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Some New Fixed Point Theorem via Shifting Distance Functions

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Abstract

In this paper, we present a new fixed point theorem involving non-compactness measures and shifting distance functions. This paper provides a generalization of the famous fixed point theorem of Banach. A fixed point theory is a gorgeous blend of mathematical analysis that explains the conditions under which maps provide excellent solutions. Numerous mathematicians later used this theory to prove their results; see, for example, the Schauder fixed point theorem, the Darbo fixed point theorem, the nonexpansive fixed point theorem, etc. Additionally, we hypothesized that a large number of known fixed point theorems can be simply deduced from the Banach theorem. Finally, we also use this fixed point theorem in Banach space to establish the existence of a solution to a fractional integral equation and to illustrate the results with an example.

Keywords: Measure of noncompactness; Shifting distance functions; Fixed point theorem

MSC 2010 No.: 45G05, 47H08, 47H09, 47H10

1. Introduction

Kuratowski et al. (1930) introduced the notion of a measure of non-compactness. In functional analysis, this idea is particularly important in operator equation theory and metric fixed point theory in Banach spaces. The theory of infinite systems of fractional integral equations plays a pivotal role in different fields, which includes various implications in scaling system theory, the theory of algorithms, etc. Many mathematicians have solved integral, differential, and functional problem classes using the measure of non-compactness notion and Darbo's fixed point theorem and have obtained significant research results (Banaś and Goebel (1980); Banaś (2012); Mursaleen and Mohiuddine (2012); Kavitha et al. (2020); Dineshkumar et al. (2021); Das et al. (2021b)).

In recent times, the fixed point theory has had applications in various scientific fields. Also, fixed point theory can be applied to seek solutions to fractional integral equations see (Raja et al. (2020); Williams et al. (2020); Raja et al. (2021); Das et al. (2021c)).

Different real-life situations which are shaped via fractional integral equations can be studied using fixed point theory and measure of non-compactness (Banaś and Mursaleen (2014); Darwish and Sadarangani (2015); Das et al. (2019a); Das et al. (2019c); Das et al. (2021a); Mursaleen and Mohiuddine (2012); Nashine et al. (2018); Rabbani et al. (2020b); Rabbani et al. (2020a)).

2. Main Result

Theorem 2.1.

Let $\mathtt{L}$ be a non-empty, bounded, closed, and convex $(\mathfrak{BCC})$ subset of a Banach Space $\mathtt{H}$, and let $\mathfrak{J} : \mathtt{L} \to \mathtt{L}$ be a continuous operator such that

$$\mu_n \left[\mathcal{W}(\sigma(\mathfrak{J}\mathtt{Q}), \Gamma(\sigma(\mathfrak{J}\mathtt{Q})))\right] \leq \nu_n \left[\mathcal{W}(\sigma(\mathtt{Q}), \Gamma(\sigma(\mathtt{Q}))), \right] \tag{1}$$

for all $\mathtt{Q} \subseteq \mathtt{L}$, where $\mu_n, \nu_n : \mathbf{R}_+ \to \mathbf{R}$ are the pair of the sequence of maps with shifting distance function, $\Gamma : \mathbf{R}_+ \to \mathbf{R}$ is a continuous function and σ is an arbitrary measure of non-compactness $(\mathcal{MNC})$ and $\mathcal{W} \in \mathbf{W}$. Then, $\mathfrak{J}$ has at least one fixed point $(\mathcal{FP})$ in $\mathtt{L}$.

Proof:

Construct a sequence $\{\mathtt{Q}_t\}$ such that $\mathtt{Q}_0 = \mathtt{Q}$ and $\mathtt{Q}_t = Conv(\mathfrak{J}\mathtt{Q}_{t+1})$ for all $t \geq 1$.Then, we obtain

$$\mathfrak{J}\mathtt{Q}_0 = \mathfrak{J}\mathtt{Q} \sqsubseteq \mathtt{Q} = \mathtt{Q}_0,$$

$$\mathtt{Q}_1 = Conv(\mathfrak{J}\mathtt{Q}_0) \sqsubseteq \mathtt{Q} \sqsubseteq \mathtt{Q}_0.$$

By repeating the process mentioned above, we have

$$\mathtt{Q}_0 \sqsupseteq \mathtt{Q}_1 \sqsupseteq \mathtt{Q}_2 \sqsupseteq \ldots \sqsupseteq \mathtt{Q}_t \sqsupseteq \ldots$$

If $\exists$ an integer $t \geq 0$ such that $\sigma(\mathtt{Q}_t) = 0$, then $\mathtt{Q}_t$ is relatively compact also since

$$\mathfrak{J}\mathtt{Q}_t \sqsubseteq Conv(\mathfrak{J}\mathtt{Q}_t) = \mathtt{Q}_{t+1} \sqsubseteq \mathtt{Q}_t.$$

This implies that $\mathfrak{J}$ has a $\mathcal{FP}$ in Schauder's sense on the set $\mathtt{Q}_t$ for all $t \geq 0$. Now, we assume that $\sigma(\mathtt{Q}_t) > 0$ for all $t \geq 0$. We get

$$\begin{aligned} \mu_n \left[\mathcal{W}(\sigma(\mathtt{Q}_{t+1}), \Gamma(\sigma(\mathtt{Q}_{t+1})))\right] &= \mu_n \left[\mathcal{W}(\sigma(Conv(\mathfrak{J}\mathtt{Q}_t)), \Gamma(\sigma(Conv(\mathfrak{J}\mathtt{Q}_t))))\right] \\ &= \mu_n \left[\mathcal{W}(\sigma(\mathfrak{J}\mathtt{Q}_t), \Gamma(\sigma(\mathfrak{J}\mathtt{Q}_t)))\right] \\ &\leq \nu_n \left[\mathcal{W}(\sigma(\mathtt{Q}_t), \Gamma(\sigma(\mathtt{Q}_t)))\right]. \end{aligned} \tag{2}$$

Thus, we get $\{\mathcal{W}(\sigma(\mathtt{Q}_t),\Gamma(\sigma(\mathtt{Q}_t)))\}$ is a decreasing sequence of positive real numbers and there exists $m \geq 0$ such that $\mathcal{W}(\sigma(\mathtt{Q}_t),\Gamma(\sigma(\mathtt{Q}_t))) \to m$ as $m \to \infty$. We have

$$\mu_n\left[\mathcal{W}(\sigma(\mathtt{Q}_{t+1}),\Gamma(\sigma(\mathtt{Q}_{t+1})))\right] \to \mu\left[\mathcal{W}(\sigma(\mathtt{Q}_{t+1}),\Gamma(\sigma(\mathtt{Q}_{t+1})))\right], \qquad \text{uniformly in } n.$$

Also,

$$\mu\left[\mathcal{W}(\sigma(\mathtt{Q}_{t+1}),\Gamma(\sigma(\mathtt{Q}_{t+1})))\right] = \mu\left[\mathcal{W}(\sigma(\mathfrak{J}\mathtt{Q}_t),\Gamma(\sigma(\mathfrak{J}\mathtt{Q}_t)))\right]. \tag{3}$$

And, if $\mathcal{W}(\sigma(\mathtt{Q}_t),\Gamma(\sigma(\mathtt{Q}_t))) \to m$ as $m \to \infty$, then $\mathcal{W}(\sigma(\mathtt{Q}_{t+1}),\Gamma(\sigma(\mathtt{Q}_{t+1}))) \to m$ as $m \to \infty$.

Hence, we have

$$\begin{aligned}\lim_{t\to\infty}\mu\left[\mathcal{W}(\sigma(\mathtt{Q}_{t+1}),\Gamma(\sigma(\mathtt{Q}_{t+1})))\right] &= \lim_{t\to\infty}\mu\left[\mathcal{W}(\sigma(\mathfrak{J}\mathtt{Q}_t),\Gamma(\sigma(\mathfrak{J}\mathtt{Q}_t)))\right]\\ &\leq \lim_{t\to\infty}\nu\left[\mathcal{W}(\sigma(\mathtt{Q}_t),\Gamma(\sigma(\mathtt{Q}_t)))\right],\\ \mu(m) \leq \nu(m).\end{aligned}$$

Now, we get $m = 0$. So, we have

$$\lim_{t\to\infty}\mathcal{W}(\sigma(\mathtt{Q}_t),\Gamma(\sigma(\mathtt{Q}_t))) = 0.$$

By the property of $\mathcal{W}$, we get

$$\lim_{t\to\infty}\sigma(\mathtt{Q}_t) = \lim_{t\to\infty}\Gamma(\sigma(\mathtt{Q}_t)) = 0.$$

So, $\lim_{t\to\infty}\sigma(\mathtt{Q}_t) = 0$. We obtain $\mathtt{Q}_\infty = \bigcap_{t=1}^{\infty}\mathtt{Q}_t$ is a non-empty $\mathfrak{BCC}$ set, invariant under the function $\mathfrak{J}$ also belongs to $ker\ \sigma$. Hence, by Schauder's Theorem $\Rightarrow \mathfrak{J}$ has a $\mathcal{FP}$. ■

The following is an immediate consequence of Theorem 2.1.

Theorem 2.2.

Let $\mathtt{L}$ be a non-empty $\mathfrak{BCC}$ subset of a Banach Space $\mathtt{H}$ and let $\mathfrak{J}:\mathtt{L}\to\mathtt{L}$ be a continuous operator such that

$$\mu_n\left[\sigma(\mathfrak{J}\mathtt{Q}) + \Gamma(\sigma(\mathfrak{J}\mathtt{Q}))\right] \leq \nu_n\left[\sigma(\mathtt{Q}) + \Gamma(\sigma(\mathtt{Q}))\right], \tag{4}$$

for all $\mathtt{Q}\subseteq\mathtt{L}$, where $\mu_n,\nu_n:\mathbf{R}_+\to\mathbf{R}$ are the pair of a sequence of maps with shifting distance function, $\Gamma:\mathbf{R}_+\to\mathbf{R}$ is a continuous mapping and σ is an arbitrary $\mathcal{MNC}$. Then, $\mathfrak{J}$ has at least one $\mathcal{FP}$ in $\mathtt{L}$.

Proof:

Take $\mathcal{W}(c_1,c_2) = c_1 + c_2$ in Theorem 2.1. Then, we get the result. ■

The statement in the following corollary follows from Theorem 2.1.

Corollary 2.1.

Let $\mathtt{L}$ be a non-empty $\mathfrak{BCC}$ subset of a Banach Space $\mathtt{H}$ and let $\mathfrak{J} : \mathtt{L} \to \mathtt{L}$ be a continuous operator such that

$$\sigma(\mathfrak{J}\mathtt{Q}) + \Gamma(\sigma(\mathfrak{J}\mathtt{Q})) \leq \varpi \left[\sigma(\mathtt{Q}) + \Gamma(\sigma(\mathtt{Q}))\right], \tag{5}$$

for all $\mathtt{Q} \subseteq \mathtt{L}$, where $\Gamma : \mathbf{R}_+ \to \mathbf{R}$ is a continuous mapping and σ is an arbitrary $\mathcal{MNC}$. Then, $\mathfrak{J}$ has at least one $\mathcal{FP}$ in $\mathtt{L}$.

Proof:

Take $\mu_n(t) = t$ and $\nu_n(t) = \varpi t$, $t \geq 0$, $\varpi \in (0, 1]$ in Theorem 2.2. Thus, we obtain the result. ■

Corollary 2.2.

Let $\mathtt{L}$ be a non-empty $\mathfrak{BCC}$ subset of a Banach Space $\mathtt{H}$ and let $\mathfrak{J} : \mathtt{L} \to \mathtt{L}$ be a continuous operator such that

$$\sigma(\mathfrak{J}\mathtt{Q}) \leq \varpi\sigma(\mathtt{Q}), \tag{6}$$

for all $\mathtt{Q} \subseteq \mathtt{L}$, where σ is an arbitrary $\mathcal{MNC}$. Then, $\mathfrak{J}$ has at least one $\mathcal{FP}$ in $\mathtt{L}$.

Proof:

Putting $\Gamma(\chi) = 0$, $\chi \geq 0$, $\varpi \in (0, 1]$ in Corollary 2.1, we obtan $\mathcal{DFPT}$. ■

3. $\mathcal{MNC}$ on $\mathcal{C}([0, T])$:

Take the space $\mathtt{H} = \mathcal{C}(\mathcal{I})$ which is the set of real continuous functions on $\mathcal{I}$, where $\mathcal{I} = [0, T]$. So, $\mathtt{H}$ is a Banach space by the norm

$$\| \kappa \| = \sup \{|\kappa(\nu)| : \nu \in \mathcal{I}\}, \ \kappa \in \mathtt{H}.$$

Let $\Lambda(\neq \phi) \subseteq \mathtt{H}$ be bounded. For $\nu \in \Lambda$ and $\varepsilon > 0$, denote by $\omega(\kappa, \varepsilon)$ the modulus of the continuity of κ, that is,

$$\omega(\kappa, \varepsilon) = \sup \{|\kappa(\nu_1) - \kappa(\nu_2)| : \nu_1, \nu_2 \in \mathcal{I}, |\nu_1 - \nu_2| \leq \varepsilon\}.$$

In fact, we define

$$\omega(\Lambda, \varepsilon) = \sup \{\omega(\kappa, \varepsilon) : \kappa \in \Lambda\},$$

and

$$\omega_0(\Lambda) = \lim_{\varepsilon \to 0} \omega(\Lambda, \varepsilon).$$

4. Solvability fractional integral equation

For $\Psi \in [c,d],\ k>0,\ \upsilon>0,\ and\ s\in \mathbf{R}\backslash\{1\}$, we the definition of the Riemann-Liouville fractional integral equation of a continuous mapping f as follows:

$$ {}_{k}^{s}\mathcal{Y}_{a}^{\upsilon} f(\Psi)=\frac{1}{\Gamma(\frac{\Psi}{k})}\int_{a}^{\Psi}\left(\Psi^{s+1}-\eta^{s+1}\right)^{\frac{\upsilon}{k}-1}\eta^{s}f(\eta)d\eta\ . $$

For $c=1,\ d=T$,

$$ {}_{k}^{s}\mathcal{Y}^{\upsilon} f(\Psi)=\frac{1}{\Gamma(\frac{\Psi}{k})}\int_{1}^{\Psi}\left(\Psi^{s+1}-\eta^{s+1}\right)^{\frac{\upsilon}{k}-1}\eta^{s}f(\eta)d\eta. $$

In this section, we'll study the fractional integral equation below:

$$ \Lambda(\Psi)=\Upsilon(\Psi,\mathcal{T}\left(\Psi,\Lambda(\Psi)\right),{}_{k}^{s}\mathcal{Y}^{\upsilon}f(\Psi))\,, \tag{7} $$

where $1>k>0,\ \upsilon>0,\ s\in\mathbf{R}\backslash\{1\}\,,\ \Psi\in\mathcal{I}=[1,T]$.

Let

$$ B_{e_0}=\{\Lambda\in \mathtt{H}:\|\ \Lambda\ \|\le e_0\}\,. $$

Suppose the following.
(D1) Let $\Upsilon:\mathcal{I}\times\mathbf{R}^2\to\mathbf{R},\ \mathcal{T}:\mathcal{I}\times\mathbf{R}\to\mathbf{R}$ be continuous and $\exists$ constants $\upsilon_1,\ \upsilon_2,\ \upsilon_3\ge 0$ satisfying

$$ \left|\Upsilon(\Psi,\mathcal{T},\mathcal{I}_1)-\Upsilon(\Psi,\bar{\mathcal{T}},\bar{\mathcal{I}}_1)\right|\le\upsilon_1\left|\mathcal{T}-\bar{\mathcal{T}}\right|+\upsilon_2\left|\mathcal{I}_1-\bar{\mathcal{I}}_1\right|,\ \Psi\in\mathcal{I};\ \mathcal{T},\mathcal{I}_1,\bar{\mathcal{T}},\bar{\mathcal{I}}_1\in\mathbf{R}, $$

and

$$ \left|\mathcal{T}(\Psi,Y_1)-\mathcal{T}(\Psi,Y_2)\right|\le\upsilon_3\left|Y_1-Y_2\right|,Y_1,Y_2\in\mathbf{R}. $$

(D2) There exists $e_0>0$ satisfying

$$ \bar{\Upsilon}=\sup\left\{\left|\Upsilon(\Psi,\mathcal{T},\mathcal{I}_1)\right|:\Psi\in\mathcal{I},\mathcal{T}\in[-\hat{\mathcal{T}},\hat{\mathcal{T}}],\mathcal{I}_1\in[-\hat{\mathcal{J}},\hat{\mathcal{J}}]\right\}\le e_0; $$

also,

$$ \upsilon_1\upsilon_3<1, $$

where,

$$ \hat{\mathcal{T}}=\sup\left\{|\mathcal{T}|:\Psi\in\mathcal{I},\Lambda(\Psi)\in[-e_0,e_0]\right\}, $$

and

$$ \hat{\mathcal{J}}=\sup\left\{|{}_{k}^{s}\mathcal{Y}^{\upsilon}\Lambda(\Psi)|:\Psi\in\mathcal{I},\Lambda(\Psi)\in[-e_0,e_0]\right\}. $$

(D3) $|\Upsilon\left(\Psi,0,0\right)|=0$.
(D4) There exists a positive solution e_0 of the inequality

$$ \upsilon_1\upsilon_3e_0+\frac{\upsilon_2e_0k}{\upsilon(s+1)\Gamma(\frac{1}{k})}\left(T^{s+1}-1\right)^{\frac{\upsilon}{k}}\le e_0. $$

Theorem 4.1.

Under assumptions (D1)-(D4), Equation (7) has a solution in $\mathtt{H} = \mathcal{C}(\mathcal{I})$.

Proof:

Define the operator $\mathcal{Q} : \mathtt{H} \to \mathtt{H}$ as follows:

$$(\mathcal{Q}\Lambda)(\Psi) = \Upsilon\left(\Psi, \mathcal{T}(\Psi, \Lambda(\Psi)), {}^s_k\mathcal{Y}^\upsilon f(\Psi)\right).$$

Step 1: We show that the mapping $\mathcal{Q}$ maps B_{e_0} into B_{e_0}. Let $\Lambda \in B_{e_0}$. We get

$$\begin{aligned}\left|\left(\mathcal{Q}\Lambda\right)(\Psi)\right| &\leq \left|\Upsilon\left(\Psi, \mathcal{T}(\Psi, \Lambda(\Psi)), {}^s_k\mathcal{Y}^\upsilon \Lambda(\Psi)\right) - \Upsilon\left(\Psi, 0, 0\right)\right| + \left|\Upsilon\left(\Psi, 0, 0\right)\right| \\ &\leq \upsilon_1 \left|\mathcal{T}(\Psi, \Lambda(\Psi)) - 0\right| + \upsilon_2 \left|{}^s_k\mathcal{Y}^\upsilon \Lambda(\Psi) - 0\right| + \left|\Upsilon\left(\Psi, 0, 0\right)\right|.\end{aligned}$$

Also,

$$\begin{aligned}\left|{}^s_k\mathcal{Y}^\upsilon \Lambda(\Psi)\right| &= \left|\frac{1}{\Gamma(\frac{\Psi}{k})}\int_1^\Psi \left(\Psi^{s+1} - \eta^{s+1}\right)^{\frac{\upsilon}{k}-1} \eta^s \Lambda(\eta) d\eta\right| \\ &\leq \frac{1}{\Gamma(\frac{\Psi}{k})}\int_1^\Psi \left(\Psi^{s+1} - \eta^{s+1}\right)^{\frac{\upsilon}{k}-1} \eta^s \left|\Lambda(\eta)\right| d\eta \\ &\leq \frac{e_0}{\Gamma(\frac{\Psi}{k})}\int_1^\Psi \left(\Psi^{s+1} - \eta^{s+1}\right)^{\frac{\upsilon}{k}-1} \eta^s d\eta \\ &\leq \frac{e_0 k}{\upsilon(s+1)\Gamma(\frac{1}{k})}\left(T^{s+1} - 1\right)^{\frac{\upsilon}{k}}.\end{aligned}$$

Hence, $\parallel \Lambda \parallel < e_0$ gives

$$\parallel \mathcal{Q}\Lambda \parallel < \upsilon_1\upsilon_3 e_0 + \frac{\upsilon_2 e_0 k}{\upsilon(s+1)\Gamma(\frac{1}{k})}\left(T^{s+1} - 1\right)^{\frac{\upsilon}{k}} \leq e_0.$$

As a result of the assumption (D4), $\mathcal{Q}$ maps B_{e_0} into B_{e_0}.

Step 2: We need to show that $\mathcal{Q}$ is continuous on B_{e_0}. Let $\varepsilon > 0$ and $\Lambda, \bar{\Lambda} \in B_{e_0}$ such that $\parallel \Lambda - \bar{\Lambda} \parallel < \varepsilon$. We obtain

$$\begin{aligned}\left|\left(\mathcal{Q}\Lambda\right)(\Psi) - \left(\mathcal{Q}\bar{\Lambda}\right)(\Psi)\right| &\leq \left|\Upsilon\left(\Psi, \mathcal{T}(\Psi, \Lambda(\Psi)), {}^s_k\mathcal{Y}^\upsilon \Lambda(\Psi)\right) - \Upsilon\left(\Psi, \mathcal{T}(\Psi, \bar{\Lambda}(\Psi)), {}^s_k\mathcal{Y}^\upsilon \bar{\Lambda}(\Psi)\right)\right| \\ &\leq \upsilon_1 \left|\mathcal{T}(\Psi, \Lambda(\Psi)) - \mathcal{T}(\Psi, \bar{\Lambda}(\Psi))\right| + \upsilon_2 \left|{}^s_k\mathcal{Y}^\upsilon \Lambda(\Psi) - {}^s_k\mathcal{Y}^\upsilon \bar{\Lambda}(\Psi)\right|.\end{aligned}$$

Also,

$$\begin{aligned}\left|{}^s_k\mathcal{Y}^\upsilon \Lambda(\Psi) - {}^s_k\mathcal{Y}^\upsilon \bar{\Lambda}(\Psi)\right| &= \left|\frac{1}{\Gamma(\frac{\Psi}{k})}\int_1^\Psi \left(\Psi^{s+1} - \eta^{s+1}\right)^{\frac{\upsilon}{k}-1} \eta^s \left(\Lambda(\eta) - \bar{\Lambda}(\eta)\right) d\eta\right| \\ &\leq \frac{1}{\Gamma(\frac{\Psi}{k})}\int_1^\Psi \left(\Psi^{s+1} - \eta^{s+1}\right)^{\frac{\upsilon}{k}-1} \eta^s \left|\Lambda(\eta) - \bar{\Lambda}(\eta)\right| d\eta \\ &< \frac{\varepsilon k}{\upsilon(s+1)\Gamma(\frac{1}{k})}\left(T^{s+1} - 1\right)^{\frac{\upsilon}{k}}.\end{aligned}$$

Hence, $\parallel \Lambda - \bar{\Lambda} \parallel < \varepsilon$ gives

$$\left|\left(\mathcal{Q}\Lambda\right)(\Psi) - \left(\mathcal{Q}\bar{\Lambda}\right)(\Psi)\right| < \upsilon_1\upsilon_3\varepsilon + \frac{\upsilon_2\varepsilon k}{\upsilon(s+1)\Gamma(\frac{1}{k})}\left(T^{s+1}-1\right)^{\frac{\upsilon}{k}}.$$

As $\varepsilon \to 0$, we obtain $\left|\left(\mathcal{Q}\Lambda\right)(\Psi) - \left(\mathcal{Q}\bar{\Lambda}\right)(\Psi)\right| \to 0$. This proves that $\mathcal{Q}$ is continuous on B_{e_0}.

Step 3: An estimate of $\mathcal{Q}$ with respect to ω_0: Assume that $\Omega(\neq \phi) \subseteq B_{e_0}$. Let $\varepsilon > 0$ be arbitrary and choose $\Lambda \in \Omega$ and $\Psi_1, \Psi_2 \in \mathcal{I}$ such that $|\Psi_2 - \Psi_1| \leq \varepsilon$ and $\Psi_2 \geq \Psi_1$.

Now,

$$\begin{aligned}
&\left|\left(\mathcal{Q}\Lambda\right)(\Psi_2) - \left(\mathcal{Q}\Lambda\right)(\Psi_1)\right| \\
&= \left|\Upsilon(\Psi_2, \mathcal{T}(\Psi_2, \Lambda(\Psi_2)), {}_k^s\mathcal{Y}^{\upsilon}\Lambda(\Psi_2)) - \Upsilon(\Psi_1, \mathcal{T}(\Psi_1, \Lambda(\Psi_1)), {}_k^s\mathcal{Y}^{\upsilon}\Lambda(\Psi_1))\right| \\
&\leq \left|\Upsilon(\Psi_2, \mathcal{T}(\Psi_2, \Lambda(\Psi_2)), {}_k^s\mathcal{Y}^{\upsilon}\Lambda(\Psi_2)) - \Upsilon(\Psi_2, \mathcal{T}(\Psi_2, \Lambda(\Psi_2)), {}_k^s\mathcal{Y}^{\upsilon}\Lambda(\Psi_1))\right| \\
&+ \left|\Upsilon(\Psi_2, \mathcal{T}(\Psi_2, \Lambda(\Psi_2)), {}_k^s\mathcal{Y}^{\upsilon}\Lambda(\Psi_1)) - \Upsilon(\Psi_2, \mathcal{T}(\Psi_1, \Lambda(\Psi_1)), {}_k^s\mathcal{Y}^{\upsilon}\Lambda(\Psi_1))\right| \\
&+ \left|\Upsilon(\Psi_2, \mathcal{T}(\Psi_1, \Lambda(\Psi_1)), {}_k^s\mathcal{Y}^{\upsilon}\Lambda(\Psi_1)) - \Upsilon(\Psi_1, \mathcal{T}(\Psi_1, \Lambda(\Psi_1)), {}_k^s\mathcal{Y}^{\upsilon}\Lambda(\Psi_1))\right| \\
&\leq \upsilon_2\left|{}_k^s\mathcal{Y}^{\upsilon}\Lambda(\Psi_2) - {}_k^s\mathcal{Y}^{\upsilon}\Lambda(\Psi_1)\right| + \upsilon_1\left|\mathcal{T}(\Psi_2, \Lambda(\Psi_2)) - \mathcal{T}(\Psi_1, \Lambda(\Psi_1))\right| + \omega_{\Upsilon}(\mathcal{I}, \varepsilon) \\
&\leq \upsilon_2\left|{}_k^s\mathcal{Y}^{\upsilon}\Lambda(\Psi_2) - {}_k^s\mathcal{Y}^{\upsilon}\Lambda(\Psi_1)\right| + \upsilon_1\upsilon_3\left|\Lambda(\Psi_2) - \Lambda(\Psi_1)\right| + \omega_{\Upsilon}(\mathcal{I}, \varepsilon),
\end{aligned}$$

where

$$\omega_{\Upsilon}(\mathcal{I}, \varepsilon) = \sup\left\{\begin{array}{c}\left|\Upsilon(\Psi_2, \mathcal{T}, \mathcal{J}_1) - \Upsilon(\Psi_1, \mathcal{T}, \mathcal{J}_1)\right| : |\Psi_2 - \Psi_1| \leq \varepsilon; \Psi_1, \Psi_2 \in \mathcal{I}; \\ \mathcal{T} \in [-\hat{\mathcal{T}}, \hat{\mathcal{T}}]; \mathcal{J}_1 \in [-\hat{\mathcal{J}}, \hat{\mathcal{J}}]\end{array}\right\}.$$

And,

$$\begin{aligned}
&\left|{}_k^s\mathcal{Y}^{\upsilon}\Lambda(\Psi_2) - {}_k^s\mathcal{Y}^{\upsilon}\Lambda(\Psi_1)\right| \\
&= \left|\frac{1}{\Gamma(\frac{\Psi_2}{k})}\int_1^{\Psi_2}\left(\Psi_2^{s+1} - \eta^{s+1}\right)^{\frac{\upsilon}{k}-1}\eta^s\Lambda(\eta)d\eta - \frac{1}{\Gamma(\frac{\Psi_1}{k})}\int_1^{\Psi_1}\left(\Psi_1^{s+1} - \eta^{s+1}\right)^{\frac{\upsilon}{k}-1}\eta^s\Lambda(\eta)d\eta\right| \\
&\leq \left|\frac{1}{\Gamma(\frac{\Psi_2}{k})}\int_1^{\Psi_2}\left(\Psi_2^{s+1} - \eta^{s+1}\right)^{\frac{\upsilon}{k}-1}\eta^s\Lambda(\eta)d\eta - \frac{1}{\Gamma(\frac{\Psi_2}{k})}\int_1^{\Psi_1}\left(\Psi_1^{s+1} - \eta^{s+1}\right)^{\frac{\upsilon}{k}-1}\eta^s\Lambda(\eta)d\eta\right| \\
&\quad + \left|\frac{1}{\Gamma(\frac{\Psi_2}{k})} - \frac{1}{\Gamma(\frac{\Psi_1}{k})}\right|\int_1^{\Psi_1}\left(\Psi_1^{s+1} - \eta^{s+1}\right)^{\frac{\upsilon}{k}-1}\eta^s\left|\Lambda(\eta)\right|d\eta \\
&\leq \frac{1}{\Gamma(\frac{\Psi_2}{k})}\left|\int_1^{\Psi_2}\left(\Psi_2^{s+1} - \eta^{s+1}\right)^{\frac{\upsilon}{k}-1}\eta^s\Lambda(\eta)d\eta - \int_1^{\Psi_1}\left(\Psi_1^{s+1} - \eta^{s+1}\right)^{\frac{\upsilon}{k}-1}\eta^s\Lambda(\eta)d\eta\right| \\
&\quad + \left|\frac{1}{\Gamma(\frac{\Psi_2}{k})} - \frac{1}{\Gamma(\frac{\Psi_1}{k})}\right| \parallel \Lambda \parallel \int_1^{\Psi_1}\left(\Psi_1^{s+1} - \eta^{s+1}\right)^{\frac{\upsilon}{k}-1}\eta^s d\eta \\
&\leq \frac{\parallel \Lambda \parallel}{(s+1)(s+2)\Gamma\left(\frac{\Psi_2}{k}\right)}\left[2(\Psi_2^{s+1} - \Psi_1^{s+1})^{\frac{\upsilon}{k}} + (\Psi_2^{s+1} - 1)^{\frac{\upsilon}{k}} - (\Psi_1^{s+1} - 1)^{\frac{\upsilon}{k}}\right] \\
&\quad + \left|\frac{1}{\Gamma\left(\frac{\Psi_2}{k}\right)} - \frac{1}{\Gamma\left(\frac{\Psi_1}{k}\right)}\right| \parallel \Lambda \parallel \frac{k}{\upsilon(s+1)}\left(T^{s+1} - 1\right)^{\frac{\upsilon}{k}}.
\end{aligned}$$

As $\varepsilon \to 0$, then $\Psi_2 \to \Psi_1$ and so, $|{}_k^s\mathcal{Y}^\upsilon\Lambda(\Psi_2) - {}_k^s\mathcal{Y}^\upsilon\Lambda(\Psi_1)| \to 0$.

Hence,

$$|(\mathcal{Q}\Lambda)(\Psi_2) - (\mathcal{Q}\Lambda)(\Psi_1)| \leq \upsilon_2 |{}_k^s\mathcal{Y}^\upsilon\Lambda(\Psi_2) - {}_k^s\mathcal{Y}^\upsilon\Lambda(\Psi_1)| + \upsilon_1\upsilon_3\omega(\Lambda,\varepsilon) + \omega_\Upsilon(\mathcal{I},\varepsilon),$$

i.e.,

$$\omega(\mathcal{Q}\Lambda,\varepsilon) \leq \upsilon_2 |{}_k^s\mathcal{Y}^\upsilon\Lambda(\Psi_2) - {}_k^s\mathcal{Y}^\upsilon\Lambda(\Psi_1)| + \upsilon_1\upsilon_3\omega(\Lambda,\varepsilon) + \omega_\Upsilon(\mathcal{I},\varepsilon).$$

By the uniform continuity of Υ on $\mathcal{I} \times [-\hat{\mathcal{T}},\hat{\mathcal{T}}] \times [-\hat{\mathcal{J}},\hat{\mathcal{J}}]$, we get $\omega_\Upsilon(\mathcal{I},\varepsilon) \to 0$, as $\varepsilon \to 0$.

Putting $\sup_{\Lambda\in\Omega}$ with $\varepsilon \to 0$ we have,

$$\omega_0(\mathcal{Q}\Omega) \leq \upsilon_1\upsilon_3\omega_0(\Omega),$$

Hence, by Corollary 2.2, $\mathcal{Q}$ has a $\mathcal{FP}$ in $\Omega \subseteq B_{e_0}$, that is, Equation (7) has a solution in $\mathtt{H}$. ∎

Example 4.1.

Take the equation below,

$$\Lambda(\Psi) = \frac{\Lambda(\Psi)}{11+\Psi^4} + \frac{{}_{\frac{1}{5}}^{\frac{1}{5}}\mathcal{Y}^{\frac{2}{5}}\Lambda(\Psi)}{800}, \tag{8}$$

for $\Psi \in [1,2] = \mathcal{I}$.

Here,

$${}_{\frac{1}{5}}^{\frac{1}{5}}\mathcal{Y}^{\frac{2}{5}}\Lambda(\Psi) = \frac{1}{\Gamma(5\Psi)}\int_1^\Psi \vartheta^{\frac{1}{5}}\left(\Psi^{\frac{6}{5}} - \vartheta^{\frac{6}{5}}\right)\Lambda(\vartheta)d\vartheta,$$

and $\Upsilon(\Psi,\mathcal{T},\mathcal{I}_1) = \mathcal{T} + \frac{\mathcal{I}_1}{800}$ with $\mathcal{T}(\Psi,\Lambda) = \frac{\Lambda}{11+\Psi^4}$. It is trivial that both Υ and $\mathcal{T}$ are continuous satisfying

$$|\mathcal{T}(\Psi,Y_1) - \mathcal{T}(\Psi,Y_2)| \leq \frac{|Y_1 - Y_2|}{12},$$

also,

$$\left|\Upsilon(\Psi,\mathcal{T},\mathcal{I}_1) - \Upsilon(\Psi,\bar{\mathcal{T}},\bar{\mathcal{I}}_1)\right| \leq \left|\mathcal{T} - \bar{\mathcal{T}}\right| + \frac{1}{800}\left|\mathcal{I}_1 - \bar{\mathcal{I}}_1\right|.$$

Therefore, $\upsilon_1 = 1,\ \upsilon_2 = \frac{1}{800},\ \upsilon_3 = \frac{1}{12}$ and $\upsilon_1\upsilon_3 = \frac{1}{12} < 1$.

If $\parallel \Lambda \parallel \leq e_0$, then,

$$\hat{\mathcal{T}} = \frac{e_0}{12},$$

and

$$\hat{\mathcal{J}} = \frac{5e_0}{12\Gamma(5)}.$$

Further,

$$|\Upsilon(\Psi, \mathcal{T}, \mathcal{I}_1)| \leq \frac{e_0}{12} + \frac{1}{800} . \frac{5e_0}{12\Gamma(5)} \leq e_0.$$

If we choose $e_0 = 3$, then

$$\hat{\mathcal{T}} = \frac{1}{4},\ \hat{\mathcal{J}} = \frac{5}{4\Gamma(5)},$$

which gives

$$\bar{\Upsilon} \leq 3.$$

On the other hand, assumption $(D4)$ is also satisfied for $e_0 = 3$.

We see that the hypothesis from all of $(D1) - (D4)$ of Theorem 4.1 is verified. From Theorem 4.1, it is obvious that Equation (8) has a solution in $\mathtt{H} = \mathcal{C}(\mathcal{I})$.

5. Conclusion

Here, the existence of a solution to a fractional integral equation using a new generalized version of Darbo's theorem using shifting distance functions has been studied. With the help of the Schauder fixed point theorem, this theorem can easily be proven. Next, we found some new theorems and corollaries of the generalized Darbo's fixed point theorem. Also, we use suitable examples to support the validity of our findings.

This method can be used to solve several sorts of integral equations requiring fractional integrals. The Banach theorem appears to have several limitations. Any continuous function transferring the unit interval onto itself seems intuitive to have a fixed point. We anticipate that our approach will be applicable to functional analysis, including normed spaces and fixed point theory. Our results generalize the analogous existing fixed point results for Banach spaces on their norm spaces. Then, all the expected results in this work will aid in our understanding of a difficult theorem's solution. In the future, we shall study Banach spaces in relation to norm spaces and physical problems.

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