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Nimai Sarkar
Madanapalle Institute of Technology & Science

Mausumi Sen
National Institute of Technology Silchar

Pratiksha Jadli
National Institute of Technology Silchar

Dipankar Saha
National Institute of Technology Silchar

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A Novel Collocation Method for Solving Second-order Volterra Integro-differential Equations

[1*]Nimai Sarkar, [2]Mausumi Sen, [3]Pratiksha Jadli and [4]Dipankar Saha

[1]Department of Mathematics
Madanapalle Institute of Technology & Science
Madanapalle, Andhra Pradesh, India

[2,3,4]Department of Mathematics
National Institute of Technology Silchar
Silchar, Assam, India
[1]nimaisarkar298@gmail.com; [2]senmausumi@gmail.com; [3]pratikshajadli65@gmail.com;
[4]nayan0507@gmail.com

*Corresponding Author

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Abstract

In this article, we present an efficient numerical methodology to solve second-order linear Volterra integro-differential equations. Further, the modified Chebyshev collocation method is used at the Gauss-Lobatto collocation points. In that context, some theoretical investigation related to error analysis is suggested through residual function. Numerical examples are also encountered to study the applicability of the present method. In order to get a vivid illustration of the efficiency, we present a comparative survey with three existing collocation methods.

Keywords: Volterra integro-differential equations; Collocation method; Gauss-Lobatto collocation points

MSC 2010 No.: 45J05, 74S99

1. Introduction

In recent years, a large community of mathematicians has put their attention on the development of effective numerical methods to deal with various kinds of integral equations, differential equations, and most prominently integro-differential equations. Specifically different types of integro-differential equations occur in numerous branches of scientific phenomenons and engineering problems, such as population dynamics (Yuzbasi et al. (2013); Bonnefon et al. (2014)), evolution theory (Caraballo et al. (2009)), spreading of the epidemic (Medlock et al. (2003); Kumar et al. (2020); Gao et al. (2020)), heat conduction (Dehghan et al. (2010)), potential theory (Caffarelli et al. (2009)), electrostatics (Hildebrandt et al. (2004)), radiative transfer (Bellman et al. (1966); Phillip et al. (1969)), mechanical engineering (Indiaminov et al. (2020)), control theory (Jothimani et al. (2019)), and many other sectors of mathematical physics. Extensive efforts have been engaged on the solvability and uniqueness of exact solutions for several types of integro-differential equations, see the literature by Sarkar and Sen (2021), Saha et al. (2020), Ravichandran et al. (2019) and references therein. Consequently, the approximate solutions are the immediate requirement once the analytical methods fail.

To investigate approximate solutions of complex models, a good number of techniques were proposed in the past, such as the finite element method (Chen et al. (2016); Chen et al. (2019)), the finite difference technique (Zhao et al. (2006)), the boundary element method (Sarkar et al. (2020)), the iterative method (Ghasemi et al. (2007); Yousefi et al. (2009)), successive approximation method (Ronto and Ronto (2009)), standard integral collocation method (Karamete et al. (2002); Yalcinbas et al. (2002); Darania et al. (2008)), Chebyshev collocation method (Akyuz-Dascioglu and Sezer (2005)) and other polynomial collocation methods found in the literature (Mohsen and El-Gamel (2010); Sarkar et al. (2021); Turkyilmazoglu (2014)).

In this present work, our main objective is to produce a simpler and efficient methodology to solve the considered class of equations. For that purpose, we have adopted power series along with the collocation scheme at modified Chebyshev-Gauss-Lobatto collocation points.

This article aims to apply an efficient method for solving the second-order integro-differential equation. The present work suggests that the proposed scheme is comparatively simpler to apply than many other existing methods, whereas the numerical results and graphical illustration depict the accuracy and superiority of the presented method. The prime attraction of present technique is displayed by the comparative study. The superior results for different input values suggest the novelty of present work and it is worthy to solve the considered type equations approximately.

The manuscript is arranged in the following manner. In Section 2, the considered class of equation is introduced and the methodology is elaborately described. Section 3 deals with the illustration of the numerical scheme through some problems, also the outcomes are compared with other existing methods. Finally, in Section 4 some concluding remarks are mentioned.

2. Main Results

In this study, the hybridization of power series approximation and collocation method is adopted to investigate the approximate solution of the considered class of equations. In the connection of numerical solution, the current article develops a hybridized numerical technique for the linear counterpart of following equation

$$A_2(x)w^{''}(x) + A_1(x)w^{'}(x) = f\Big(x, w(x), \int_{x_0}^{x} K(x,s)w(s)ds\Big), \tag{1}$$

with initial conditions

$$w(x_0) = w_0, w^{'}(x_0) = w_0^1, \tag{2}$$

where $w(x)$ is the unknown function, the kernel $K(x,s)$ is non singular function and $S = \Big\{(x,s) : x_0 \leq s \leq x \leq T\Big\}$. Our main focus is to apply the proposed methodology to the linear counterpart of the equation (1).

2.1. Description of the methodology

We proceed with the linear counterpart of (1) as the following equation:

$$A_2(x)w^{''}(x) + A_1(x)w^{'}(x) + A_0(x)w(x) = F(x) + \int_{x_0}^{x} K(x,s)w(s)ds. \tag{3}$$

In this study, the approximate solution is investigated under the consideration that the solution of considered class of equation is analytic on the prescribed domain. We assume that approximate solution of (3) is of the following form,

$$w_M(x) = \sum_{i=0}^{M} \alpha_i x^i, i \geq 0, \tag{4}$$

where x^is are monomial bases and α_is are real coefficients. Implementing (4) and its derivatives in both sides of Equation (3) we have the following estimates,

$$A_2(x)\sum_{i=0}^{M} i(i-1)\alpha_i x^{i-2} + A_1(x)\sum_{i=0}^{M} i\alpha_i x^{i-1} + A_0(x)\sum_{i=0}^{M}\alpha_i x^i$$
$$= F(x) + \sum_{i=0}^{M}\alpha_i \int_{x_0}^{x} s^i K(x,s)ds, \tag{5}$$

with

$$F(x) = \sum_{i=0}^{M}\alpha_i\Big(A_2(x)i(i-1)x^{i-2} + A_1(x)ix^{i-1} + A_0(x)x^i - \int_{x_0}^{x} s^i K(x,s)ds\Big), \tag{6}$$

where $F(x)$, $A_0(x)$, $A_1(x)$, $A_2(x)$ and $K(x,s)$ are prescribed smooth functions. Thus, for fixed M, above illustration gives rise to a system of $M+1$ linear algebraic equations,

$$\Sigma_{k=0}^{M}\alpha_k\sigma_k(x_l) = F(x_l), l = 0, 1, 2, ..., M \tag{7}$$

where $\sigma_k(x_l)$ is given by

$$\sigma_k(x_l) = A_2(x_l)k(k-1){x_l}^{k-2} + A_1(x_l)k{x_l}^{k-1} + A_0(x_l){x_l}^k - \int_{x_0}^{x_l} s^i K(x_l,s)ds.$$

Afterwards, collocation method is used at the modified Chebyshev-Gauss-Lobatto points,

$$x_l = \frac{1}{2}\left[1 + \cos\left(\frac{(M-l)\pi}{M}\right)\right], l = 0, 1, 2, ..., M. \tag{8}$$

The system (7) is written in matrix notation as

$$\mathbf{UD} = \mathbf{F}, \tag{9}$$

where

$$\mathbf{U} = \begin{pmatrix} \sigma_0(x_0) & \sigma_1(x_0) & \sigma_2(x_0) & ... & \sigma_M(x_0) \\ \sigma_0(x_1) & \sigma_1(x_1) & \sigma_2(x_1) & ... & \sigma_M(x_1) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \sigma_0(x_{M-1}) & \sigma_1(x_{M-1}) & \sigma_2(x_{M-1}) & ... & \sigma_M(x_{M-1}) \\ \sigma_0(x_M) & \sigma_1(x_M) & \sigma_2(x_M) & ... & \sigma_M(x_M) \end{pmatrix},$$

$$\mathbf{D} = \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{M-1} \\ \alpha_M \end{pmatrix} \text{ and } \mathbf{F} = \begin{pmatrix} F(x_0) \\ F(x_1) \\ \vdots \\ F(x_{M-1}) \\ F(x_M) \end{pmatrix}.$$

The consistency of (9) is assured by rank($\mathbf{U}$) $=$ rank($\mathbf{U}|\mathbf{F}$) $= M+1$ and subsequently $\mathbf{D}$ is determined. Finally, the approximate solution follows from (4).

Now we establish a brief description on the upper bound of error estimation.

2.2. Boundedness of error estimation

In this section, we develop the residual error and the classical absolute error methods. After substituting the approximate solution in equation (3), the resulting relation is obtained as,

$$\mathcal{R}_M(x_l) = w_M^{''}(x_l) + \frac{1}{A_2(x_l)}\Big(A_1(x_l)w_M^{'}(x_l) + A_0(x_l)w_M(x_l) - \int_{x_0}^{x_l} K(x_l,s)w(s)ds - F(x_l)\Big), \tag{10}$$

where x_l's are given by (8). Consider $\mathcal{R}_M(x_l) \leq 10^{\gamma_l}$ such that γ_l is any positive number. If maximum of $10^{-\gamma_l} = 10^{-\gamma}$, then the truncation limit M is increased until $\mathcal{R}_M(x_l)$ at each point is smaller than the desired accuracy $10^{-\gamma}$. For sufficiently large M, $\mathcal{R}_M(x_l) \to 0$; in that case, error decreases rapidly.

However, the boundedness of residual function can be estimated as follows. Adopting standard inequality from integral calculus, we have

$$\left| \int_{x_0}^{x} \mathcal{R}_M(t)dt \right| \leq \int_{x_0}^{x} |\mathcal{R}_M(t)|dt,$$

and implementing integral mean value theorem, the upper bound of mean error is obtained as

$$\mathcal{R}_M(\theta) \leq \frac{\int_{x_0}^{x} |\mathcal{R}_M(t)|dt}{x-x_0}; x_0 < \theta < x.$$

3. Numerical Results

In this section the proposed methodology has been employed to solve two sample problems. The absolute error is defined by

$$|w(x_l) - w_M(x_l)|, l = 0, 1, 2, ..., M, \tag{11}$$

where $w(x_l)$ and $w_M(x_l)$ stand for exact and approximate solution at $x = x_l$, respectively. Moreover, the representative tables demonstrate comparative study among the developed method and the other three standard methods.

Example 3.1.

$$w^{''}(x) - w^{'}(x) = -\frac{\left(1 + xe^{x+1}\right)}{(x+1)^2} + \int_0^1 te^{xt}w(t)dt, \tag{12}$$

with initial conditions, $w(0) = 1$ and $w^{'}(0) = 2 - e + \int_0^1 w(t)dt$.

Table 1. Solution for $M = 2$

input(x_l)	0	0.2	0.4	0.6	0.8	1.0
Exact solution	1.10e-2	1.18e-2	1.28e-2	1.40e-2	1.55e-2	1.73e-2
M-I	1.10e-2	1.03e-2	0.73e-2	0.05e-2	0.03e-2	0.02e-2
M-II	0.01e-2	0.37e-2	0.95e-2	0.07e-2	0.67e-2	0.39e-2
M-III	0.01e-2	0.18e-2	0.41e-2	0.70e-2	1.05e-2	1.49e-2
Proposed method	1.10e-2	1.19e-2	1.31e-2	1.46e-2	1.60e-2	1.76e-2

For the first example, Table 1 through Table 3 (and for the second example, Table 6 through Table 8), illustrate that the approximate solution is compatible with the exact solution. A comparative study with M-I (Legendre polynomial collocation method), M-II (Chebyshev polynomial collocation method), M-III (Laguree polynomial collocation method) suggests the superiority of the proposed method. Table 5 and Table 4 are there in support of the error analysis for the developed scheme.

Table 2. Solution for $M = 3$

input(x_l)	0	0.2	0.4	0.6	0.8	1.0
Exact solution	1.10e-2	1.18e-2	1.28e-2	1.40e-2	1.55e-2	1.73e-2
M-I	0.01e-2	0.75e-2	0.06e-2	0.75e-2	0.36e-2	0.20e-2
M-II	0.01e-2	0.47e-2	0.33e-2	1.54e-2	1.07e-2	0.38e-2
M-III	0.01e-2	0.80e-2	0.48e-2	1.55e-2	1.79e-2	1.20e-2
Proposed method	1.10e-2	1.18e-2	1.28e-2	1.41e-2	1.56e-2	1.74e-2

Table 3. Solution for $M = 4$

input(x_l)	0	0.2	0.4	0.6	0.8	1.0
Exact solution	1.10e-2	1.18e-2	1.28e-2	1.40e-2	1.55e-2	1.73e-2
M-I	1.10e-2	1.08e-2	0.05e-2	0.08e-2	0.02e-2	0.87e-2
M-II	1.10e-2	0.46e-2	0.12e-2	0.62e-2	0.21e-2	0.57e-2
M-III	0.01e-2	0.56e-2	0.05e-2	0.50e-2	0.95e-2	0.44e-2
Proposed method	1.10e-2	1.18e-2	1.28e-2	1.40e-2	1.55e-2	1.73e-2

Example 3.2.

$$w^{''}(x) = \left(2 + \frac{(x-2)e^x + s + 2}{x^3}\right) + \int_0^1 e^{xt} w(t)dt, \tag{13}$$

with initial conditions, $w(0) = 0$ and $w^{'}(0) = -1$.

Table 4. Absolute error for $M = 3, 4, 5$

input(x_l)	$M = 3$	$M = 4$	$M = 5$
0	0.01e-2	0	0
0.2	0.01e-2	0	0
0.4	0.01e-2	0.01e-2	0
0.6	0.01e-2	0.01e-2	0
0.8	0.01e-2	0.01e-2	0
1.0	0	0.01e-2	0.01e-2

Figure 1 through Figure 6 represents a graphical illustration of the outcomes corresponding to Table 1 through Table 7, respectively. The error estimations are shown in Figures 7 and 8.

4. Conclusion

This manuscript aims to apply an efficient method for solving the second-order integro-differential equation. The present article suggests that the proposed scheme is comparatively simpler than many other existing methods, whereas the numerical results and graphical illustration depict the accuracy and superiority of the presented method. The main benefit of this hybridized scheme is its accuracy, particularly in the second-order class. Results obtained from considered two numerical examples show that error reduces as M increases, which strongly supports the efficiency of the present method in the considered class of integro-differential equation.

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Appendix

Table 5. Absolute error for $M = 2, 3, 4$

input(x_l)	$M = 2$	$M = 3$	$M = 4$
0	0	0	0
0.2	0.01e-2	0.01e-2	0
0.4	0.04e-2	0	0
0.6	0.05e-2	0.01e-2	0
0.8	0.06e-2	0.01e-2	0
1.0	0.02e-2	0.02e-2	0

Table 6. Solution for $M = 3$

input(x_l)	0	0.2	0.4	0.6	0.8	1.0
Exact solution	0	-1.60e-2	-2.40e-2	-2.40e-2	-1.60e-2	0
M-I	0	-0.41e-2	-0.57e-2	-0.53e-2	-0.32e-2	0
M-II	0	-1.77e-2	-1.69e-2	-0.71e-2	-0.18e-2	0
M-III	0	-0.24e-2	-0.36e-2	-0.36e-2	-0.24e-2	0
Proposed Method	0	-1.6e-2	-2.4e-2	-2.4e-2	-1.59e-2	0.01e-2

Table 7. Solution for $M = 4$

input(x_l)	0	0.2	0.4	0.6	0.8	1.0
Exact solution	0	-1.60e-2	-2.41e-2	-2.41e-2	-1.63e-2	0
M-I	0	-0.38e-2	-0.58e-2	-0.54e-2	-0.30e-2	0
M-II	0	-0.15e-2	-0.22e-2	-0.22e-2	-0.15e-2	0
M-III	0	-0.57e-2	-0.16e-2	-0.99e-2	-0.20e-2	0
Proposed Method	0	-1.61e-2	-2.43e-2	-2.42e-2	-1.61e-2	0

Table 8. Solution for $M = 5$

input(x_l)	0	0.2	0.4	0.6	0.8	1.0
Exact solution	0	-1.60e-2	-2.42e-2	-2.40e-2	-1.61e-2	0
M-I	0	-0.33e-2	-0.62e-2	-0.72e-2	-0.60e-2	0
M-II	0	-0.15e-2	-0.22e-2	-0.22e-2	-0.14e-2	0
M-III	0	-0.10e-2	-0.06e-2	-0.02e-2	-0.05e-2	0
Proposed Method	0	-1.60e-2	-2.40e-2	-2.41e-2	-1.63e-2	0

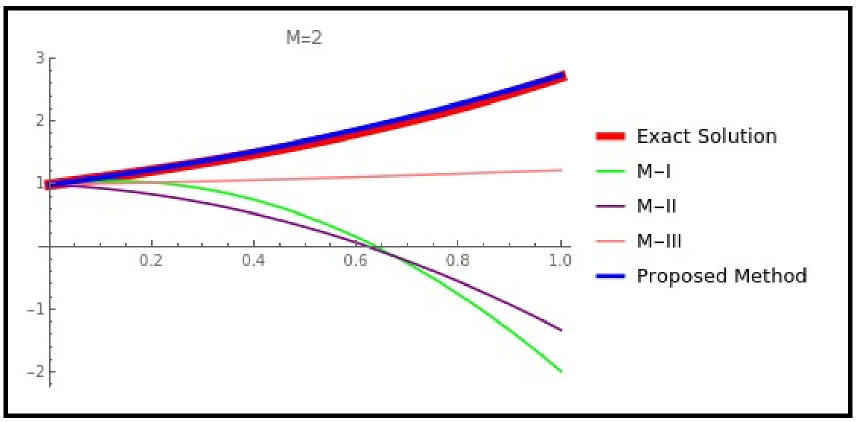

Figure 1. Graphical view for $M = 2$

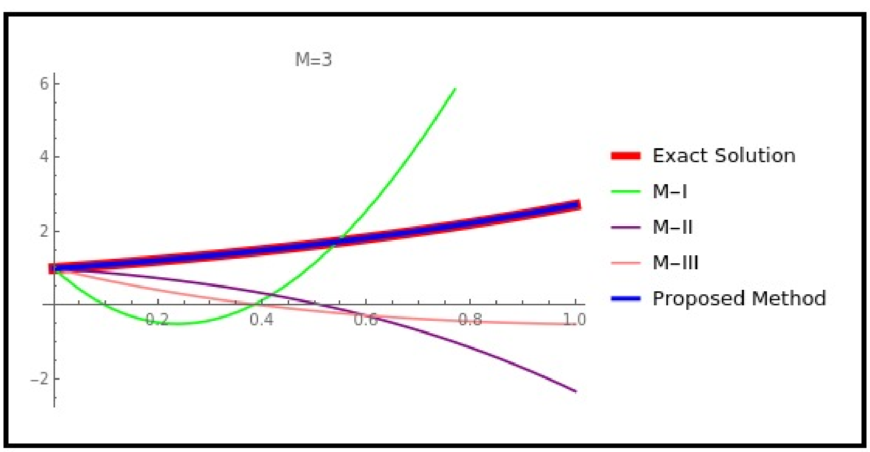

Figure 2. Graphical view for $M = 3$

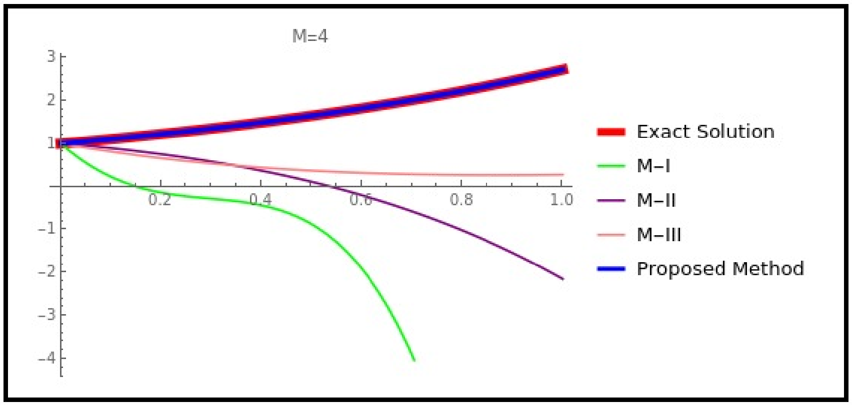

Figure 3. Graphical view for $M = 4$

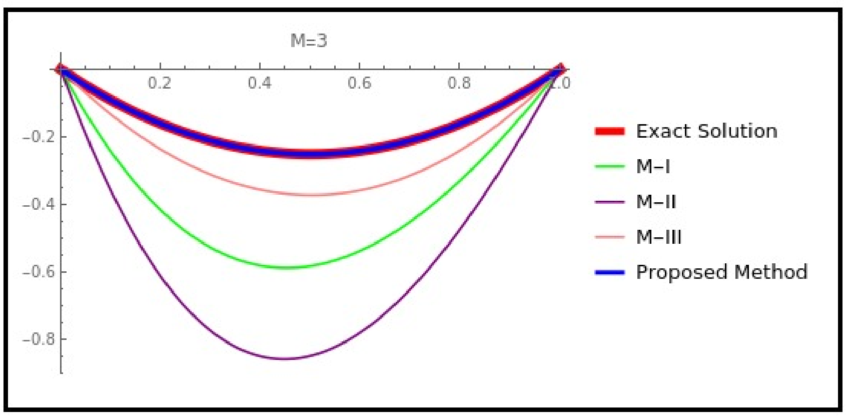

Figure 4. Graphical view for $M = 3$

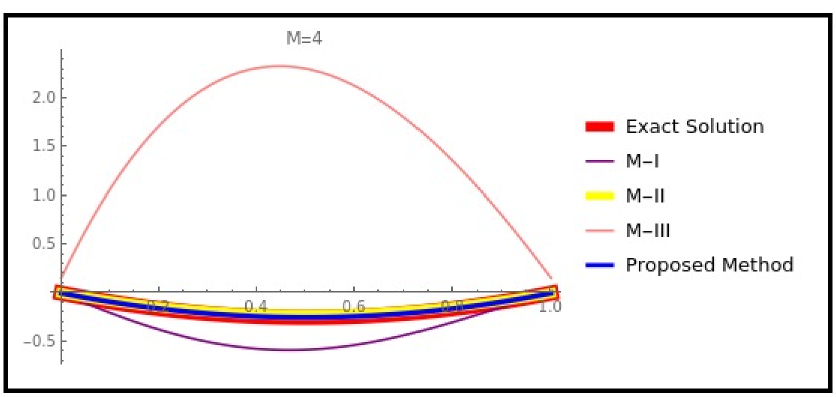

Figure 5. Graphical view for $M = 4$

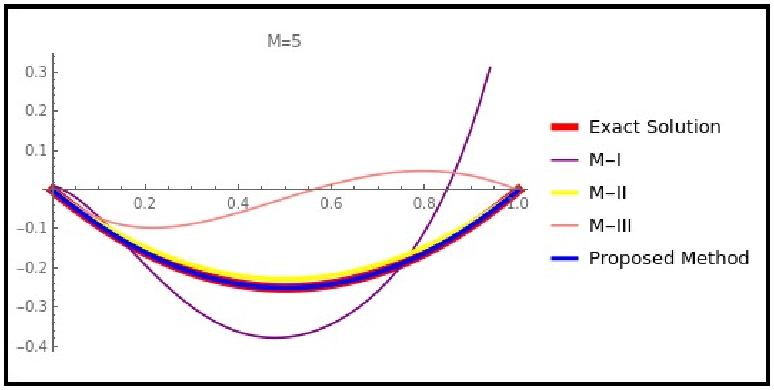

Figure 6. Graphical view for $M = 5$

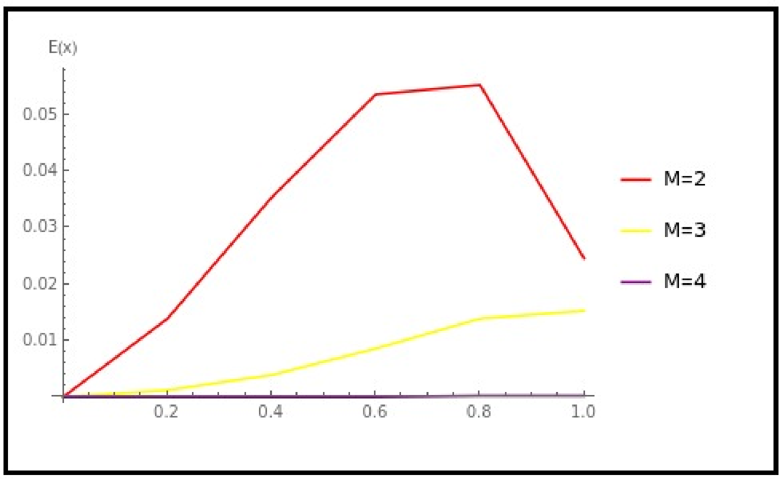

Figure 7. Error plot for example 1

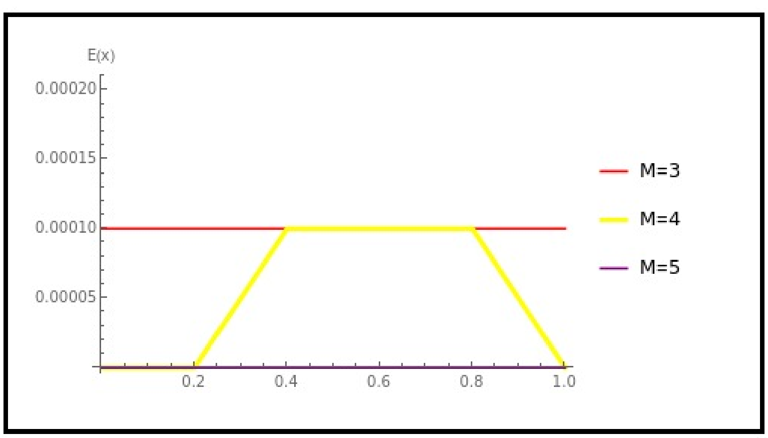

Figure 8. Error plot for example 2