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Number of Automorphisms of Some Non-Abelian p-Groups Of Order p^4

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Abstract

The automorphism of a group is a way of mapping the object to itself while preserving all of its structure, and the set of automorphisms of an object forms a group called the automorphism group. It is simply a bijective homomorphism. One of the earliest group automorphism was given by Irish mathematician William Rowan Hamilton in 1856, in his icosian calculus where he discovered an order two automorphism. In this paper, we compute the automorphisms of some non-Abelian groups of order p^4, where p is an odd prime and GAP (Group Algorithm Programming) software has been used for the verification of number of automorphisms. We use Burnside's categorization for the classification of all non-isomorphic groups of order p^4. There are fifteen groups of order p^4, out of which five are Abelian and the rest are non-Abelian. The automorphisms of all five Abelian groups are already computed by researchers. We find out the automorphisms by using the structure description of group and elementary calculations. By assuming any arbitrary automorphisms, we find out the constraints for group automorphism by using these results and the property of automorphisms. This research work is good in nature and helpful in the further studies of algebra.

Keywords: p-Groups; Automorphism; Semi-direct product; Kernel of homomorphism; Non-Abelian groups; Structural description of groups; Constraints for group automorphism; Order of automorphism; Groups of order p^4

MSC 2010 No.: 20D45, 20D60

1. Introduction

Let G be a p-group of order p^4, where p is a prime not equal to two and $Aut(G)$ be the group of all automorphisms of a group G. There are many research papers in the literature related to the automorphisms of p-groups, for instance Arora et al. (2017), Liebeck (1966), Helleloid (2007), etc. In Gustav et al. (2010), the authors defined the term semi direct product of p-groups and computed automorphisms of groups of order p^3. The authors also found the subgroups of p-groups of order p^3 and p^4. In Arora et al. (2017), the authors investigated the automorphisms of non-Abelian groups of order p^3 and Abelian groups of order p^4 and then found the fusion classes of these groups. They also found the probability that a group element is fixed by an arbitrarily chosen automorphism in terms of fusion classes. In Arora (2021), the author determined the automophisms of finite Abelian groups and computed the fusion classes. They also found the probability, that a group element is fixed by an automorphism in terms of fusion classes. In Salahshour et al. (2019), the authors defined the structure of finite Abelian groups as the product of groups and they also proved that, every endomorphism becomes the automorphism after fulfill some conditions. Further, they found the size of automorphisms of finite Abelian groups. In Muniya et al. (2022), the authors computed the automorphism groups of several non-Abelian finite $p-$groups of order p^4. They computed the automorphism group of semi direct product of direct product of cyclic group of order p^2 with cyclic group of order p and cyclic group of order p. They also computed the automorphism group of semi direct product of semi-direct product of cyclic group of order p^2 with cyclic group of order p and cyclic group of order p. In Curran (1988), the authors worked on p-groups of order p^5. They found that there exists only a single group whose automorphism group is also p-group. The order of that automorphism group is p^6.

In Herstein (1975), the author stated that any automorphism of a group preserve the relation of elements of that group. The author also defined that if an element of a group is of finite order then the order of automorphic image of that element is same as the order of element. In Zainal et al. (2014), the authors determined the capability of several non-Abelian groups, whose order is p^4. A group is known as a capable group if there exists any another group so that the group is isomorphic to central factor group. They worked on six non-Abelian groups of order p^4 and found that only two groups out of these six groups are capable. In Hillar et al. (2007), the authors discussed about the capability of some finite groups of order p^4.They worked on three non-Abelian groups and found that all three are not capable.

The present study is an extension of the research work in Arora et al. (2017). In this paper, we shall compute the automorphisms of some non-Abelian groups of order p^4. All groups of order p^4 are classified in Burnside (1987) and we shall use this classification to compute the automorphisms of groups of order p^4. There are fifteen (15) groups of order p^4, five of which are Abelian and rest are non-Abelian.

2. Automorphisms of Some Non-Abelian Groups of Order p^4

The automorphisms of five Abelian groups, namely $Z_{p^4},\ Z_{p^3}\times Z_p,\ Z_{p^2}\times Z_{p^2},\ Z_{p^2}\times Z_p\times Z_p$ and $Z_p\times Z_p\times Z_p\times Z_p$ of order p^4 are already computed in the paper by Arora and Karan (2017). In this paper, we compute the automorphisms of some non-Abelian p-groups of order p^4, particularly $Z_{p^3}\rtimes_\phi Z_p,\ Z_{p^2}\rtimes_\phi Z_{p^2},\ (Z_p\times Z_p)\rtimes Z_{p^2}$ and $(Z_{p^2}\times Z_p)\rtimes Z_p$.

2.1. Computation of the automorphisms of $Z_{p^3}\rtimes Z_p$

$Z_{p^3}\rtimes Z_p \cong < u_1,u_2 : u_1^{p^3}=u_2^p=1,\ u_2u_1=u_1^{1+p^2}u_2 >$.

From the structural description of the group $Z_{p^3}\rtimes Z_p$, we deduce some useful relations by using relations of generators and elementary calculations,

$$u_2^ju_1^i=u_1^{i+jip^2}u_2^j,$$
$$(u_1^iu_2^j)^n=u_1^{ni+\frac{n(n-1)}{2}ijp^2}u_2^{nj}.$$

With these relations, we begin our study of automorphisms of $Z_{p^3}\rtimes Z_p$. Let us consider a map $\psi\in Aut(Z_{p^3}\rtimes Z_p)$, defined by:

$$\psi:\begin{cases}u_1\to u_1^iu_2^j; & i\in Z_{p^3},\ j\in Z_p,\\ u_2\to u_1^ku_2^l; & k\in Z_{p^3},\ l\in Z_p.\end{cases}$$

Since order of $u_1\in Z_{p^3}\rtimes Z_p$ is p^3, the image of u_1, that is , $\psi(u_1)=u_1^iu_2^j$, must have the same order p^3.

Therefore,

$$(u_1^iu_2^j)^p=u_1^{pi+\frac{p(p-1)}{2}ijp^2}u_2^{pj}=u_1^{pi}\neq 1$$
$$\implies p\nmid i.$$

Thus, the choices for i for which ψ is automorphism as

$$\phi(p^3)=p^3-p^2,\ \text{where}\,\phi\ \text{is Euler-}\ \phi\ \text{function.}$$

As order of $u_2\in Z_{p^3}\rtimes Z_p$ is p, then $\psi(u_2)=u_1^ku_2^l$ must have order equal to p.

Therefore,

$$(u_1^ku_2^l)^p=u_1^{pk+\frac{p(p-1)}{2}klp^2}u_2^{pl}=u_1^{pk}\equiv 1(mod\,p^2)$$
$$\implies p^2/k$$
$$\implies k=p^2m.$$

Thus, there are p choices of k for which ψ is an automorphism.

Since ψ is an automorphism, ψ should preserve the relation $u_2u_1 = u_1^{1+p^2}u_2$, that is,

$$\psi(u_2)\psi(u_1) = (\psi(u_1))^{1+p^2}\psi(u_2)$$

$$\implies u_1^k u_2^l u_1^i u_2^j = (u_1^i u_2^j)^{1+p^2} u_1^k u_2^l$$

$$\implies u_1^{(i+k+ilp^2)} u_2^{(l+j)} = u_1^{((1+p^2)i+k+jkp^2)} u_2^{(j+l)}$$

$$\implies u_1^{(ilp^2)} = u_1^{(ip^2+jkp^2)}$$

$$\implies u_1^{ilp^2} = u_1^{ip^2}$$

$$\implies ilp^2 \equiv ip^2 (mod\, p^3)$$

$$\implies il \equiv i (mod\, p).$$

Since $i \not\equiv 0(mod\, p)$, we have $l \equiv 1(mod\, p)$.

In order for ψ to be an automorphism it should be bijective, so we have

$$< \psi(u_2) > \cap < \psi(u_1) > = \ 1.$$

But this does not give anything new for the constraints already deduced as any element in $<\psi(u_1)>$ of order p is of form u_1^{pr} for some r and no element in $<\psi(u_2)>$ is of that form. Hence, we have

$$\psi : \begin{cases} u_1 \to u_1^i u_2^j; & i \in Z_{p^3} \text{ and } p \nmid i, \\ u_2 \to u_1^{p^2 m} u_2; & j,\ m \in Z_p. \end{cases}$$

and

$$|Aut(Z_{p^3} \rtimes Z_p)| = p^4(p-1).$$

2.2. Computation of the automorphisms of $Z_{p^2} \rtimes Z_{p^2}$

$Z_{p^2} \rtimes Z_{p^2} \ \cong < u_1, u_2 : u_1^{p^2} = u_2^{p^2} = 1, u_2u_1 = u_1^{1+p}u_2 >.$

The relations obtained by using generators relations from structural description of $Z_{p^2} \rtimes Z_{p^2}$ and elementary calculations are

$$u_2^j u_1^i = u_1^{i+ijp} u_2^j,$$

$$(u_1u_2)^n = u_1^{n+\frac{n(n-1)}{2}p} u_2^n,$$

$$(u_1^i u_2^j)^n = u_1^{ni+\frac{n(n-1)}{2}ijp} u_2^{nj}.$$

With these relations, we compute the automorphism group of $Z_{p^2} \rtimes Z_{p^2}$.

Let $\psi \in Aut(Z_{p^2} \rtimes Z_{p^2})$ defined by:

$$\psi : \begin{cases} u_1 \to u_1^i u_2^j; & i,\, j \in Z_{p^2}, \\ u_2 \to u_1^k u_2^l; & k,\, l \in Z_{p^2}. \end{cases}$$

As order of $u_1 \in Z_{p^2} \rtimes Z_{p^2}$ is p^2, $\psi(u_1)$ must have the same order. Therefore, $(u_1^i u_2^j)^p = u_1^{pi+\frac{p(p-1)}{2}jip} u_2^{pj} = u_1^{pi} u_2^{pj} \neq 1$.

Further order of $u_2 \in Z_{p^2} \rtimes Z_{p^2}$ is p^2, $\psi(u_2)$ also must have the same order.

Thus, $(u_1^k u_2^l)^p = u_1^{pk+\frac{p(p-1)}{2}klp} u_2^{pl} = u_1^{pk} u_2^{pl} \neq 1$.

Since ψ is an automorphism, so ψ should preserve the relation $u_2 u_1 = u_1^{1+p} u_2$.

Therefore,

$$\psi(u_2)\psi(u_1) = \psi(u_1)^{1+p}\psi(u_2)$$

$$\Rightarrow u_1^k u_2^l u_1^i u_2^j = (u_1^i u_2^j)^{1+p} u_1^k u_2^l$$

$$\Rightarrow u_1^{(i+k+ilp)} u_2^{(l+j)} = u_1^{(i+k+ip)} u_2^{(j+l+jp)}$$

$$\Rightarrow u_1^{(ilp)} = u_1^{(ip)} u_2^{(jp)}$$

$$\Rightarrow ilp \equiv ip (mod\, p^2) \text{ and } 0 \equiv jp (mod\, p^2)$$

$$\Rightarrow il \equiv i (mod\, p) \qquad \text{and}$$

$$0 \equiv j (mod\, p)$$

$$\Rightarrow p/j$$

$$\Rightarrow j = pm, \;\; m \in Z_p.$$

We also have,

$$u_1^{pi} u_2^{pj} \neq 1$$

$$\Rightarrow u_1^{pi} u_2^{p^2 m} \neq 1$$

$$\Rightarrow u_1^{pi} \neq 1$$

$$\Rightarrow p \nmid i$$

$$\Rightarrow i \not\equiv 0 (mod\, p).$$

Thus, we get $il \equiv i(mod\, p)$ but $i \not\equiv 0(mod\, p)$.

Therefore, $l \equiv 1(mod\, p)$.

Now we shall find the choices of the parameters for which ψ is bijective. Since all the groups under consideration are finite groups, it is sufficient to show that ψ is injective or to show that kernel is trivial.

Let $u_1^x u_2^y \in ker(\psi)$

$$\Rightarrow \psi(u_1^x u_2^y) = 1$$

$$\Rightarrow \psi(u_1)^x \psi(u_2)^y = 1$$

$$\Rightarrow (u_1^i u_2^j)^x (u_1^k u_2^l)^y = 1$$

$$\Rightarrow u_1^{(xi+yk+\frac{y(y-1)}{2}lkp)} u_2^{(ly)} = 1$$

$$\Rightarrow xi + yk + \frac{y(y-1)}{2}lkp \equiv 0(mod\, p^2) \qquad \text{and}$$

$$ly \equiv 0(mod\, p^2).$$

As

$$l \not\equiv 0(mod\, p)$$

$$\Rightarrow \; l \not\equiv 0(mod\, p^2)$$

$$\Rightarrow y = 0,$$

and $\quad xi + yk + \frac{y(y-1)}{2}lkp \equiv 0(mod\, p^2)$

$$\Rightarrow xi \equiv 0(mod\, p^2).$$

But $\quad p \nmid i$

$$\Rightarrow x = 0.$$

So $u_1^x u_2^y = 1$.

Therefore, the kernel has only identity element of group and ψ is an automorphism with the constraints we have already deduced. Now it is easy to determine the order of automorphism group.

There are $p^2 - p$ choices of i, p choices for both j & l and p^2 choices of k for automorphisms. Hence,

$$\psi : \begin{cases} u_1 \to u_1^i u_2^{pm}; & p \nmid i, i \in Z_{p^2}, m \in Z_p, \\ u_2 \to u_1^k u_2^l; & k, l \in Z_{p^2}, l \equiv 1 (mod\, p). \end{cases}$$

and

$$|Aut(Z_{p^2} \rtimes Z_{p^2})| = p^5(p-1).$$

2.3. Computation of the automorphisms of $(Z_p \times Z_p) \rtimes Z_{p^2}$

$(Z_p \times Z_p) \rtimes Z_{p^2} \cong < u_1, u_2, u_3 : u_1^p = u_2^p = u_3^{p^2} = 1, u_1u_2 = u_2u_1, u_2u_3 = u_3u_2, u_3u_1 = u_1u_2u_3 >$.

The relations we obtain from the structure of $(Z_p \times Z_p) \rtimes Z_{p^2}$ are

$$u_3^j u_1^i = u_1^i u_2^{ij} u_3^j,$$

$$(u_1^i u_2^j)^n = u_1^{(ni)} u_2^{\frac{n(n-1)}{2}ij} u_3^{(nj)}.$$

With these relations, we compute the automorphism group of $(Z_p \times Z_p) \rtimes Z_{p^2}$.

Let $\psi \in Aut((Z_p \times Z_p) \rtimes Z_{p^2})$ be defined by:

$$\psi : \begin{cases} u_1 \to u_1^i u_2^j u_3^k; & i,\ j \in Z_p,\ k \in Z_p^2, \\ u_2 \to u_1^l u_2^m u_3^n; & l,\ m \in Z_p,\ n \in Z_p^2, \\ u_3 \to u_1^q u_2^r u_3^s; & q,\ r \in Z_p,\ s \in Z_p^2. \end{cases}$$

Now order of $u_1 \in (Z_p \times Z_p) \rtimes Z_{p^2}$ is p, so that order of $\psi(u_1)$ is also p.

Therefore,

$$(u_1^i u_2^j u_3^k)^p = 1$$

$$\Rightarrow u_1^{(ip)} u_2^{(jp + \frac{p(p-1)}{2}ik)} u_3^{(kp)} = 1$$

$$\Rightarrow u_3^{kp} = 1$$

$$\Rightarrow p/k$$

$$\Rightarrow k = pt, \text{ for some } t \in Z_p.$$

Also order of $u_2 \in (Z_p \times Z_p) \rtimes Z_{p^2}$ is p, so that order of $\psi(u_2)$ is also p.

Therefore,

$$(u_1^l u_2^m u_3^n)^p = 1$$

$$\Rightarrow u_3^{np} = 1$$

$$\Rightarrow p/n$$

$$\Rightarrow n = pu, \text{ for some } u \in Z_p.$$

And order of $u_3 \in (Z_p \times Z_p) \rtimes Z_{p^2}$ is p^2, so the order of $(\psi(u_3)) = p^2$.

Therefore,

$$(u_1^q u_2^r u_3^s)^p \neq 1$$

$$\Rightarrow u_1^{(qp)} u_2^{(rp+\frac{p(p-1)}{2}qs)} u_3^{(sp)} \neq 1$$

$$\Rightarrow u_3^{sp} \neq 1$$

$$\Rightarrow p \nmid s.$$

Further, ψ is an automorphism. It should preserve the relation $u_3u_1 = u_1u_2u_3$. Hence,

$$\psi(u_3)\psi(u_1) = \psi(u_1)\psi(u_2)\psi(u_3)$$

$$\Rightarrow u_1^q u_2^r u_3^s u_1^i u_2^j u_3^k = u_1^i u_2^j u_3^k u_1^l u_2^m u_3^n u_1^q u_2^r u_3^s$$

$$\Rightarrow u_1^{(q+i)} u_2^{(r+is+j)} u_3^{(s+k)} = u_1^{(i+l+q)} u_2^{(j+kl+m+kq+nq+r)} u_3^{(k+n+s)}$$

$$\Rightarrow u_2^{(is)} = u_1^l u_2^{(kl+m+kq+nq)} u_3^n$$

$$\Rightarrow l \equiv 0 (mod\, p),$$

$$si \equiv kl + m + kq + nq (mod\, p) \quad \text{and}$$

$$n \equiv 0 (mod\, p^2).$$

Now

$$si \equiv kl + m + kq + nq (mod\, p)$$
$$\Rightarrow si \equiv m (mod\, p) \text{ or } m \equiv si (mod\, p).$$

And $m \not\equiv 0 (mod\, p)$, as if $m \equiv 0 (mod\, p)$, then $o(u_2) \neq p$.

So,

$$m = si \not\equiv 0(mod\, p)$$
$$\Rightarrow i \not\equiv 0(mod\, p).$$

Now for bijective, as $(Z_p \times Z_p) \rtimes Z_{p^2}$ is finite, it is sufficient to show that ψ is injective, i.e., kernel is trivial.

Let $u_1^x u_2^y u_3^z \in (Z_p \times Z_p) \rtimes Z_{p^2}$ such that $\psi(u_1^x u_2^y u_3^z) = 1$. Then we get

$$\psi(u_1^x u_2^y u_3^z) = \psi(u_1)^x \psi(u_2)^y \psi(u_3)^z = 1$$

$$\Rightarrow (u_1^i u_2^j u_3^k)^x (u_1^l u_2^m u_3^n)^y (u_1^q u_2^r u_3^s)^z = 1$$

$$\Rightarrow u_1^{(ix+qz)} u_2^{(jx+\frac{x(x-1)}{2}ik+my+kxqz+rz+\frac{z(z-1)}{2}qs)} u_3^{(kx+sz)} = 1$$

$$\Rightarrow ix + qz = 0, \;\; jx + \frac{x(x-1)}{2}ik + my + kxqz + rz + \frac{z(z-1)}{2}qs = 0, \;\; kx + sz = 0.$$

First and last homogeneous equations have zero solution only if the determinant $is - kq \neq 0$.

Thus, we have $x = z = 0$ and from second equation we get $y = 0$. Hence, the kernel is trivial. So ψ is automorphism with calculated restrictions.

Thus, for any map ψ to be an automorphism, there are $p-1$ choices for i, p choices for all $j, \; k, \; q, \; r$ and $p^2 - p$ choices for s. Thus,

$$\psi : \begin{cases} u_1 \rightarrow u_1^i u_2^j u_3^k; & p \nmid i, \; p/k, \; i, \, j \in Z_p, \; k \in Z_{p^2}, \\ u_2 \rightarrow u_2^{is}, & \\ u_3 \rightarrow u_1^q u_2^r u_3^s; & p \nmid s, \; q, \, r \in Z_p, \; s \in Z_{p^2}. \end{cases}$$

and

$$|Aut((Z_p \times Z_p) \rtimes Z_{p^2})| = p^5(p-1)^2.$$

2.4. Computation of the automorphisms of $(Z_{p^2} \times Z_p) \rtimes Z_p$

$(Z_{p^2} \times Z_p) \rtimes Z_p \cong < u_1, u_2, u_3; u_1^{p^2} = u_2^p = u_3^p = 1, u_3 u_2 = u_1^p u_2 u_3, u_2 u_1 = u_1 u_2, u_1 u_3 = u_3 u_1 > .$

From the structure description of $(Z_{p^2} \times Z_p) \rtimes Z_p$ and using elementary calculations, we can deduce some useful relations,

$$u_3^i u_2^j = u_1^{ijp} u_2^j u_3^i,$$

$$(u_2 u_3)^n = u_1^{\frac{n(n-1)}{2}p} u_2^n u_3^n,$$

$$(u_2^i u_3^j)^n = u_1^{\frac{n(n-1)}{2}ijp} u_2^{ni} u_3^{nj}.$$

By using these results, we begin our study of automorphism group of $(Z_{p^2} \times Z_p) \rtimes Z_p$.

Let $\phi \in (Z_{p^2} \times Z_p) \rtimes Z_p$ be defined as

$$\phi : \begin{cases} u_1 \to u_1^i u_2^j u_3^k; & i \in Z_{p^2}, j, k \in Z_p, \\ u_2 \to u_1^l u_2^m u_3^n; & l \in Z_{p^2}, m, n \in Z_p, \\ u_3 \to u_1^q u_2^r u_3^s; & q \in Z_{p^2}, r, s \in Z_p. \end{cases}$$

If order of $u_1 \in (Z_{p^2} \times Z_p) \rtimes Z_p$ is p^2, then the order of $\phi(u_1)$ is also p^2.

Therefore,

$$(u_1^i u_2^j u_3^k)^p \neq 1$$

$$\Rightarrow u_1^{pi} \neq 1$$

$$\Rightarrow p \nmid i.$$

Also order of $u_2 \in (Z_{p^2} \times Z_p) \rtimes Z_p$ is p, so order of $\phi(u_2)$ is also p.

Thus,

$$(u_1^l u_2^m u_3^n)^p = 1$$

$$\Rightarrow u_1^{lp} = 1$$

$$\Rightarrow p/l$$

$$\Rightarrow l = pt, \text{ for some } t \in Z_p.$$

And order of $u_3 \in (Z_{p^2} \times Z_p) \rtimes Z_p$ is p. Therefore, order of $\phi(u_3)$ is p. Therefore,

$$(u_1^q u_2^r u_3^s)^p = 1$$

$$\Rightarrow u_1^{qp} = 1$$

$$\Rightarrow p/q$$

$$\Rightarrow q = pu, \quad \text{for some } u \in Z_p.$$

From non-Abelian relation $u_3 u_2 = u_1^p u_2 u_3$, we have

$$\phi(u_3)\phi(u_2) = \phi(u_1)^p \phi(u_2)\phi(u_3)$$

$$\Rightarrow u_1^q u_2^r u_3^s u_1^l u_2^m u_3^n = (u_1^i u_2^j u_3^k)^p u_1^l u_2^m u_3^n u_1^q u_2^r u_3^s$$

$$\Rightarrow u_1^{q+l+smp}u_2^{r+m}u_3^{s+n} = u_1^{pi+l+q+nrp}u_2^{m+r}u_3^{n+s}$$

$$\Rightarrow u_1^{smp} = u_1^{pi+nrp}$$

$$\Rightarrow smp \equiv pi + nrp (mod\ p^2)$$

$$\Rightarrow sm \equiv i + nr (mod\ p)$$

$$\Rightarrow i \equiv sm - nr (mod\ p).$$

From Abelian relation $u_1u_2 = u_2u_1$, we get

$$\phi(u_1)\phi(u_2) = \phi(u_2)\phi(u_1)$$

$$\Rightarrow u_1^i u_2^j u_3^k u_1^l u_2^m u_3^n = u_1^l u_2^m u_3^n u_1^i u_2^j u_3^k$$

$$\Rightarrow u_1^i u_1^l u_2^j u_3^k u_2^m u_3^n = u_1^l u_1^i u_2^m u_3^n u_2^j u_3^k$$

$$\Rightarrow u_1^{i+l+kmp}u_2^{j+m}u_3^{k+n} = u_1^{l+i+njp}u_2^{m+j}u_3^{n+k}$$

$$\Rightarrow u_1^{kmp} = u_1^{njp}$$

$$\Rightarrow kmp \equiv njp (mod\ p^2)$$

$$\Rightarrow km \equiv nj (mod\ p)$$

$$\Rightarrow km - nj \equiv 0 (mod\ p). \tag{1}$$

And $u_1u_3 = u_3u_1$, then

$$\phi(u_1)\phi(u_3) = \phi(u_3)\phi(u_1)$$

$$\Rightarrow u_1^i u_2^j u_3^k u_1^q u_2^r u_3^s = u_1^q u_2^r u_3^s u_1^i u_2^j u_3^k$$

$$\Rightarrow u_1^{q+i+krp}u_2^{j+r}u_3^{k+s} = u_1^{q+i+sjp}u_2^{r+j}u_3^{k+s}$$

$$\Rightarrow u_1^{krp} = u_1^{sjp}$$

$$\Rightarrow krp \equiv sjp (mod\ p^2)$$

$$\Rightarrow kr \equiv sj (mod\ p)$$

$$\Rightarrow kr - sj \equiv 0 (mod\ p). \tag{2}$$

(1) and (2) are two homogeneous equations with $sm - nr \neq 0$ determinant. So, they have only trivial solution

$$\Rightarrow j = k = 0.$$

Therefore, we have $j = k = 0$, $i = sm - nr \neq 0$, p/l and p/q are the only constraints. We have satisfied the relations of the group for homomorphism. We are going to prove that it is bijective. As the group is finite, so it is sufficient to show that it is one-one or kernel is trivial.

Let $u_1^x u_2^y u_3^z \in (Z_{p^2} \times Z_p) \rtimes Z_p$ and is mapped to identity element of group, i.e.,

$$\phi(u_1^x u_2^y u_3^z) = 1$$

$$\Rightarrow \phi(u_1)^x \phi(u_2)^y \phi(u_3)^z = 1$$

$$\Rightarrow (u_1^i)^x (u_1^l u_2^m u_3^n)^y (u_1^q u_2^r u_3^s)^z = 1$$

$$\Rightarrow u_1^{ix+ly+\frac{y(y-1)}{2}mnp+qz+\frac{z(z-1)}{2}rsp+nryzp} u_2^{my+rz} u_3^{ny+sz} = 1$$

$$\Rightarrow ix + ly + \frac{y(y-1)}{2}mnp + qz + \frac{z(z-1)}{2}rsp + nryzp \equiv 0(mod\ p^2), \tag{3}$$

$$my + rz \equiv 0(mod\ p) \quad and \tag{4}$$

$$ny + sz \equiv 0(mod\ p). \tag{5}$$

(4) and (5) are two homogeneous equations with $sm - nr \neq 0$ determinant. Hence, these equations have only trivial solution.

$$\Rightarrow y = z = 0.$$

By using values of y and z in (3), we get

$$ix \equiv 0(mod\ p^2).$$

But $i \not\equiv 0(mod\ p) \Rightarrow i \not\equiv 0(mod\ p^2)$.

So, $x \equiv 0(modp^2)$.

Therefore, $u_1^x u_2^y u_3^z = 1$, which implies that the kernel is trivial and ϕ is an automorphism with the constraints we have already derived. Now we can determine the order of automorphism group. There are p choices for both l and q and $(p^2 - 1)(p^2 - p)$ choices for m, n, r, s.

Hence, $\phi \in (Z_{p^2} \times Z_p) \rtimes Z_p$ is really defined as

$$\phi : \begin{cases} u_1 \to u_1^{sm-nr}; & sm - nr \neq 0, \\ u_2 \to u_1^{pt} u_2^m u_3^n; & t, m, n \in Z_p, \\ u_3 \to u_1^{pu} u_2^r u_3^s; & u, r, s \in Z_p, \end{cases}$$

and

$$|Aut((Z_{p^2} \times Z_p) \rtimes Z_p)| = p^3(p-1)(p^2-1).$$

3. Conclusion

In this research paper, we worked on some non-Abelian groups of order p^4. We find out the automorphisms and size of automorphisms of some non-Abelian groups, whose order is p^4. The automorphism group and size of automorphism group of semi direct product of cyclic group of order p^3 and cyclic group of order p are computed. The automorphism group and size of automorphism group of semi direct product of cyclic group of order p^2 and cyclic group of order p^2 are investigated. The automorphism group and size of automorphism group of semi direct product of direct product of cyclic group of order p with cyclic group of order p and cyclic group of order p^2 are calculated. The automorphism group and size of automorphism group of semi direct product of direct product of cyclic group of order p^2 with cyclic group of order p and cyclic group of order p are computed. The size of automorphism group depends on prime. The result derived in this research paper can be used to find the fusion classes of these groups. The fusion classes are related to automorphisms of group. For any group, two elements are fused if there exists an automorphism of that group such that it maps one element to another. The fusion relation is an equivalence relation. These results can be used to find the probability that a group element fixed by an automorphism.

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