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$M/M/1$ Retrial Queue with Working Vacation and Interruption in Bernoulli Schedule under N-Control Pattern

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Abstract

An $M/M/1$ retrial queue with working vacation and interruption in Bernoulli schedule under N-control pattern is investigated in this article. To describe the system, we employ a QBD analogy. The model's stability condition is deduced. The stationary probability distribution is generated by utilizing the matrix-analytic technique. The performance measures and special cases are designed. The model's firmness is demonstrated numerically.

Keywords: Retrial queue; Working vacation; N-control pattern; Conditional stochastic decomposition; Bernoulli schedule

MSC 2010 No.: 60K25, 90B22

1. Introduction

Wallace (1969) investigated the Quasi Birth-Death process (QBD) in Queueing theory using a Markov chain with a tridiagonal generator. Numerical techniques can be used to analyze congestion situations when it is impossible to achieve a explicit solution for queueing problems. The Matrix

Geometric technique is ideal for this type of solution. Neuts (1981), Latouche and Ramaswami (1999) proposed the matrix geometric solution to the QBD process. Control policies are important for managing queue levels at different epochs. Yadin and Naor (1963) first propose the N - policy.

The queueing system with attendant vacation is noteworthy and can be referred in Tian and Zhang (2006). Servi and Finn (2002) created a modern vacation policy, termed as Working Vacation (WV), where the attendant delivers a lesser rate of service than during the engaged period. Wu and Takagi (2006) worked on $M/G/1/MWV$. Kalyanaraman and Pazhani Bala Murugan (2008) have worked on the retrial queue with vacation, Pazhani Bala Murugan and Santhi (2013b) have worked on WV.

Liu (2007) analysed the stochastic decompositions in the $M/M/1/WV$ queue. The $M/M/1/WV$ queue and WV interruptions was analysed by Li and Tian (2007). Tian (2008) considered $M/M/1/SWV$ queue. Analysis for the $M/M/1/MWV$ queue and N-policy was studied by Zhang and Xu (2008). Ye and Liu (2015) discussed the analysis of the $M/M/1$ Queue with two vacation policies

Recently retrial queues have been studied widely and it was different from normal queues. Due to limited waiting space in the retrial queue the customers are forced to stay in the orbit. Whenever the approaching customers finds that the attendant is engaged, they join the orbit and requests service from the orbit. An $M/M/1$ retrial queue with general retrial times was studied by Choi et al. (1993). The retrial queue and WV was simultaneously considered by Do (2010).

At the time of service completion in the WV period, if the orbit contains N customers, then the server will opt to terminate the WV (that means WV interruption happens) with a certain probability or to continue in the WV with a complementary probability. This is termed as Bernoulli schedule WV interruption. Rao (1965) and Majid and Manoharan (2019) considered vacation interruption queues. Tao et al. (2012) discussed the $M/M/1$ retrial queue with WV interruption collision under N - Policy. Manoharan and Ashok (2018) discussed an $M/M/1/WV$ and vacation interruption under Bernoulli schedule. Li et al. (2018) considered $M/G/1$ retrial queue with balking customers and Bernoulli WV interruption.

2. QBD process model

We examine a Markovian retrial queue with WV and interruption in Bernoulli schedule under N-Control. With the parameter λ, the customer's inter-arrival times are exponentially distributed. The retrial requests from the orbit follows a Poisson process with rate α. The attendant will take a WV when the system gets clear, which is exponentially distributed with parameter θ. The service is exponentially distributed with parameter μ at the time of the regular busy period. When comparing to the service offered throughout engaged period, the service provided during the WV is at a slower rate. WV service is exponentially distributed with parameters η $(\eta < \mu)$. At the time of service completion during the WV period, if the attendant identifies not less than N customers in the orbit, then the attendant will opt to terminate WV (WV interruption happens) with probability

$p(0 \leq p \leq 1)$, otherwise the attendant will carry on with the WV period with the complementary probability $q(= 1 - p)$. This is termed as Bernoulli Schedule WV interruption. When a WV ends, if the attendant identifies not less than N customers in the orbit, then the attendant will return to engaged period, otherwise will start another WV. Inter-arrival times, inter-retrial periods, service periods, and WV periods are all presumed to be independent of one another. Let the number of customers in the orbit at time t is indicated by $Q(t)$ and $H(t)$ represents attendant's position at time t. The single attendant might exist in four different states at time t,

$$H(t) = \begin{cases} 0 \text{ - attendant is on WV and is unoccupied,} \\ 1 \text{ - attendant is on WV and is engaged,} \\ 2 \text{ - attendant is on engaged period and is unoccupied,} \\ 3 \text{ - attendant is on engaged period and is engaged.} \end{cases}$$

Clearly, $\{(Q(t), H(t)); t \geq 0\}$ is a Markov process with state space

$$\Omega = \{(m, h) : m \geq 0, h = 0, 1, 2, 3\}.$$

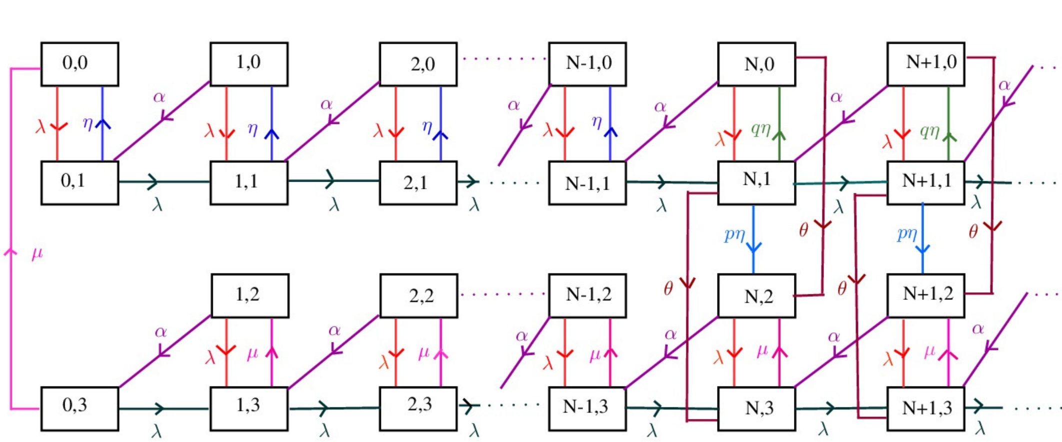

Figure 1. Transition Diagram of the States

The states infinitesimal generator can be described by using lexicographical sequence as follows:

$$\widetilde{Q} = \begin{bmatrix} D_0 & F & & & & & & & \\ E & D_1 & F & & & & & & \\ & E & D_1 & F & & & & & \\ & & E & D_1 & F & & & & \\ & & & \vdots & \vdots & \vdots & & & \\ & & & E & D_1 & F & & & \\ & & & & E & D & F & & \\ & & & & & E & D & F & \\ & & & & & & \vdots & \vdots & \vdots \end{bmatrix},$$

where

$$D_0 = \begin{bmatrix} -\lambda & \lambda & 0 & 0 \\ \eta & -\eta-\lambda & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \mu & 0 & 0 & -\mu-\lambda \end{bmatrix}, \qquad F = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix},$$

$$D_1 = \begin{bmatrix} -\alpha-\lambda & \lambda & 0 & 0 \\ \eta & -\eta-\lambda & 0 & 0 \\ 0 & 0 & -\alpha-\lambda & \lambda \\ 0 & 0 & \mu & -\mu-\lambda \end{bmatrix}, \qquad E = \begin{bmatrix} 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$D = \begin{bmatrix} -\alpha-\lambda-\theta & \lambda & \theta & 0 \\ q\eta & -\lambda-\eta-\theta & p\eta & \theta \\ 0 & 0 & -\alpha-\lambda & \lambda \\ 0 & 0 & \mu & -\mu-\lambda \end{bmatrix}.$$

Due to the block structure of matrix $\widetilde{Q}$, $\{(Q(t), H(t));\ t \geq 0\}$ is called a *QBD* process.

Pr{that the attendant is engaged and does not offer a service to a customer while there is no customer in the orbit}= 0.

3. The Model's Stability Condition and Rate Matrix (R)

Theorem 3.1.

The *QBD* process $\{(Q(t), H(t)); t \geq 0\}$ is positive recurrent if and only if $\alpha(\mu-\lambda) > \lambda^2$.

Proof:

Consider

$$S_m = E + D + F = \begin{bmatrix} -\alpha-\lambda-\theta & \alpha+\lambda & \theta & 0 \\ q\eta & -\theta-\eta & p\eta & \theta \\ 0 & 0 & -\alpha-\lambda & \alpha+\lambda \\ 0 & 0 & \mu & -\mu \end{bmatrix}.$$

Theorem $7.3.1$ in Latouche and Ramaswami (1999) offers requirements for positive recurrence of the QBD process because matrix S_m is reducible. After permutation of rows and columns, and hence, the *QBD* is positive recurrent if and only if $\pi \begin{bmatrix} 0 & \alpha \\ 0 & 0 \end{bmatrix} e > \pi \begin{bmatrix} 0 & 0 \\ 0 & \lambda \end{bmatrix} e$.

Here, all the elements of the column vector $e = 1$ and π is a unique solution of the system $\pi \begin{bmatrix} -\alpha-\lambda & \alpha+\lambda \\ \mu & -\mu \end{bmatrix}$ = 0, $\pi e = 1$. The *QBD* process is positive recurrent if and only if $\alpha(\mu-\lambda) > \lambda^2$ after some algebraic manipulations. ∎

Theorem 3.2.

If $\alpha(\mu-\lambda) > \lambda^2$, the matrix quadratic equation $R^2E+RD+F=0$ has the minimal non-negative solution $R = \begin{bmatrix} 0 & 0 & 0 & 0 \\ r_1 & r_2 & r_3 & r_4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & r_5 & r_6 \end{bmatrix}$, where

$$r_1 = \frac{r_2 q\eta}{(\lambda+\alpha+\theta)}, \qquad r_2 = \frac{t-\sqrt{t^2-4\alpha\lambda q\eta(\lambda+\alpha+\theta)}}{2\alpha q\eta},$$

and $t = [(\lambda+\alpha+\theta)(\lambda+\theta+\eta) - q\eta\lambda]$,

$$r_3 = \frac{r_1\theta + r_4\mu + r_2 p\eta}{(\lambda+\alpha)}, \qquad r_4 = \frac{\alpha r_2 r_1\theta + r_1\theta\lambda + r_2\theta(\lambda+\alpha) + r_2 p\eta(r_2\alpha+\lambda)}{(\lambda+\mu)(\lambda+\alpha) - \alpha r_2\mu - \alpha r_5(\lambda+\alpha) - \mu\lambda},$$

$$r_5 = \frac{\lambda}{\alpha}, \qquad r_6 = \frac{\lambda(\lambda+\alpha)}{\mu\alpha}.$$

Proof:

We can consider $R = \begin{bmatrix} R_{11} & R_{12} \\ 0 & R_{22} \end{bmatrix}$, from the matrices E, D, F where R_{11}, R_{12} and R_{22} are all 2x2 matrices. Substituting R into $R^2E+RD+F=0$, we get

$$R_{11}^2 \begin{bmatrix} 0 & \alpha \\ 0 & 0 \end{bmatrix} + R_{11} \begin{bmatrix} -\alpha-\lambda-\theta & \lambda \\ q\eta & -\lambda-\eta-\theta \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$(R_{11}R_{12} + R_{12}R_{22}) \begin{bmatrix} 0 & \alpha \\ 0 & 0 \end{bmatrix} + R_{11} \begin{bmatrix} \theta & 0 \\ p\eta & \theta \end{bmatrix} + R_{12} \begin{bmatrix} -\alpha-\lambda & \lambda \\ \mu & -\mu-\lambda \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$R_{22}^2 \begin{bmatrix} 0 & \alpha \\ 0 & 0 \end{bmatrix} + R_{22} \begin{bmatrix} -\alpha-\lambda & \lambda \\ \mu & -\mu-\lambda \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

From the above set of equations with some computations, we get R_{11}, R_{22}, and R_{12}, respectively as

$$R_{11} = \begin{bmatrix} 0 & 0 \\ r_1 & r_2 \end{bmatrix}, R_{22} = \begin{bmatrix} 0 & 0 \\ r_5 & r_6 \end{bmatrix} \text{ and } R_{12} = \begin{bmatrix} 0 & 0 \\ r_3 & r_4 \end{bmatrix}.$$

■

4. The Model's Stationary Probability Distribution

If $\alpha(\mu-\lambda) > \lambda^2$, assign (Q, H) be the stationary probability distribution of the process $\{(Q(t), H(t)); t \geq 0\}$. Represent,

$$\pi_m = (\pi_{m,0}, \pi_{m,1}, \pi_{m,2}, \pi_{m,3}), \qquad m \geq 0;$$

$$\pi_{m,h} = P\{Q = m, H = h\} = \lim_{t\to\infty} P\{Q(t) = m, H(t) = h\}, \quad (m,h) \in \Omega.$$

Here, $\pi_{0,2} = 0$.

Theorem 4.1.

If $\alpha(\mu - \lambda) > \lambda^2$, the stationary probability distribution of (Q, H) is indicated by

$$\pi_{m,0} = \pi_{N-1,1} r_1 r_2^{m-N}, \qquad m \geq N, \tag{1}$$

$$\pi_{m,1} = \pi_{N-1,1} r_2^{m+1-N}, \qquad m \geq N, \tag{2}$$

$$\pi_{m,2} = \pi_{N-1,1}\left[r_3 r_2^{m-N} + \frac{r_4 r_5}{r_6 - r_2}\left(r_6^{m-N} - r_2^{m-N}\right)\right] + \pi_{N-1,3} r_5 r_6^{m-N}, \qquad m \geq N, \tag{3}$$

$$\pi_{m,3} = \pi_{N-1,1}\frac{r_4}{r_6 - r_2}(r_6^{m+1-N} - r_2^{m+1-N}) + \pi_{N-1,3} r_6^{m+1-N}, \qquad m \geq N, \tag{4}$$

$$\pi_{m,0} = \frac{\eta}{\lambda+\alpha}\pi_{0,1} + \frac{\eta}{\lambda+\alpha}(\pi_{1,1} - \pi_{0,1})\frac{1-q_1^m}{1-q_1}, \qquad 2 \leq m \leq N-2, \tag{5}$$

$$\pi_{m,1} = \pi_{0,1} + (\pi_{1,1} - \pi_{01})\frac{1-q_1^m}{1-q_1}, \qquad 2 \leq m \leq N-2, \tag{6}$$

$$\pi_{m,2} = \frac{\mu}{\lambda+\alpha}\pi_{0,3} + \frac{\mu}{\lambda+\alpha}(\pi_{1,3} - \pi_{0,3})\frac{1-q_2^m}{1-q_2}, \qquad 2 \leq m \leq N-2, \tag{7}$$

$$\pi_{m,3} = \pi_{0,3} + (\pi_{1,3} - \pi_{0,3})\frac{1-q_2^m}{1-q_2}, \qquad 2 \leq m \leq N-2, \tag{8}$$

$$\pi_{N-1,0} = \frac{-\lambda\eta}{[\lambda\eta + (r_1\alpha - \lambda - \eta)(\lambda+\alpha)]}\pi_{N-2,1}, \tag{9}$$

$$\pi_{N-1,1} = \frac{\lambda+\alpha}{\eta}\pi_{N-1,0}, \tag{10}$$

$$\pi_{N-1,2} = r_3\pi_{N-1,1} + \frac{\lambda}{\alpha}\pi_{N-2,3}, \tag{11}$$

$$\pi_{N-1,3} = \frac{\lambda+\alpha}{\mu}\pi_{N-1,2}, \tag{12}$$

where $\quad q_1 = \dfrac{\lambda(\lambda+\alpha)}{\alpha\eta} \quad$ and $\quad q_2 = \dfrac{\lambda(\lambda+\alpha)}{\alpha\mu}$,

$$\pi_{1,1} = -K^{-1}\left[\frac{\lambda(\lambda+\alpha+\eta)}{\lambda+\alpha} + \Delta - K\right]\pi_{0,1}, \tag{13}$$

$$\pi_{1,0} = \frac{\eta}{\lambda+\alpha}\pi_{1,1}, \tag{14}$$

$$\pi_{0,0} = \frac{\lambda+\eta}{\lambda}\pi_{0,1} - \frac{\alpha}{\lambda}\pi_{1,0}, \tag{15}$$

$$\pi_{0,3} = \frac{\lambda}{\mu}\pi_{0,0} - \frac{\eta}{\mu}\pi_{0,1}, \tag{16}$$

$$\pi_{1,2} = \frac{\lambda+\mu}{\alpha}\pi_{0,3}, \tag{17}$$

$$\pi_{1,3} = \frac{\lambda+\alpha}{\mu}\pi_{1,2}, \tag{18}$$

where $\quad \Delta = \dfrac{-\lambda\alpha\eta}{[\lambda\eta + (r_1\alpha - \lambda - \eta)(\lambda+\alpha)]} - \lambda - \eta \quad$ and

$$K = \left[\lambda\frac{1-q_1^{N-3}}{1-q_1} + \left(\Delta + \frac{\lambda\eta}{\lambda+\alpha}\right)\frac{1-q_1^{N-2}}{1-q_1}\right].$$

The normalization condition can finally be used to determine $\pi_{0,1}$.

Proof:

Using the technique from Neuts (1981), we have

$$\pi_m = (\pi_{m,0}, \pi_{m,1}, \pi_{m,2}, \pi_{m,3}) = \pi_{N-1}R^{m+1-N}$$

$$= (\pi_{N-1,0}, \pi_{N-1,1}, \pi_{N-1,2}, \pi_{N-1,3})R^{m+1-N}, \qquad m \geq N.$$

For $m \geq N$,

$$R^{m+1-N} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ r_1 r_2^{m-N} & r_2^{m+1-N} & r_3 r_2^{m-N} + \dfrac{r_4 r_5 (r_6^{m-N} - r_2^{m-N})}{r_6 - r_2} & \dfrac{r_4(r_6^{m+1-N} - r_2^{m+1-N})}{r_6 - r_2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & r_5 r_6^{m-N} & r_6^{m+1-N} \end{bmatrix},$$

substituting R^{m+1-N} into the above equation, we get $(1-4)$. However, $\pi_0, \pi_1, ..., \pi_{N-1}$ satisfies the equation $(\pi_0, \pi_1, ..., \pi_{N-1}$)B[R]=0, where

$$B[R] = \begin{bmatrix} D_0 & F & & & \\ E & D_1 & F & & \\ & E & D_1 & F & \\ & & \vdots & \vdots & \vdots \\ & & E & D_1 & F \\ & & & E & RE + D_1 \end{bmatrix},$$

and

$$RE + D_1 = \begin{bmatrix} -(\lambda+\alpha) & \lambda & 0 & 0 \\ \eta & r_1\alpha-\lambda-\eta & 0 & r_3\alpha \\ 0 & 0 & -(\lambda+\alpha) & \lambda \\ 0 & 0 & \mu & r_5\alpha-\lambda-\mu \end{bmatrix}.$$

The following equations are computed from $B[R]$,

$$-\lambda\pi_{0,0}+\eta\pi_{0,1}+\mu\pi_{0,3}=0, \tag{19}$$

$$\lambda\pi_{0,0}-(\lambda+\eta)\pi_{0,1}+\alpha\pi_{1,0}=0, \tag{20}$$

$$-(\lambda+\mu)\pi_{0,3}+\alpha\pi_{1,2}=0, \tag{21}$$

$$-(\lambda+\alpha)\pi_{m,0}+\eta\pi_{m,1}=0, \qquad 1\le m\le N-2, \tag{22}$$

$$\lambda\pi_{m-1,1}+\lambda\pi_{m,0}-(\lambda+\eta)\pi_{m,1}+\alpha\pi_{m+1,0}=0, \qquad 1\le m\le N-2, \tag{23}$$

$$-(\lambda+\alpha)\pi_{m,2}+\mu\pi_{m,3}=0, \qquad 1\le m\le N-2, \tag{24}$$

$$\lambda\pi_{m-1,3}+\lambda\pi_{m,2}-(\lambda+\mu)\pi_{m,3}+\alpha\pi_{m+1,2}=0, \qquad 1\le m\le N-2, \tag{25}$$

$$-(\lambda+\alpha)\pi_{N-1,0}+\eta\pi_{N-1,1}=0, \tag{26}$$

$$\lambda\pi_{N-2,1}+\lambda\pi_{N-1,0}+(r_1\alpha-\lambda-\eta)\pi_{N-1,1}=0, \tag{27}$$

$$-(\lambda+\alpha)\pi_{N-1,2}+\mu\pi_{N-1,3}=0, \tag{28}$$

$$\lambda\pi_{N-2,3}+r_3\alpha\pi_{N-1,1}+\lambda\pi_{N-1,2}+(r_5\alpha-\lambda-\mu)\pi_{N-1,3}=0, \tag{29}$$

From (19) to (29), we get (5) to (18) , where $\sum_{h=0}^{3}\sum_{m=0}^{\infty}\pi_{m,h}=1$, finally we can get $\pi_{0,1}$. ∎

5. The Model's Conditional Stochastic Decomposition

Lemma 5.1.

If $\alpha(\mu-\lambda)>\lambda^2$, let Q_0 be the conditional line length of an M/M/1 retrial queue in the orbit when the attendant is engaged, then Q_0 has a PGF $G_{Q_0}(z)=\dfrac{1-r_6}{1-r_6 z}$.

Proof:

Consider a Markovian retrial queue. Two inter-valued random variables are used to explain the system at time t. Let $Q^{\bullet}(t)$ be the number of customers in the orbit at time t,

$$H^{\bullet}(t) = \begin{cases} 0 \text{ - attendant is unoccupied,} \\ 1 \text{ - attendant is engaged.} \end{cases}$$

Then, $\{(Q^{\bullet}(t), H^{\bullet}(t)); t \geq 0\}$ is a Markov process with state space $\{(m,h) : m \geq 0, h = 0, 1\}$. The *infinitesimal generator* can be expressed as

$$\widetilde{Q^{\bullet}} = \begin{bmatrix} D_0 & F & & & \\ E & D & F & & \\ & E & D & F & \\ & & \vdots & \vdots & \vdots \end{bmatrix},$$

where

$$D_0 = \begin{bmatrix} -\lambda & \lambda \\ \mu & -\mu-\lambda \end{bmatrix}, \quad F = \begin{bmatrix} 0 & 0 \\ 0 & \lambda \end{bmatrix}, \quad E = \begin{bmatrix} 0 & \alpha \\ 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} -\alpha-\lambda & \lambda \\ \mu & -\mu-\lambda \end{bmatrix}.$$

The *QBD* process $\{(Q^{\bullet}(t), H^{\bullet}(t)); t \geq 0\}$ is positive recurrent iff $(\mu - \lambda)\alpha > \lambda^2$. Express,

$$\pi_{m,h} = P\{Q^{\bullet} = m, H^{\bullet} = h\} = \lim_{t \to \infty} P\{Q^{\bullet}(t) = m, H^{\bullet}(t) = h\}.$$

The stationary probability distribution is

$$\widetilde{\pi}_{m,0} = \widetilde{\pi}_{0,1} r_5 r_6^{m-1}, \qquad m \geq 1,$$

$$\widetilde{\pi}_{m,1} = \widetilde{\pi}_{0,1} r_6^{m}, \qquad m \geq 0,$$

$$\widetilde{\pi}_{0,0} = \left(1 + \frac{1+r_5}{1-r_6}\frac{\lambda}{\mu}\right)^{-1},$$

$$\widetilde{\pi}_{0,1} = \frac{\lambda}{\mu}\widetilde{\pi}_{0,0}.$$

The normalization condition is used to determine the value of $\pi_{0,0}$.

Therefore,

$$G_{Q_0}(z) = \sum_{m=0}^{\infty} z^m P\{Q_0 = m\} = \frac{\sum_{m=0}^{\infty} \widetilde{\pi}_{0,1} r_6^m z^m}{\sum_{m=0}^{\infty} \widetilde{\pi}_{0,1} r_6^m} = \frac{1-r_6}{1-r_6 z}.$$

Establishing Q^N = {Difference of Q and N such that the state of the attendant is either 1 *or* 3 and $Q \geq N$} and Q^N is the line length which depends on the condition that the attendant is engaged and there are not less than N customers in the orbit.

Let $P_b^\bullet$ denotes *Pr{the server is occupied given that atleast N customers present in the orbit}*,

$$
\begin{aligned}
P_b^\bullet = P\{Q \geq N, H = 1 \text{ or } 3\} &= \sum_{m=N}^{\infty} \pi_{m,1} + \sum_{m=N}^{\infty} \pi_{m,3} \\
&= \sum_{m=N}^{\infty} \pi_{N-1,1} r_2^{m+1-N} + \sum_{m=N}^{\infty} \frac{r_4}{r_6 - r_2}(r_6^{m+1-N} - r_2^{m+1-N})\pi_{N-1,1} + \sum_{m=N}^{\infty} r_6^{m+1-N}\pi_{N-1,3} \\
&= \frac{r_4 + r_2(1-r_6)}{(1-r_2)(1-r_6)}\pi_{N-1,1} + \frac{r_6}{(1-r_6)}\pi_{N-1,3}.
\end{aligned}
$$

This completes the proof. ∎

Theorem 5.1.

If $(\mu - \lambda)\alpha > \lambda^2$, then we can disintegrate $Q^N = Q_0 + Q_c$, where Q_0 go along with a geometric distribution with specification $1 - r_6$. Subsidiary line length Q_c has a distribution

$$
P\{Q_c = 0\} = \frac{1}{P_b^\bullet} \frac{(r_2 + r_4)\pi_{N-1,1} + r_6\pi_{N-1,3}}{1 - r_6},
$$

$$
P\{Q_c = m\} = \frac{\pi_{N-1,1}}{P_b^\bullet} \frac{r_2(r_2 + r_4 - r_6)}{1 - r_6} r_2^{m-1}, \qquad m \geq 1.
$$

Proof:

The PGF of Q^N is given below:

$$
\begin{aligned}
G_{Q^N}(z) = \sum_{m=0}^{\infty} z^m P\{Q^N = m\} &= \frac{1}{p_b^\bullet}\left(\sum_{m=0}^{\infty} z^m \pi_{N+m,1} + \sum_{m=0}^{\infty} z^m \pi_{N+m,3}\right), \\
&= \frac{1}{p_b^\bullet}\left[\pi_{N-1,1}\frac{r_2}{1 - r_2 z} + \pi_{N-1,1}\frac{r_4}{(1-r_2 z)(1-r_6 z)} + \pi_{N-1,3}\frac{r_6}{1 - r_6 z}\right], \\
&= \frac{1}{p_b^\bullet}\frac{1-r_6}{1-r_6 z}\left[\pi_{N-1,1}\frac{r_2(1-r_6 z)}{(1-r_2 z)(1-r_6)} + \pi_{N-1,1}\frac{r_4}{(1-r_2 z)(1-r_6)} + \pi_{N-1,3}\frac{r_6}{1-r_6}\right], \\
&= \frac{1}{p_b^\bullet}\frac{1-r_6}{1-r_6 z}\left[\frac{(r_2 + r_4)\pi_{N-1,1} + r_6\pi_{N-1,3}}{1-r_6} + \pi_{N-1,1}\frac{r_2(r_2 + r_4 - r_6)z}{(1-r_2 z)(1-r_6)}\right], \\
&= \frac{1-r_6}{1-r_6 z}\left[\frac{1}{p_b^\bullet}\frac{(r_2 + r_4)\pi_{N-1,1} + r_6\pi_{N-1,3}}{1-r_6} + \pi_{N-1,1}\frac{1}{p_b^\bullet}\frac{r_2(r_2 + r_4 - r_6)z}{(1-r_2 z)(1-r_6)}\right], \\
&= G_{Q_0}(z) G_{Q_c}(z).
\end{aligned}
$$

∎

6. The Model's Performance Measures

From Theorem 4.1, we have

Pr{that the attendant is engaged} = P_b

$$
\begin{aligned}
P_b &= \sum_{m=0}^{\infty}\pi_{m,1} + \sum_{m=0}^{\infty}\pi_{m,3},\\
&= (N-1)\left(\frac{\pi_{1,1}}{1-q_1} - \frac{q_1\pi_{0,1}}{1-q_1}\right) - \frac{\pi_{1,1}-\pi_{0,1}}{(1-q_1)^2}(1-q_1^{N-1})\\
&\quad + (N-1)\left(\frac{\pi_{1,3}}{1-q_2} - \frac{q_2\pi_{0,3}}{1-q_2}\right) - \frac{\pi_{1,3}-\pi_{0,3}}{(1-q_2)^2}(1-q_2^{N-1})\\
&\quad + \frac{1-r_6+r_4}{(1-r_2)(1-r_6)}\pi_{N-1,1} + \frac{1}{(1-r_6)}\pi_{N-1,3},
\end{aligned}
$$

Pr{that the attendant is unoccupied}= $P_f = \sum_{m=0}^{\infty}\pi_{m,0} + \sum_{m=1}^{\infty}\pi_{m,2} = 1 - P_b.$

Assume that $E[L]$ *denotes the mean number of customers in the orbit*, then

$$
\begin{aligned}
E[L] &= \sum_{m=1}^{\infty} m(\pi_{m,0}+\pi_{m,1}+\pi_{m,2}+\pi_{m,3}),\\
&= \sum_{m=1}^{N-1} m(\pi_{m,0}+\pi_{m,2}) + \sum_{m=1}^{N-2} m(\pi_{m,1}+\pi_{m,3}) + (N-1)\pi_{N-1,3}\frac{1+r_5}{1-r_6}\\
&\quad + (N-1)\pi_{N-1,1}\frac{(1+r_1+r_3)(1-r_6)+r_4(1+r_5)}{(1-r_2)(1-r_6)} + \pi_{N-1,3}\frac{r_5+r_6}{(1-r_6)^2}\\
&\quad + \pi_{N-1,1}\frac{(r_1+r_2+r_3)(1-r_6)^2 + r_4r_5(2-r_2-r_6) + r_4(1-r_2r_6)}{(1-r_6)^2(1-r_2)^2}.
\end{aligned}
$$

Let $E[L_s]$ *be the mean number of customers in the system*, then

$$
E[L_s] = \sum_{m=1}^{\infty} m(\pi_{m,0}+\pi_{m,2}) + \sum_{m=0}^{\infty}(m+1)(\pi_{m,1}+\pi_{m,3}).
$$

We have the following assumptions and results.

Let $E[W]$ be the expected sojourn time of a orbit customers, using Little's formula

$$
E[W] \;=\; \frac{E[L]}{\lambda}.
$$

Let $E[W_s]$ be the customer's expected sojourn time in the system

$$
E[W_s] \;=\; \frac{E[L_s]}{\lambda}.
$$

We have the following result for our model,

$$\pi_{0,0} \ = \ \frac{E[T_{0,0}]}{E[T]+1/\lambda},$$

where:

T - engaged period.

$E[T_{0,0}]$ - duration of time while the system is in the state $(0,0)$ in a regenerative cycle.

Also, $E[T_{0,0}] = \dfrac{1}{\lambda}$ and $E[T] = (\pi_{0,0}^{-1} - 1)\lambda^{-1}$.

7. Special Cases

(a) If $p = 0, q = 1$, this model is remodeled as "M/M/1 retrial queue with multiple working vacations under N-policy."
(b) If $\alpha \to \infty$, this model is remodeled as "Analysis for the M/M/1 queue with multiple working vacations and N-policy."
(c) If $\alpha \to \infty$, $\eta = 0$, this model is remodeled as"An M/M/1 queue with multiple vacation under N-policy."
(d) If $\alpha \to \infty$, $\eta = 0$, $\theta = 0$ this model is remodeled as "Standard M/M/1 queue under N-policy."

8. Numerical Results

By fixing the values of $N = 2$, $\mu = 8.9$, $\theta = 2.1$, $\eta = 1.3$, $p = 0.8$, $q = 0.2$ and extending the value of λ from 1.0 to 2.0 incremented with 0.2 and extending the values of α from 3.2 to 5.2 insteps of 1.0 subject to the stability condition, the values of $E(L)$ are calculated and tabulated in Table 1 and the corresponding line graphs are drawn in the Figure 2. From the graph it is inferred that as λ rises $E(L)$ rises as expected.

Table 1. λ versus $E(L)$

λ	$\alpha = 3.2$	$\alpha = 4.2$	$\alpha = 5.2$
1.0	0.1696	0.1419	0.1242
1.2	0.2352	0.1958	0.1709
1.4	0.3112	0.2579	0.2245
1.6	0.3997	0.3293	0.2857
1.8	0.5037	0.4121	0.3561
2.0	0.6281	0.5091	0.4375

By fixing the values of $N = 2$, $\mu = 9.4$, $\theta = 2$, $\alpha = 4.1$, $p = 0.7$, $q = 0.3$ and extending the value of λ from 1.0 to 2.0 incremented with 0.5 and extending the values of η from 0.5 to 2.5 insteps of 1 subject to the stability condition, the values of $E(L)$ are calculated and tabulated in Table 2 and the corresponding line graphs are drawn in the Figure 3. From the graph it is inferred that as λ rises $E(L)$ rises as expected.

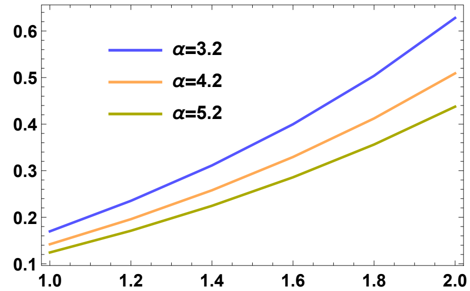

Figure 2. λ versus $E(L)$

Table 2. λ versus $E(L)$

λ	$\eta = 0.5$	$\eta = 1.5$	$\eta = 2.5$
1.0	0.1726	0.1313	0.1046
1.2	0.2286	0.1818	0.1494
1.4	0.2912	0.2401	0.2025
1.6	0.3617	0.3069	0.2649
1.8	0.4418	0.3842	0.3381
2.0	0.5339	0.4739	0.4242

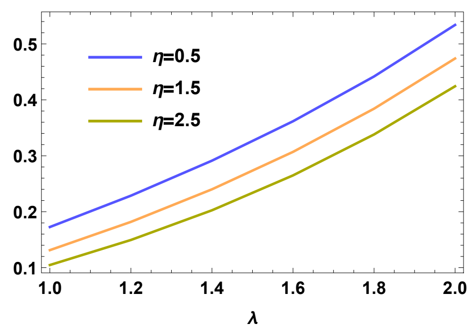

Figure 3. λ versus $E(L)$

By fixing the values of $N = 2$, $\mu = 5$, $\theta = 0.3$, $\eta = 0.2$, $p = 0.9$, $q = 0.1$ and extending the value of λ from 1.0 to 2.0 incremented with 0.2 and extending the values of α from 3.0 to 4 insteps of 0.5 subject to the stability condition, the values of P_b are calculated and tabulated in Table 3 and the corresponding line graphs are drawn in the Figure 4. From the graph it is inferred that as λ rises P_b rises as expected.

Table 3. λ versus P_b

λ	$\alpha = 3.0$	$\alpha = 3.5$	$\alpha = 4.0$
1.0	0.5108	0.5194	0.5259
1.2	0.5134	0.5243	0.5327
1.4	0.5158	0.5289	0.5391
1.6	0.5183	0.5333	0.5451
1.8	0.5207	0.5376	0.5511
2.0	0.5231	0.5418	0.5566

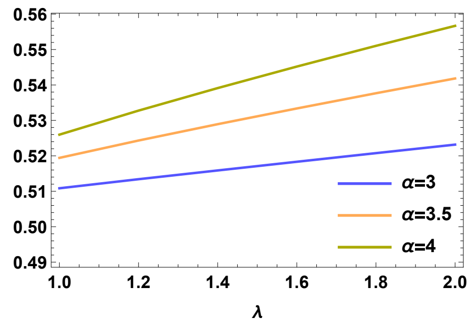

Figure 4. λ versus P_b

By fixing the values of $N = 2$, $\mu = 11$, $\theta = 3.5$, $\eta = 5$, $p = 0.1$, $q = 0.9$ and extending the values of λ from 1.0 to 2.0 incremented with 0.2 and extending the values α from 0.5 to 1.5 insteps of 0.5 subject to the stability condition, the values of P_f are calculated and tabulated in Table 4 and the corresponding line graphs are drawn in the Figure 5. From the graph it is inferred that as λ rises P_f falls as expected.

Table 4. λ versus P_f

λ	$\alpha = 0.5$	$\alpha = 1$	$\alpha = 1.5$
1.0	0.8537	0.8439	0.8404
1.2	0.8391	0.8236	0.8179
1.4	0.8278	0.8062	0.7981
1.6	0.8196	0.7915	0.7804
1.8	0.8138	0.7791	0.7648
2.0	0.8098	0.7686	0.7512

9. Conclusion

In this article, Markovian retrial queue with WV and interruption in Bernoulli schedule under N-Control is evaluated. We calculate stability condition and rate matrix of the model. We went on the stationary probability distribution by adopting the matrix-analytic methods. We also derive the conditional stochastic decomposition and performance measures. We perform some special cases.

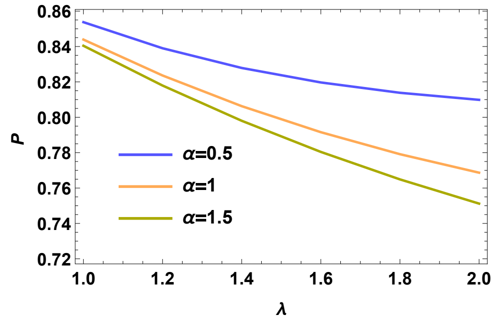

Figure 5. λ versus P_f

We illustrate some numerical examples under the stability condition.

REFERENCES

Artalejo, J. and Corral, A. (2008). *Retrial Queueing Systems*, Springer, Berlin.

Choi, B., Park, K. and Pearce, C. (1993). An M/M/1 retrial queue with control policy and general retrial times, Queueing Syst., 14, pp. 275-292.

Do, T. (2010). M/M/1 retrial queue with working vacations, Acta Inform., Vol. 47, pp. 67-75.

Haghighi, A.M., Medhi, J. and Mohanty, S. G. (1986). On a multiserver Markovian queueing system with balking and reneging, Comput. and Ops. Res., Vol. 13, No. 4, pp. 421-425.

Haghighi, A.M. and Mishev, D.P. (2016). *Delayed and Network Queues*, John Wiley and Sons Inc., New Jersy, USA.

Kalyanaraman, R. and Pazhani Bala Murugan, S. (2008). A single server retrial queue with vacation, J.Appl.Math.and Informatics, Vol. 26, No. (3-4), pp. 721-732.

Latouche, G. and Ramaswami, V. (1999). *Introduction to Matrix Analytic Methods in Stochastic Modelling*, ASA-SIAM Series on Applied Probability, USA.

Li, J. and Tian, N. (2007). The M/M/1 queue with working vacations and vacation interruptions, J. Syst. Sci. Syst. Eng., Vol. 16, pp. 121-127.

Li, T., Zhang, L. and Gao, S. (2018). An M/G/1 retrial queue with balking customers and Bernoulli working vacation interruption, Quality Technology and Quantitative Management, pp. 1-20.

Liu, W.-Y. (2007). Stochastic decompositions in the M/M/1 queue with working vacations, Operation Research Letters, Vol. 35, pp. 595-600.

Majid, S. and Manoharan, P. (2019). M/M/1 queue with working vacation and vacation interruption, Appl. Appl. Math., Vol. 14, No. 1 pp. 19-33.

Manoharan, P. and Ashok, A. (2020). An M/M/1 queue with working vacation and vacation interruption under Bernoulli schedule, International Journal of Engineering and Technology, Vol. 7, No. (4.10), pp. 448-454.

Manoharan, P. and Majid, S. (2019). Analysis of M/M/1 queue with working vacation and vacation interruption, Appl. Appl. Math., Vol. 14, No. 1, pp. 19-33.

Neuts, M.F. (1981). *Matrix - Geometric Solutions in Stochastic Models*, Johns Hopkins University

Press, Baltimore.

Pazhani Bala Murugan, S. and Santhi, K. (2013a). An M/G/1 queue with two stage heterogeneous service and single working vacation, International Mathematical Forum, Vol. 8, No. 27, pp. 1323-1336.

Pazhani Bala Murugan, S. and Santhi, K. (2013b). An M^X/G/1 queue with two stage heterogeneous service and multiple working vacation, International Journal of Mathematics Sciences and Applications, Vol. 3, No. 1, pp. 233-249.

Pazhani Bala Murugan, S. and Santhi, K. (2017). An M/G/1 retrial queue with single working vacation, Appl. Appl. Math., Vol. 12, No. 1, pp. 1-22.

Rao, S.S. (1965). Queueing models with balking, reneging and interruptions, Operations Research, Vol. 13, pp. 596.

Servi, L. and Finn, S. (2002). M/M/1 queue with working vacations (M/M/1/WV), Perform. Eval., Vol. 50, pp. 41-52.

Tao, L., Liu, Z. and Wang, Z. (2012). M/M/1 retrial queue with collisions and working vacation interruption under N-policy, RAIRO Operations Research, Vol. 46, pp. 355-371.

Tian, N. (2008). The M/M/1 queue with single working vacation, International Journal of Information and Management Sciences, Vol. 19, No. 14, pp. 621-634.

Tian, N. and Zhang, Z. (2006). *Vacation Queueing Models - Theory and Applications*, Springer-Verlag, New York.

Wallace, P. (1969). The solution of birth and death processes arising from multiple access computer systems, Ph.D. dissertation, Systems Engineering Laboratory, University of Michigan. Tech. Rept., 07742- 6 T.

Wu, D. and Takagi, H. (2006). M/G/1 queue with multiple working vacations, Perform. Eval., Vol. 63, pp. 654-681.

Yadin, M. and Naor, P. (1963). Queueing system with a removable service station, Operational Research Quarterly, Vol. 14, No. 4, pp. 393-405.

Ye, Q. and Liu, L. (2015). The analysis of the M/M/1 queue with two vacation policies, International Journal of Computer Mathematics, Vol. 1, No. 20.

Zhang, M. and Hou, Z. (2010). Performance analysis of M/G/1 queue with working vacations and vacation interruption, J. Comput. Appl. Math, Vol. 234, pp. 2977-2985.

Zhang, Z. and Xu, X. (2008). Analysis for the M/M/1 queue with multiple working vacations and N-policy, Infor. Manag. Sci., Vol. 19, pp. 495-506.