^{n} \hat{c}_{rr} = \sum_{r=1}^{n}\sum_{s=1}^{n} \hat{a}_{rs}\hat{b}_{sr},$$

and

$$tr(\hat{B}\hat{A}) = \sum_{r=1}^{n} \hat{d}_{rr} = \sum_{r=1}^{n}\sum_{t=1}^{n} \hat{a}_{rt}\hat{b}_{tr}.$$

So, we get $tr(\hat{A}\hat{B}) = tr(\hat{B}\hat{A})$. ■

5. Real Matrix Representation of Dual Quaternion Matrices

This section will discuss the relationships between dual quaternion matrices and their real matrix counterparts.

Let $\hat{A} = A\,\hat{1} + B\,\hat{I} + C\,\hat{J} + D\,\hat{K}$ be a dual quaternion matrix. We shall establish the linear map $\Re_{\hat{A}}$ as $\Re_{\hat{A}} :\ M_n(H_{\mathbb{D}}) \to M_n(H_{\mathbb{D}})$ such that $\Re_{\hat{A}}(\hat{B}) = \hat{A}\,\hat{B}$. With the basis $\{\hat{1}, \hat{I}, \hat{J}, \hat{K}\}$ of the module $M_n(H_{\mathbb{D}})$ and this operator, we may find

$$\begin{aligned}
\Re_{\hat{A}}(\hat{1}) &= \hat{A}\,\hat{1} = A\,\hat{1} + B\,\hat{I} + C\,\hat{J} + D\,\hat{K},\\
\Re_{\hat{A}}(\hat{I}) &= \hat{A}\,\hat{I} = 0\,\hat{1} + A\,\hat{I} + 0\,\hat{J} + 0\,\hat{K},\\
\Re_{\hat{A}}(\hat{J}) &= \hat{A}\,\hat{J} = 0\,\hat{1} + 0\,\hat{I} + A\,\hat{J} + 0\,\hat{K},\\
\Re_{\hat{A}}(\hat{K}) &= \hat{A}\,\hat{K} = 0\,\hat{1} + 0\,\hat{I} + 0\,\hat{J} + A\,\hat{K}.
\end{aligned}$$

Later, the real matrix representation is obtained and shown below:

$$\Re_{\hat{A}} = \begin{bmatrix} A & 0_n & 0_n & 0_n \\ B & A & 0_n & 0_n \\ C & 0_n & A & 0_n \\ D & 0_n & 0_n & A \end{bmatrix}_{4n\times 4n} \in S_{4n}(\mathbb{R}), \tag{14}$$

where $S_{4n}(\mathbb{R}) \subset M_{4n}(\mathbb{R})$ and $S_{4n}(\mathbb{R})$ is a special subset of $M_{4n}(\mathbb{R})$.

Example 5.1.

$\hat{1}, \hat{I}, \hat{J}$ and $\hat{K}$'s real matrix counterparts are

$$\Re_{\hat{1}} = \begin{bmatrix} I_n & 0_n & 0_n & 0_n \\ 0_n & I_n & 0_n & 0_n \\ 0_n & 0_n & I_n & 0_n \\ 0_n & 0_n & 0_n & I_n \end{bmatrix}_{4n\times 4n}, \Re_{\hat{I}} = \begin{bmatrix} 0_n & 0_n & 0_n & 0_n \\ I_n & 0_n & 0_n & 0_n \\ 0_n & 0_n & 0_n & 0_n \\ 0_n & 0_n & 0_n & 0_n \end{bmatrix}_{4n\times 4n},$$

$$\Re_{\hat{J}} = \begin{bmatrix} 0_n & 0_n & 0_n & 0_n \\ 0_n & 0_n & 0_n & 0_n \\ I_n & 0_n & 0_n & 0_n \\ 0_n & 0_n & 0_n & 0_n \end{bmatrix}_{4n\times 4n}, \Re_{\hat{K}} = \begin{bmatrix} 0_n & 0_n & 0_n & 0_n \\ 0_n & 0_n & 0_n & 0_n \\ 0_n & 0_n & 0_n & 0_n \\ I_n & 0_n & 0_n & 0_n \end{bmatrix}_{4n\times 4n}$$

where $\hat{1}, \hat{I}, \hat{J}, \hat{K} \in M_n(H_{\mathbb{D}})$, $\Re_{\hat{1}}$, $\Re_{\hat{I}}$, $\Re_{\hat{J}}$, $\Re_{\hat{K}} \in S_{4n}(\mathbb{R})$. Besides, these matrices satisfy $\Re_{\hat{I}}\Re_{\hat{J}} = \Re_{\hat{J}}\Re_{\hat{I}} = \Re_{\hat{J}}\Re_{\hat{K}} = \Re_{\hat{K}}\Re_{\hat{J}} = \Re_{\hat{K}}\Re_{\hat{I}} = \Re_{\hat{I}}\Re_{\hat{K}} = 0_{4n}$, $\Re_{\hat{1}}^2 = I_{4n}$, $\Re_{\hat{I}}^2 = \Re_{\hat{J}}^2 = \Re_{\hat{K}}^2 = 0_{4n}$.

Example 5.2.

Let $\hat{A} = \begin{bmatrix} 1+7\,i-3\,j+8\,k & 4-2i+\,j+6\,k \\ 4+3\,i+5\,j-3\,k & 3+5\,i-4\,j+4\,k \end{bmatrix} \in M_2(H_{\mathbb{D}})$.

After that, the real matrix representation of $\hat{A}$ is

$$\Re_{\hat{A}} = \begin{bmatrix} 1 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 7 & -2 & 1 & 4 & 0 & 0 & 0 & 0 \\ 3 & 5 & 4 & 3 & 0 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 & 1 & 4 & 0 & 0 \\ 5 & -4 & 0 & 0 & 4 & 3 & 0 & 0 \\ 8 & 6 & 0 & 0 & 0 & 0 & 1 & 4 \\ -3 & 4 & 0 & 0 & 0 & 0 & 4 & 3 \end{bmatrix}.$$

MATLAB is used to determine the actual representation of the dual quaternion matrix in Example 5.2. In the box below is the MATLAB command followed by the output.

```
≫ m=2;
A=[1 4; 4 3];
B=[7 -2;3 5];
C=[-3 1; 5 -4];
D=[8 6; -3 4];
Zero=zeros(m);
Re1=vertcat(A, B, C, D);
Re2=vertcat(Zero, A, Zero, Zero);
Re3=vertcat(Zero, Zero, A, Zero);
Re4=vertcat(Zero, Zero, Zero, A);
Re=horzcat(Re1, Re2, Re3, Re4)
Re =
1  4 0 0 0 0 0 0
4  3 0 0 0 0 0 0
7 -2 1 4 0 0 0 0
3  5 4 3 0 0 0 0
-3 1 0 0 1 4 0 0
5 -4 0 0 4 3 0 0
8  6 0 0 0 0 1 4
-3 4 0 0 0 0 4 3
```

Theorem 5.1.

Let $\mu \in \mathbb{R}$, $\hat{A}$, $\hat{B} \in M_n(H_{\mathbb{D}})$. Thus, next conditions are met:

(i) $\Re_{\hat{I}_n} = \hat{I}_{4n}$,
(ii) $\Re_{\hat{A}+\hat{B}} = \Re_{\hat{A}} + \Re_{\hat{B}}$,
(iii) $\Re_{\hat{A}\,\hat{B}} = \Re_{\hat{A}}\Re_{\hat{B}}$,
(iv) $\Re_{\mu\,\hat{A}} = \mu\,\Re_{\hat{A}}$,
(v) $(\Re_{\hat{A}})^{-1} = \Re_{\hat{A}^{-1}}$ if $\hat{A}^{-1}$ and $\Re_{\hat{A}}^{-1}$ exist,
(vi) $\Re_{\hat{A}^T} \neq (\Re_{\hat{A}})^T$ in general,
(vii) $\overline{\Re_{\hat{A}}} \neq \Re_{\overline{\hat{A}}}$ in general,
(viii) $\Re_{\overline{\hat{A}}} = 2\,\Re_A - \Re_{\hat{A}}$,
(ix) $\Re_{\hat{A}}\,\Re_{\overline{\hat{A}}} = \Re_{A\,\overline{\hat{A}}} + \Re_{\hat{A}\,A} - \Re_{A^2}$.

Proof:

(i), (ii), (iv), (v), (vi), (vii), (viii) and (ix) are easily demonstrable. We shall demonstrate (iii).

Let $\hat{A} = A + B\,i + C\,j + D\,k,\ \hat{B} = A_2 + B_2\,i + C_2\,j + D_2\,k \in M_n(H_{\mathbb{D}})$. Then, the real matrix counterparts of $\hat{A}$ and $\hat{B}$ are

$$\Re_{\hat{A}} = \begin{bmatrix} A & 0_n & 0_n & 0_n \\ B & A & 0_n & 0_n \\ C & 0_n & A & 0_n \\ D & 0_n & 0_n & A \end{bmatrix}, \quad \Re_{\hat{B}} = \begin{bmatrix} A_2 & 0_n & 0_n & 0_n \\ B_2 & A_2 & 0_n & 0_n \\ C_2 & 0_n & A_2 & 0_n \\ D_2 & 0_n & 0_n & A_2 \end{bmatrix}, \quad \Re_{\hat{A}}\,\Re_{\hat{B}} = \begin{bmatrix} A\,A_2 & 0_n & 0_n & 0_n \\ A\,B_2 + B\,A_2 & A\,A_2 & 0_n & 0_n \\ A\,C_2 + C\,A_2 & 0_n & A\,A_2 & 0_n \\ A\,D_2 + D\,A_2 & 0_n & 0_n & A\,A_2 \end{bmatrix}.$$

On the other hand, $\hat{A}\,\hat{B} = A\,A_2 + (A\,B_2 + B\,A_2)\,i + (A\,C_2 + C\,A_2)\,j + (A\,D_2 + D\,A_2)\,k$. Therefore, we find

$$\Re_{\hat{A}\,\hat{B}} = \begin{bmatrix} A\,A_2 & 0_n & 0_n & 0_n \\ A\,B_2 + B\,A_2 & A\,A_2 & 0_n & 0_n \\ A\,C_2 + C\,A_2 & 0_n & A\,A_2 & 0_n \\ A\,D_2 + D\,A_2 & 0_n & 0_n & A\,A_2 \end{bmatrix}.$$

Thus, we get $\Re_{\hat{A}\,\hat{B}} = \Re_{\hat{A}}\Re_{\hat{B}}$. ■

Definition 5.1.

Let $\hat{A} = A + B\,i + C\,j + D\,k \in M_n(H_{\mathbb{D}})$ and $Q \in M_n(\mathbb{R})$, where $A,\ B,\ C,\ D \in M_n(\mathbb{R})$. The Kronecker product of an $n \times n$ dual quaternion matrix with $4n \times 4n$ real matrix is defined as follows:

$$Q \otimes \Re_{\hat{A}} = Q \otimes \begin{bmatrix} A & 0_n & 0_n & 0_n \\ B & A & 0_n & 0_n \\ C & 0_n & A & 0_n \\ D & 0_n & 0_n & A \end{bmatrix} = \begin{bmatrix} Q\,A & 0_n & 0_n & 0_n \\ Q\,B & Q\,A & 0_n & 0_n \\ Q\,C & 0_n & Q\,A & 0_n \\ Q\,D & 0_n & 0_n & Q\,A \end{bmatrix}. \tag{15}$$

Theorem 5.2.

Let $Q \in M_n(\mathbb{R})$, $\hat{A},\ \hat{B} \in M_n(H_{\mathbb{D}})$, and $\Re_{\hat{A}}, \Re_{\hat{B}}, \Re_Q \in S_{4n}(\mathbb{R})$. Thus, the next conditions are met:

(i) $Q \otimes \Re_{\hat{A}} = \Re_{Q\,\hat{A}} = \Re_Q\,\Re_{\hat{A}}$,
(ii) $Q \otimes (\Re_{\hat{A}} + \Re_{\hat{B}}) = Q \otimes \Re_{\hat{A}} + Q \otimes \Re_{\hat{B}}$,
(iii) $(Q_1 + Q_2) \otimes \Re_{\hat{A}} = Q_1 \otimes \Re_{\hat{A}} + Q_2 \otimes \Re_{\hat{A}}$, for $Q_1,\ Q_2 \in M_n(\mathbb{R})$,
(iv) $(Q_1\,Q_2) \otimes \Re_{\hat{A}} = Q_1 \otimes (Q_2 \otimes \Re_{\hat{A}})$, for $Q_1,\ Q_2 \in M_n(\mathbb{R})$.

Proof:

(ii), (iii) and (iv) are easily demonstrated. Now shall prove (i).

Let $\hat{A} = A + B\,i + C\,j + D\,k \in M_n(H_{\mathbb{D}})$, $\Re_{\hat{A}} \in S_{4n}(\mathbb{R})$ and $Q \in M_n(\mathbb{R})$.

Also, $\Re_Q = \begin{bmatrix} Q & 0_n & 0_n & 0_n \\ 0_n & Q & 0_n & 0_n \\ 0_n & 0_n & Q & 0_n \\ 0_n & 0_n & 0_n & Q \end{bmatrix}$. From Equation (15) and Equation (4), we get

$$Q \otimes \Re_{\hat{A}} = \begin{bmatrix} Q\,A & 0_n & 0_n & 0_n \\ Q\,B & Q\,A & 0_n & 0_n \\ Q\,C & 0_n & Q\,A & 0_n \\ Q\,D & 0_n & 0_n & Q\,A \end{bmatrix} = \Re_{Q\,\hat{A}} = \Re_Q\,\Re_{\hat{A}}.$$ ■

6. The Determinant of the Dual Quaternion Matrix

The determinant of a dual quaternion matrix $\hat{A}$ is a dual quaternion number and is denoted by $det(\hat{A})$. Let S_n be the symmetric group on $\{1, 2, ..., n\}$, $S(\sigma)$ be the sign of permutation σ and n be a positive integer.

Definition 6.1.

Let $\hat{A} = [\hat{a}_{rs}] \in M_n(H_{\mathbb{D}})$ and the determinant of $\hat{A}$ dual quaternion matrix is

$$det(\hat{A}) = \sum_{\sigma \in S_n} S(\sigma)\hat{a}_{\sigma(1)1}\hat{a}_{\sigma(2)2}...\hat{a}_{\sigma(n)n},$$

where the sum is over all permutations of n elements.

Remark 6.1.

If $\hat{A}$ is an identity dual quaternion matrix, then $det(\hat{A}) = 1$. $det(\hat{A})$ is identical to the normal determinant and possesses the same qualities as the normal determinant.

Example 6.1.

Let $\hat{A} = \begin{bmatrix} 4 - i + 2j + k & -1 + i + 2k \\ 2 + 3j - k & 2 + i - j \end{bmatrix} \in M_2(H_{\mathbb{D}})$. Then, by using the definition, we find the determinant of $\hat{A}$ as $det(\hat{A}) = 10 + 3j - 3k$.

Example 6.2.

Let $\hat{A} = \begin{bmatrix} 1 - i & 2 - j & 3 - k \\ -i + j & 1 - k & 2 + 2j \\ 3 - i & 2 + i & -i - j - k \end{bmatrix} \in M_3(H_{\mathbb{D}})$. Then, we find the determinant of $\hat{A}$ as

$$det(\hat{A}) = -1 - 6i + 7j + 11k.$$

Theorem 6.1.

Let $\hat{A} = A + B\,i + C\,j + D\,k \in M_n(H_{\mathbb{D}})$ be a matrix that is invertible. Thus, we obtain

$$det(\hat{A}) = det(A)(1 + tr(A^{-1}B)i + tr(A^{-1}C)j + tr(A^{-1}D)k). \tag{16}$$

Proof:

Since $\hat{A}$ is an invertible matrix, we can write $\hat{A}$ as $\hat{A} = A(I_n + (A^{-1}B)i + (A^{-1}C)j + (A^{-1}D)k)$. Then,

$$det(\hat{A}) = det(A)\, det(I_n + (A^{-1}B)i + (A^{-1}C)j + (A^{-1}D)k). \tag{17}$$

If we use determinant expansion, we get

$$det(I_n + (A^{-1}B)i + (A^{-1}C)j + (A^{-1}D)k) = 1 + tr(A^{-1}B)i + tr(A^{-1}C)j + tr(A^{-1}D)k. \tag{18}$$

From Equation (17) and Equation (18), we get

$$det(\hat{A}) = det(A)(1 + tr(A^{-1}B)i + tr(A^{-1}C)j + tr(A^{-1}D)k).$$

■

Example 6.3.

The determinant of a dual quaternion matrix can be found in the Example 6.2 with Theorem 6.1.

$A^{-1}B = \begin{bmatrix} 3 & -1 & -1 \\ -5 & 2 & -2 \\ 2 & -1 & 1 \end{bmatrix}$, $A^{-1}C = \begin{bmatrix} -6 & -4 & -11 \\ 9 & 6 & 16 \\ -4 & -3 & -7 \end{bmatrix}$, $A^{-1}D = \begin{bmatrix} 0 & 6 & -3 \\ 0 & -9 & 4 \\ 0 & 4 & -2 \end{bmatrix}$ and $det(A) = -1$.

Then, we obtain the determinant of $\hat{A}$ by Equation (16) as $det(\hat{A}) = -1 - 6i + 7j + 11k$.

Theorem 6.2.

Let $\hat{A} = A + B\,i + C\,j + D\,k \in M_n(H_{\mathbb{D}})$, where $A,\ B,\ C,\ D \in M_n(\mathbb{R})$, $\Re_{\hat{A}} \in S_{4n}(\mathbb{R})$, then we get,

$$det(\Re_{\hat{A}}) = \left\|det(\hat{A})\right\|^4 = (det(A))^4. \tag{19}$$

Proof:

From Equation (14) and Equation (16), we can obtain Equation (19). ■

MATLAB enables us to determine the determinant of a dual quaternion matrix, as seen in Example 6.2. In the box below is the MATLAB command followed by the output.

```
≫ A=[1 2 3; 0 1 2; 3 2 0];
B=[-1 0 0; -1 0 0; -1 1 -1];
C=[0 -1 0; 1 0 2; 0 0 -1];
D=[0 0 -1; 0 -1 0; 0 0 -1];
X1=det(A);
X2= det(A)*trace(inv(A)*B);
X3= det(A)*trace(inv(A)*C);
X4= det(A)*trace(inv(A)*D);
disp(['Det= 'num2str(X1)' 'num2str(X2)' i + 'num2str(X3)' j + 'num2str(X4)' k ' ])
Det= -1 -6i + 7j + 11k
```

Theorem 6.3.

Let $\hat{A} = A + B\,i + C\,j + D\,k \in M_n(H_{\mathbb{D}})$ matrix be invertible. Thus,

$$det(\hat{A}) = det(A) + (\sum_{s=1}^{n} det([A|B]_s))i + (\sum_{s=1}^{n} det([A|C]_s))j + (\sum_{s=1}^{n} det([A|D]_s))k. \tag{20}$$

Proof:

Let $\hat{A} = A + B\,i + C\,j + D\,k \in M_n(H_{\mathbb{D}})$. By $det([A|B]_s), det([A|C]_s)$, $det([A|D]_s)$ and usual determinant of A, Equation (20) could be found, easily. ■

Remark 6.2.

The matrices $[A|B]_s, [A|C]_s$ and $[A|D]_s$ mean changing $s.$ column components of the matrices B, C and D instead of $s.$ column components of matrix A.

```
≫ m=3;
A=[1 2 3; 0 1 2; 3 2 0];
B=[-1 0 0; -1 0 0; -1 1 -1];
C=[0 -1 0; 1 0 2; 0 0 -1];
D=[0 0 -1; 0 -1 0; 0 0 -1];
Aoriginal=A;
X1=det(A);
X2=0;
X3=0;
X4=0;
for r = 1:m
A(:,r) = B(:,r);
X2 = X2+det(A);
A=Aoriginal;
end
```

```
≫ for r = 1:m
A(:,r) = C(:,r);
X3 = X3+det(A);
A=Aoriginal;
end
for r = 1:m
A(:,r) = D(:,r);
X4 = X4+det(A);
A=Aoriginal;
end
disp(['Det= 'num2str(X1)' 'num2str(X2)' i + 'num2str(X3)' j +'num2str(X4)' k ' ])
Det= -1 -6i + 7j + 11k
```

Determinant of the dual quaternion matrix could be found in Example 6.2 by Equation (20) with MATLAB. In the box below is the MATLAB command followed by the output.

Remark 6.3.

Applying the determinant in three distinct methods to the displayed matrix in Example 6.2 yielded equal outcomes. It's not easy to determine the determinant of a dual quaternion matrix directly. The determinant can be easily calculated by the methods in Theorem 6.1 and Theorem 6.3.

Definition 6.2.

Let $\hat{A} = [\hat{a}_{rs}] \in M_n(H_{\mathbb{D}})$ be a dual quaternion matrix. Minor of $\hat{A} = [\hat{a}_{rs}]$ is indicated by $det(\hat{M}_{rs})$ and is defined to be the determinant of the submatrix that remains after the r^{th} row and s^{th} column deleted from $\hat{A}$. The number $(-1)^{r+s} det(\hat{M}_{rs})$ is denoted by $\hat{A}_{rs}$ and is called the cofactor of $[\hat{a}_{rs}]$. The adjoint of $\hat{A}$ is the matrix's transposition obtained by replacing each element of $\hat{A} = [\hat{a}_{rs}]$ with its cofactor. It is denoted as $adj(\hat{A})$.

$$adj(\hat{A}) = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} & \dots & \hat{A}_{1n} \\ \hat{A}_{21} & \hat{A}_{22} & \dots & \hat{A}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{A}_{n1} & \hat{A}_{n2} & \dots & \hat{A}_{nn} \end{bmatrix}^T .$$

Then, the following theorem and corollary can be stated without proof.

Theorem 6.4.

Let $\hat{A} = [\hat{a}_{rs}] \in M_n(H_{\mathbb{D}})$ be an invertible dual quaternion matrix. We get

$$\hat{A}\, adj(\hat{A}) = adj(\hat{A})\hat{A} = det(\hat{A})\hat{I}_n.$$

Proof:

$$\hat{A}\, adj(\hat{A}) = \begin{bmatrix} \hat{a}_{11} & \hat{a}_{12} & \dots & \hat{a}_{1n} \\ \hat{a}_{21} & \hat{a}_{22} & \dots & \hat{a}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{a}_{n1} & \hat{a}_{n2} & \dots & \hat{a}_{nn} \end{bmatrix} \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} & \dots & \hat{A}_{1n} \\ \hat{A}_{21} & \hat{A}_{22} & \dots & \hat{A}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{A}_{n1} & \hat{A}_{n2} & \dots & \hat{A}_{nn} \end{bmatrix}^T = \begin{bmatrix} \sum_{k=1}^{n} \hat{a}_{1k}\hat{A}_{1k} & \sum_{k=1}^{n} \hat{a}_{1k}\hat{A}_{2k} & \dots & \sum_{k=1}^{n} \hat{a}_{1k}\hat{A}_{nk} \\ \sum_{k=1}^{n} \hat{a}_{2k}\hat{A}_{1k} & \sum_{k=1}^{n} \hat{a}_{2k}\hat{A}_{2k} & \dots & \sum_{k=1}^{n} \hat{a}_{2k}\hat{A}_{nk} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{k=1}^{n} \hat{a}_{nk}\hat{A}_{1k} & \sum_{k=1}^{n} \hat{a}_{nk}\hat{A}_{2k} & \dots & \sum_{k=1}^{n} \hat{a}_{nk}\hat{A}_{nk} \end{bmatrix}$$

$$= \begin{bmatrix} det(\hat{A}) & 0 & \dots & 0 \\ 0 & det(\hat{A}) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & det(\hat{A}) \end{bmatrix} = det(\hat{A}) \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} = det(\hat{A})\hat{I}_n$$

Similarly, if $adj(\hat{A})\hat{A}$ is calculated, then we obtain $\hat{A}\, adj(\hat{A}) = adj(\hat{A})\hat{A} = det(\hat{A})\hat{I}_n$. ∎

Corollary 6.1.

Let $\hat{A} = [\hat{a}_{rs}] \in M_n(H_{\mathbb{D}})$ be an invertible dual quaternion matrix. We obtain

$$\hat{A}^{-1} = \frac{1}{det(\hat{A})}\, adj(\hat{A}). \tag{21}$$

Proof:

From Theorem 6.4, we have

$$\hat{A}\, adj(\hat{A}) = det(\hat{A})\hat{I}_n \tag{22}$$

Since $det(\hat{A}) \neq 0$, we can divide Equation (22) by $det(\hat{A})$, we get $\hat{A}\frac{1}{det(\hat{A})}\, adj(\hat{A}) = \hat{I}_n$. So, we obtain $\hat{A}^{-1} = \frac{1}{det(\hat{A})}\, adj(\hat{A})$. ■

Example 6.4.

We may obtain the inverse of $\hat{A}$ by using Corollary 6.1 on Example 4.1. Then, we find the adjoint matrix

$$adj(\hat{A}) = \begin{bmatrix} -4-3\,i+5\,j-\,k & 6+5\,i+2\,j & 1+2\,j+4\,k \\ 6-2\,i+6\,j & -9+2\,i-j+2\,k & -2-i+j \\ -3-i+2\,j+3\,k & 4+i-3\,j & 1+i-2\,j-k \end{bmatrix},$$

and $det(\hat{A}) = -1-6\,i+7\,j+11\,k$. So, we find the inverse matrix by using Equation (21)

$$\hat{A}^{-1} = \begin{bmatrix} 4-21\,i+33\,j+45\,k & -6+31\,i-44\,j-66\,k & -1+6\,i-9\,j-15\,k \\ -6+38\,i-48\,j-66\,k & 9-56\,i+64\,j+97\,k & 2-11\,i+13\,j+22\,k \\ 3-17\,i+19\,j+30\,k & -4+25\,i-25\,j-44\,k & -1+5\,i-5\,j-10\,k \end{bmatrix}.$$

Example 6.5.

$det(\Re_{\hat{1}}) = 1,\ det(\Re_{\hat{I}}) = det(\Re_{\hat{J}}) = det(\Re_{\hat{K}}) = 0.$

Hence, we may find the inverse of $\hat{A}$ on Example 4.1 using Theorem 5.1 property (v). The following example demonstrates how to invert a dual quaternion matrix using the inverse of its real matrix representation.

Example 6.6.

The real matrix representation of $\hat{A}$ on Example 4.1 is $\Re_{\hat{A}} = \begin{bmatrix} 1 & 2 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 2 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & -1 & 3 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 1 & 2 & 3 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 3 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 2 & 0 \end{bmatrix}$. Here,

$det(\Re_{\hat{A}}) = 1 \neq 0$. Hence, $\hat{A}$ is invertible. $(\Re_{\hat{A}})^{-1}$ is found using MATLAB as below:

$$(\Re_{\hat{A}})^{-1} = \begin{bmatrix} 4 & -6 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -6 & 9 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & -4 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -21 & 31 & 6 & 4 & -6 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 38 & -56 & -11 & -6 & 9 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -17 & 25 & 5 & 3 & -4 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 33 & -44 & -9 & 0 & 0 & 0 & 4 & -6 & -1 & 0 & 0 & 0 \\ -48 & 64 & 13 & 0 & 0 & 0 & -6 & 9 & 2 & 0 & 0 & 0 \\ 19 & -25 & -5 & 0 & 0 & 0 & 3 & -4 & -1 & 0 & 0 & 0 \\ 45 & -66 & -15 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & -6 & -1 \\ -66 & 97 & 22 & 0 & 0 & 0 & 0 & 0 & 0 & -6 & 9 & 2 \\ 30 & -44 & -10 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & -4 & -1 \end{bmatrix}.$$

By using the inverse of $\Re_{\hat{A}}$, we find

$$\hat{A}^{-1} = \begin{bmatrix} 4-21\,i+33\,j+45\,k & -6+31\,i-44\,j-66\,k & -1+6\,i-9\,j-15\,k \\ -6+38\,i-48\,j-66\,k & 9-56\,i+64\,j+97\,k & 2-11\,i+13\,j+22\,k \\ 3-17\,i+19\,j+30\,k & -4+25\,i-25\,j-44\,k & -1+5\,i-5\,j-10\,k \end{bmatrix}.$$

Corollary 6.2.

Let $\hat{A} \in M_n(H_{\mathbb{D}})$ be an invertible dual quaternion matrix and $adj(\hat{A})$ be the adjoint matrix of $\hat{A}$. Then, we get $det(adj(\hat{A})) = (det(\hat{A}))^{n-1}$.

Proof:

Taking determinants on both sides of the Equation (22), we get $det(det(\hat{A})\hat{I}_n) = det(\hat{A}\,adj(\hat{A}))$ and $det(\hat{A})^n = det(\hat{A})\,det(adj(\hat{A}))$. Finally, we obtain, $det(adj(\hat{A})) = \frac{(det(\hat{A}))^n}{det(\hat{A})} = (det(\hat{A}))^{n-1}$. ■

The main results presented in this paper concern the study of the algebras $H_{\mathbb{D}}$ and $M_n(H_{\mathbb{D}})$. We gave the comparison of real and dual quaternions in Table 1. Moreover, we compared our results and methods for dual quaternion matrices with real quaternion matrices in Table 2.

Table 1. Comparison of Real and Dual Quaternion

Real Quaternion ($H_{\mathbb{R}}$)	**Dual Quaternion** ($H_{\mathbb{D}}$)
$i^2 = j^2 = k^2 = -1\,,\ \ i\,j = -j\,i = k\,,$ $j\,k = -k\,j = i\,,\quad k\,i = -i\,k = j.$	$i^2 = j^2 = k^2 = 0,$ $i\,j = j\,i = j\,k = k\,j = k\,i = i\,k = 0.$
Multiplication is non-commutative.	Multiplication is commutative.
The set $H_{\mathbb{R}}$ is a division algebra (Lewis, 2006; Zhang, 1997).	The set $H_{\mathbb{D}}$ is not a division algebra.
All non-zero elements of quaternions are invertible (Lewis, 2006).	Pure dual quaternions are not invertible.
The set $H_{\mathbb{R}}$ is a ring (Lewis, 2006).	The set $H_{\mathbb{D}}$ is a commutative unital ring under dual quaternion multiplication.
The set $H_{\mathbb{R}}$ is a 4-dimensional vector space on $\mathbb{R}$ and isomorphic to the $\mathbb{R}^4$ (Zhang, 1997).	The set $H_{\mathbb{D}}$ is a 4-dimensional vector space on $\mathbb{R}$ and isomorphic to the Galilean space G^4.

Table 2. Comparison of Real and Dual Quaternion Matrices

Real Quaternion Matrices ($M_n(H_{\mathbb{R}})$)	**Dual Quaternion Matrices** ($M_n(H_{\mathbb{D}})$)
The set $M_n(H_{\mathbb{R}})$ is a vector space on $H_{\mathbb{R}}$ (Zhang, 1997).	The set $M_n(H_{\mathbb{D}})$ is an n^2 dimensional module on $H_{\mathbb{D}}$.
—	The set $M_n(H_{\mathbb{D}})$ is a $4\,n^2$ dimensional vector space on $\mathbb{R}$.
—	The set $M_n(H_{\mathbb{D}})$ is a 4 dimensional module on $M_n(\mathbb{R})$.
The set $M_n(H_{\mathbb{R}})$ is a division algebra.	The set $M_n(H_{\mathbb{D}})$ is not a division algebra.
The set $M_n(H_{\mathbb{R}})$ is a ring.	The set $M_n(H_{\mathbb{D}})$ is a ring.
No general formula is found in the literature for the power of real quaternion matrices.	The power of the dual quaternion matrix can be found. Let $\hat{A} = A + B\,i + C\,j + D\,k \in M_n(H_{\mathbb{D}})$ and $m > 0$. $\hat{A}^m = A^m + \sum\limits_{r=0}^{m-1} A^{m-r-1}(B\,i + C\,j + D\,k)\,A^r$.
Inverse of quaternion matrices could be found. But no general formula is found for the inverse of quaternion matrices in the literature.	Let $\hat{A} = A + B\,i + C\,j + D\,k \in M_n(H_{\mathbb{D}})$. If $det(A) \neq 0$, then dual quaternion matrix $\hat{A}$ is invertible and inverse of this matrix can be found as follows: $\hat{A}^{-1} = A^{-1}\,\overline{\hat{A}}\,A^{-1}$.
The generalization of the adjoint matrix's inverse approach to quaternion matrices is quite challenging.	We generalized the inverse method of the adjoint matrix to dual quaternion matrices $\hat{A}^{-1} = \frac{1}{det(\hat{A})}\,adj(\hat{A})$.
It is very difficult to generalize the determinant of the real matrix to quaternion matrices.	$det(\hat{A})$ is the same as the usual determinant and has the same properties as the usual determinant.
No formula is found by using trace for determinant calculation.	Let $\hat{A} = A + B\,i + C\,j + D\,k \in M_n(H_{\mathbb{D}})$ be an invertible matrix. $det(\hat{A}) = det(A)(1 + tr(A^{-1}B)i + tr(A^{-1}C)j + tr(A^{-1}D)k)$.
Let $\tilde{A}, \tilde{B}$ be real quaternion matrices. Real quaternion matrices satisfy the properties listed below: $\overline{\tilde{A}\,\tilde{B}} \neq \overline{\tilde{A}}\,\overline{\tilde{B}},\ for\ \tilde{A} \in M_{m\times n}(H_{\mathbb{R}}),\ \tilde{B} \in M_{n\times p}(H_{\mathbb{R}})$, $(\tilde{A}\,\tilde{B})^T \neq \tilde{B}^T\,\tilde{A}^T,\ for\ \tilde{A},\ \tilde{B} \in M_n(H_{\mathbb{R}})$, (in general) $(\overline{\tilde{A}})^{-1} \neq (\overline{\tilde{A}^{-1}})$, if $\tilde{A}$ is invertible, (in general) $(\tilde{A}^T)^{-1} \neq (\tilde{A}^{-1})^T$, if $\tilde{A}$ is invertible. (in general)	Let $\hat{A}, \hat{B}$ be dual quaternion matrices. Dual quaternion matrices satisfy the properties listed below: $\overline{\hat{A}\,\hat{B}} = \overline{\hat{A}}\,\overline{\hat{B}},\ for\ \hat{A} \in M_{m\times n}(H_{\mathbb{D}}),\ \hat{B} \in M_{n\times p}(H_{\mathbb{D}})$, $(\hat{A}\,\hat{B})^T = \hat{B}^T\,\hat{A}^T,\ for\ \hat{A},\ \hat{B} \in M_n(H_{\mathbb{D}})$, $(\overline{\hat{A}})^{-1} = (\overline{\hat{A}^{-1}})$, if $\hat{A}$ is invertible, $(\hat{A}^T)^{-1} = (\hat{A}^{-1})^T$, if $\hat{A}$ is invertible.

7. Conclusion

Our work is to examine dual quaternion matrices theoretically and include practical examples for those working in applied fields. Users who do not use MATLAB can be inspired by applied MATLAB programs and adapt them to the programs they use. It will be easier to solve examples with large matrix sizes with MATLAB. This study is critical because it is an applied study related to quaternion matrices. Real quaternions and real quaternion matrices have been the subject of significant research in academic journals. There is just a handful of published research on dual quaternions. In this study, the novel aspect of our research, a set of dual quaternion matrices, will be defined for the very first time, and its features will be described in detail for the first time. In addition to this, the study's significance lies in the fact that it is an applied investigation into dual quaternion matrices. Within the confines of this investigation, not only will the dual quaternion matrices that were the subject of our investigation be defined for the very first time, but also the intricacies of their characteristics will be investigated for the very first time. Our investigation has led to the discovery of this one-of-a-kind and very important addition. In addition, the value of this study cannot be overstated because it is an applied piece of research that addresses dual quaternion matrices, and the fact that it does so alone makes it an incredibly essential piece of work. Users who do not work with MATLAB can take concepts from applications built with it and implement them in the software with which they work.

We found that many methods that cannot be generalized with real quaternion matrices can be generalized with dual quaternion matrices. This study investigated the basic and additional properties of dual quaternion matrices. In addition, real matrix representations of these matrices and the features of these matrices were constructed. These were employed in determining the determinants and inverses of the dual quaternion matrices. Also, several kinds of determinants and inverses of dual quaternion matrices were devised, and MATLAB apps were built to make solving instances with these approaches easier. The characteristics of real quaternions and dual quaternions have been compared and contrasted with one another.

Moreover, our results and methods for dual quaternion matrices with real quaternion matrices were compared. Obtained results and methods were contrasted with the fundamental features of the quaternion matrix. In future studies, we will analyze additional features of real quaternion matrices and examine these aspects for dual quaternion matrices.

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