Applications and Applied Mathematics: An International Journal (AAM)

Volume 19 | Issue 1 | Article 10

12-2022

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Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466

Applications and Applied Mathematics:
An International Journal
(AAM)

Vol. 19, Issue 1 (June 2024), 21 pages

Analysis of the Auto-Oscillation Of a Perturbed SIR Epidemiological Model

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Received: August 5, 2022; Accepted: April 12, 2023

Abstract

In this paper, we study a class of compatimental epidemiological models consisting of Susceptible, Infected, and Removed (SIR) individuals with a perturbation factor or exterior effects such as noise, climate change, pollution, etc. We prove the existence and uniqueness of a limit cycle confined in a nonempty closed and convex set by relying on a recent result of Lobanova and Sadovskii. Moreover, we study the existence of Hopf and Bogdanov-Takens bifurcations by applying respectively Poincare-Andronov-Hopf bifurcation theorem and Bogdanov-Takens theorem. Eventually, using Scilab, we illustrate the validity of our results with numerical simulations and also interpret them.

Keywords: SIR epidemiology model; Perturbation factor; Closed convex set; Metric projection; Hopf bifurcation; Bogdanov-Takens bifurcation; Limit cycle

MSC 2010 No.: 37G15, 92D30

1. Introduction

The compartmental epidemiological models **SIR** originating from the work by Kermack and McKendrick in Kermack and Mckendrick (1927), Kermack and Mckendrick (1991), and

Martcheva (2015) on the plague epidemic in India, are the simplest epidemiological models often used to describe a disease spread in a population over time. In this model, the whole population under consideration is divided into the three following disjoint categories (compartments) of individuals:

- **S**: consisting of healthy individuals who may be infected (susceptible).
- **I**: consisting of infected and infectious individuals with a probability β (representing the rate) of transmission of the disease from an infected person to a healthy person.
- **R**: consisting of individuals removed or recovered, not likely to be infected because they are cured and immune or have died, with the probability ν representing the cure rate, i.e., the inverse of the average duration of symptoms (Villani (2020); Sallet (2010)).

Throuhgout this paper, we assume that individuals **R** can lose their immunity with probability γ without dying and that the total population is constant during the epidemic. We also assume that susceptible and infected individuals are confined to a closed convex environment (no immigration of the two groups outside the environment, e.g., the confinement of populations during the COronaVIrus Disease 2019 (COVID-19) pandemic). Furthermore, we have taken into account external effects on this environment, e.g., the effects of climate change on the environment or environmental pollution.

This paper is organzed as follows: Section 1 is devoted to the introduction. In Section 2, we present our mathematical model. The existence and number of equilibrium points and the local dynamics of this system are studied in Section 3. In Section 4, we prove the existence of Hopf bifurcation and Bogdanov-Takens bifurcation. The existence and uniqueness of a limit cycle of which orbit is stable are studied in Section 5 by using Lobanova-Sadovskii theorem (Lobanova and Sadovskii (2007)) in the line of Appell, Merentez and Sanche (Appell et al. (2017)). Moreover, we illustrate the validity of the results with numerical simulation using Scilab. Finally, we conclude in Section 6.

2. Mathematical model

The total population $N(t) = S(t) + I(t) + R(t)$ of the epidemiological model SIR described in the introduction is assumed constant $N(t) = N_0$ throughout the duration of the epidemic. Thus, this three variable model can be reduced to a two variable model $S(t) = x(t)$ and $I(t) = y(t)$. Moreover, taking into account the possible external perturbations, we obtain the following model,

$$\begin{cases} \dot{x} = \gamma\,(N_0 - y) - (\beta y + \gamma)\,x + \alpha(x - x_*), \\ \\ \dot{y} = y\,(\beta x - \nu) + \alpha(y - y_*), \end{cases} \tag{1}$$

where

$$(x_*,\ y_*) := \left(\frac{\nu}{\beta}\ ,\ \ \frac{\gamma(\beta N_0 - \nu)}{\beta(\gamma + \nu)}\right), \tag{2}$$

is the positive equilibrium point of (1) for $\alpha = 0$, and

$$u_\alpha(x\ ,\ y) = \alpha(x - x_*\ ,\ y - y_*),\ \ \alpha \in (0\ ,+\infty), \tag{3}$$

is the perturbation term.

Note that, since $N_0 = x + y + R$, we have:

$$(x,\ y) \in K := \Big\{(x\ ,\ y) \in \big[0\ ,\ N_0\big]^2,\ \ x + y \le N_0\Big\}.$$

Let $Z = (z_1,\ z_2) \in \mathbb{R}^2$ such that $Z + U^* \in K$. So, $Z \in K - U^*$.

In the sequel, we consider the following set

$$Q := \Big\{(z_1\ ,\ z_2) \in \big[-x_*\ ,\ N_0 - x_*\big] \times \big[-y_*\ ,\ N_0 - y_*\big];\ -(y_* + x_*) \le z_1 + z_2 \le N_0 - (y_* + x^*)\Big\}.$$

Let $Z = (z_1\ ,\ z_2) \in Q$. Then, there exists $(x,\ y) \in K$ such that $z_1 = x - x_*$ and $z_2 = y - y_*$.

By substituting $x = z_1 + x_*$ and $y = z_2 + y_*$ in the system (1), we obtain on Q the following system,

$$\begin{cases} \dot{z}_1 = f_1(Z), \\ \dot{z}_2 = f_2(Z), \end{cases} \tag{4}$$

where the map $f = (f_1\ ,\ f_2)$ is defined from Q to $\mathbb{R}^2$ by

$$f_1(Z) = (\alpha - \gamma - \beta y_*)z_1 - (\gamma + \nu)z_2 - \beta z_1 z_2 \ \text{ and } \ f_2(Z) = \beta y_* z_1 + \alpha z_2 + \beta z_1 z_2.$$

In the sequel, we shall consider the system (4).

3. Local behavior of the system (4)

In this section, we study the dynamic behavior of the system (4) in a neighborhood of their equilibrium points.

3.1. Equilibrium points of (4)

In this subsection, we study the existence of equilibrium points for the system (4).

Let $\beta,\ \nu,\ \gamma,\ N_0$ and α be some positive real numbers and let us set:

$$\Delta_0 = \frac{(\gamma\beta N_0 - (\gamma+\nu)^2)^2 + \gamma\nu(\gamma^2 + 3\gamma\nu + 2\nu^2)}{(\gamma+\nu)^2}, \quad \Delta_1 = (\beta N_0 - \gamma - 2\nu)^2 - 4\nu(\gamma + \nu),$$

$$\overline{\alpha}_0 = (\gamma + \nu)(\gamma + \nu - \beta N_0)(\gamma + 2\nu - \beta N_0), \quad \overline{\alpha}_1 = \frac{(\gamma + 2\nu - \beta N_0)\nu}{\gamma + \nu}, \quad \overline{\alpha}_2 = \frac{\gamma + \nu + \beta y_* - \sqrt{\Delta_0}}{\gamma + \nu},$$

$$\overline{\alpha}_3 = \frac{\gamma + \nu + \beta y_* + \sqrt{\Delta_0}}{\gamma + \nu},\ \ \overline{\alpha}_4 = \frac{\gamma + \beta N_0 - \sqrt{\Delta_1}}{\gamma + \nu},\ \ \overline{\alpha}_5 = \frac{\gamma + \beta N_0 + \sqrt{\Delta_1}}{\gamma + \nu},\ \ A(\alpha) = (\alpha - \overline{\alpha}_2)(\alpha - \overline{\alpha}_3),$$

$$\overline{\alpha} = \gamma - \frac{\nu(\beta N_0 - \nu)}{\gamma + \nu},\ \ B(\alpha) = \alpha\big(\alpha - \overline{\alpha}\big),\ C(\alpha) = (\alpha - \overline{\alpha}_4)(\alpha - \overline{\alpha}_5),\ \ E(\alpha) = \frac{\alpha}{\overline{\alpha}_1}\big(\alpha - \overline{\alpha}_0\big),$$

$$\mathcal{R}_1 = \left\{(\beta\ ,\ \nu\ ,\ \gamma\ ,\ \alpha\ ,\ N_0) \in \mathbb{R}^5_+, \quad \alpha < \overline{\alpha}, \quad A(\alpha) > 0, \quad C(\alpha) > 0 \ \text{ and } \ E(\alpha) < 0\right\},$$

and

$$\mathcal{R}_2 = \left\{(\beta\ ,\ \nu\ ,\ \gamma\ ,\ \alpha\ ,\ N_0) \in \mathbb{R}^5_+, \quad \gamma + \nu < \alpha, \quad A(\alpha) < 0, \quad C(\alpha) < 0 \ \text{ and } \ E(\alpha) < 0\right\}.$$

Moreover, let $\mathcal{S} = \left\{Z = (z_1\ ,\ z_2) \in Q; \ \ f(Z) = O\right\}$ be the set of equilibrium points of (4).

Lemma 3.1.

(1) If $(\beta\ ,\ \nu\ ,\ \gamma\ ,\ \alpha\ ,\ N_0) \in \mathcal{A}_1 = \mathcal{R}_1 \cup \mathcal{R}_2$, then

$$\mathcal{S} = \left\{(0\ ,\ 0)\ ,\ \left(\frac{\alpha(\alpha-\gamma-\beta y_*)+\gamma(\beta N_0-\nu)}{\beta(\gamma-\alpha)}\ ,\ \frac{\alpha(\gamma-\alpha)}{\beta(\gamma+\nu-\alpha)} - y_*\right)\right\},$$

with $\alpha \neq \gamma$ and $\alpha \neq \gamma + \nu$.

(2) Otherwise, $\mathcal{S} = \left\{\left(0\ ,\ 0\right)\right\}$.

Proof:

Let $Z = (z_1,\ z_2) \in Q$.

$$f(Z) = O \iff \begin{cases} (\alpha + \beta z_1) z_2 = -\beta y_* z_1, \\ \big((\alpha-\gamma)z_1 + \alpha(\alpha-\gamma-\beta y_*) + \gamma(\beta N_0 - \nu)\big) z_1 = 0, \end{cases}$$

$$\iff z_1 = z_2 = 0 \ \text{ or } \ \begin{cases} (\alpha-\gamma)z_1 + \alpha(\alpha-\gamma-\beta y_*) + \gamma(\beta N_0 - \nu) = 0, \\ \beta(\gamma+\nu-\alpha)(z_2 + y_*) = \alpha(\gamma-\alpha), \end{cases}$$

$$\iff z_1 = z_2 = 0 \ \text{ or } \ \begin{cases} \alpha \neq \gamma \ \text{ and } \ \alpha \neq \gamma + \nu, \\ z_1 = \frac{\alpha(\alpha-\gamma-\beta y_*)+\gamma(\beta N_0-\nu)}{\beta(\gamma-\alpha)}, \\ z_2 = \frac{\alpha(\gamma-\alpha)}{\beta(\gamma+\nu-\alpha)} - y_*. \end{cases}$$

The equilibrium point $Z_0 = (0\ ,\ 0)$ is always inside Q.

The equilibrium point Z_1 is inside Q if and only if its components z_1 and z_2 satisfy the following condition:

$$0 \leqslant z_1 + x_* \leqslant N_0, \ \ 0 \leqslant z_2 + y_* \leqslant N_0, \quad \text{and} \quad 0 \leqslant z_2 + y_* + z_1 + x_* \leqslant N_0. \tag{5}$$

We have

$$z_1 + x_* = \frac{A(\alpha)}{\beta(\gamma-\alpha)}, \quad z_1 + x_* - N_0 = \frac{B(\alpha)}{\beta(\gamma-\alpha)}, \ \ z_2 + y_* - N_0 = \frac{C(\alpha)}{\beta(\alpha-\gamma-\nu)},$$

and

$$z_1 + z_2 + x_* + y_* - N_0 = \frac{E(\alpha)}{\beta(\gamma-\alpha)(\gamma+\nu-\alpha)}.$$

Thus, $Z_1 = (z_1 \ , \ z_2) \in Q$ if and only if

$$\frac{A(\alpha)}{\beta(\gamma-\alpha)} \geq 0, \quad \frac{B(\alpha)}{\beta(\gamma-\alpha)} \leq 0, \quad \frac{C(\alpha)}{\beta(\alpha-\gamma-\nu)} \leq 0 \quad \text{and} \quad \frac{E(\alpha)}{\beta(\gamma-\alpha)(\gamma+\nu-\alpha)} \leq 0. \tag{6}$$

That is, $(\beta \ , \ \nu \ , \ \gamma \ , \ \alpha \ , \ N_0) \in \mathcal{A}_1 = \mathcal{R}_1 \cup \mathcal{R}_2$.

Hence, $Z = (z_1, \ z_2) \notin Q$ if and only if $(\beta, \ \nu, \ \gamma, \ \alpha, \ N_0) \in \mathcal{A}_2 = (\mathcal{R}_1 \cup \mathcal{R}_2)^c$, where

$\mathcal{A}_2 = (\mathcal{R}_1 \cup \mathcal{R}_2)^c$ is the complementary of $\mathcal{A}_1 = \mathcal{R}_1 \cup \mathcal{R}_2$ in $\mathbb{R}^5_+$. The proof is completed. ■

3.2. Local dynamic behavior of (4)

In this subsection, we are interested in the behavior of system (4) in a neighborhood of its equilibrium points. Note that the local dynamic behavior of a dynamical system in a neighborhood of each equilibrium point depends on the signs of the trace, determinant and of the discriminant of the characteristic equation of its Jacobian matrix at this equilibrium point.

3.2.1. *Local dynamic behavior of (4) in a neighborhood of* $Z_0 = (0, 0)$

Theorem 3.1.

(1) If $\beta_1 \leqslant \beta \leqslant \beta_2$, then
   - (a) $Z_0 = (0 \ , \ 0)$ is a stable focus if and only if $\alpha < \frac{\gamma+\beta y_*}{2}$.
   - (b) $Z_0 = (0 \ , \ 0)$ is an unstable focus if and only if $\alpha > \frac{\gamma+\beta y_*}{2}$.

(2) If $\beta \in (0 \ ; \ \beta_1) \cup (\beta_2 \ ; \ +\infty)$, then
   - (a) $Z_0 = (0 \ , \ 0)$ is a stable focus if and only if $\alpha < \alpha_1$.
   - (b) $Z_0 = (0 \ , \ 0)$ is an unstable focus if and only if $\alpha > \alpha_2$.
   - (c) $Z_0 = (0 \ , \ 0)$ is a saddle point if and only if $\alpha_1 < \alpha < \alpha_2$.

Remark 3.1.

If $\alpha = \frac{\gamma+\beta y_*}{2}$ and $\beta_1 < \beta < \beta_2$, then $Z_0 = (0 \ , \ 0)$ can be a focus or center.

If $\beta \in \left\{\beta_1 \ ; \ \beta_2\right\}$ and $\alpha = \frac{\gamma+\beta y_*}{2}$, then the system (4) can exibit a Bogdanov-takens bifurcation at the neigborhood of $Z_0 = (0 \ , \ 0)$, where

$$\alpha_1 = \tfrac{\gamma+\beta y_* - \sqrt{u(\beta)}}{2}, \ \ \alpha_2 = \tfrac{\gamma+\beta y_* + \sqrt{u(\beta)}}{2}, \ u(\beta) = \gamma^2(\beta-\beta_1)(\beta-\beta_2), \ \ \beta_1 = \tfrac{X_1(\gamma+\nu)+\nu}{N_0},$$

$$\beta_2 = \tfrac{X_2(\gamma+\nu)+\nu}{N_0}, \ \ X_1 = \tfrac{\gamma+2\nu-\sqrt{\nu(\nu+2\gamma)}}{\gamma} \ \ \text{and} \ \ X_2 = \tfrac{\gamma+2\nu+\sqrt{\nu(\nu+2\gamma)}}{\gamma}.$$

Proof:

For all $(\beta\ ,\ \nu\ ,\ \gamma\ ,\ \alpha\ ,\ N_0) \in (0, +\infty)^5,\ Z_0 = (0\ ,\ 0)$ is a equilibrium point of the system (4).

The Jacobian matrix of the system (4) at $Z_0 = (0\ ,\ 0)$ is

$$M_0(\alpha) = \begin{pmatrix} \alpha - \gamma - \beta y_* & -\gamma - \nu \\ \\ \beta y_* & \alpha \end{pmatrix},$$

of which trace and determinant are, respectively,

$$T_0(\alpha) = 2\alpha - \gamma - \beta y_* \quad \text{and} \quad D_0(\alpha) = \big(\alpha - \frac{\gamma + \beta y_*}{2}\big)^2 - \frac{u(\beta)}{4}.$$

Let $\Delta_0(\alpha) = T_0(\alpha)^2 - 4D_0(\alpha) = u(\beta)$ be the discriminant of the characteristic equation of $M_0(\alpha)$.

• If $\beta_1 \leqslant \beta \leqslant \beta_2$ and $\alpha < \frac{\gamma+\beta y^*}{2}$, we have $T_0(\alpha) < 0,\ D_0(\alpha) > 0$ and $\Delta_0(\alpha) < 0$. Then, Z_0 is a stable focus.

• If $\beta_1 \leqslant \beta \leqslant \beta_2$ and $\alpha > \frac{\gamma+\beta y^*}{2}$, we have $T_0(\alpha) > 0,\ D_0(\alpha) > 0$ and $\Delta_0(\alpha) < 0$. Then, Z_0 is an unstable focus.

• If $\beta \in (0,\ \beta_1) \cup (\beta_2,\ +\infty)$ and $\alpha < \alpha_1$, we have $T_0(\alpha) < 0,\ D_0(\alpha) > 0$, and $\Delta_0(\alpha) < 0$. Then, Z_0 is a stable focus.

• If $\beta \in (0,\ \beta_1) \cup (\beta_2,\ +\infty)$ and $\alpha > \alpha_2$, we have $T_0(\alpha) > 0,\ D_0(\alpha) > 0$, and $\Delta_0(\alpha) < 0$. Then, Z_0 is an unstable focus.

• If $\beta \in (0,\ \beta_1) \cup (\beta_2,\ +\infty)$ and $\alpha_1 < \alpha < \alpha_2$, we have $D_0(\alpha) < 0$. Then, Z_0 is a saddle point. Hence, the proof of Theorem 3.1 is completed. ■

3.2.2. *Local dynamic Behavior of (4) at a neigborhood of Z_0 and Z_1*

Let $(\beta\ ,\ \nu\ ,\ \gamma\ ,\ \alpha\ ,\ N_0) \in \mathcal{A}_1$. Assume that $(\beta\ ,\ \nu\ ,\ \gamma\ ,\ \alpha\ ,\ N_0)$ fulfills one of the following conditions:

$$\gamma + \nu < \alpha \quad \text{and} \quad \beta = \beta^*, \tag{7}$$

$$\beta < \beta^* \quad \text{and} \quad \gamma + \nu < \alpha < \gamma + \frac{\nu}{1 - A_0}, \tag{8}$$

$$\beta > \beta^*,\ \gamma + \nu < \alpha,\ \ \nu \in \left(0,\ \frac{N_0}{5}\right) \cup (N_0,\ +\infty) \quad \text{and} \quad \left|\gamma + \frac{3\nu - N_0}{2}\right| \leq \gamma_*, \tag{9}$$

$$\alpha < \nu,\ \ \beta = \beta^*,\ \nu \in \left(0\ ,\frac{N_0}{5}\right) \cup (N_0,\ +\infty) \quad \text{and} \quad \left|\gamma + \frac{3\nu - N_0}{2}\right| \leq \gamma_*, \tag{10}$$

$$\alpha < \gamma \quad \text{and} \quad \beta < \beta^*, \tag{11}$$

$$\beta > \beta^*,\ \gamma + \frac{\nu}{1-A_0} < \alpha < \gamma,\ \nu \in \left(0\ ,\frac{N_0}{5}\right) \cup (N_0,\ +\infty) \quad \text{and} \quad \left|\gamma + \frac{3\nu - N_0}{2}\right| \leq \gamma_*, \quad (12)$$

$$\beta < \beta^* \quad \text{and} \quad \gamma + \frac{\nu}{1-A_0} < \alpha, \quad (13)$$

$$\beta_1 \leq \beta \leq \beta_2 \quad \text{and} \quad \alpha \neq \frac{\gamma + \beta y_*}{2}, \quad (14)$$

$$\beta \in (0,\ \beta_1) \cup (\beta_2,\ +\infty) \quad \text{and} \quad \alpha \in (0,\ \alpha_1) \cup (\alpha_2,\ +\infty), \quad (15)$$

$$\beta \in (0,\ \beta_1) \cup (\beta_2,\ +\infty) \quad \text{and} \quad \alpha_1 < \alpha < \alpha_2, \quad (16)$$

where $\alpha_1,\ \alpha_2,\ \beta_1$ and β_2 are defined in Subsection 3.2.1 and $\beta^* = \frac{\gamma\nu + (\gamma+\nu)^2}{\gamma N_0},\ A_1(\alpha) = \frac{\alpha-\gamma-\nu}{\alpha-\gamma}$,

$A_0 = \sqrt{\frac{\gamma+\nu}{\beta y_*}},\ \gamma_* = \frac{\sqrt{(\nu-N_0)(5\nu-N_0)}}{2},\ D_0(\alpha) = \alpha^2 - (\gamma+\beta N_0)\alpha + \gamma(\beta N_0 - \nu)$,

$V(\alpha) = \frac{\beta^2 y_*^2}{A(\alpha)^2}\Big(A_1(\alpha)^2 - A_0^2\Big)^2 + 4D_0(\alpha)$.

Theorem 3.2.

$i)$ If the parameters $\beta,\ \nu,\ \gamma,\ \alpha$ and N_0 satisfy one of the conditions (7) to (9) and the condition (16) with $V(\alpha) < 0$, then the nontrivial equilibrium point of the system (4) is a stable focus and the origin is a saddle point.

$ii)$ If the parameters $\beta,\ \nu,\ \gamma,\ \alpha$ and N_0 satisfy one of the conditions (7) to (9) and the condition (16) with $V(\alpha) > 0$, then the nontrivial equilibrium point of the system (4) is a stable node and the origin is a saddle point.

$iii)$ If the parameters $\beta,\ \nu,\ \gamma,\ \alpha$ and N_0 satisfy one of the conditions (10) to (13) and the condition (16) with $V(\alpha) < 0$, then the nontrivial equilibrium point of the system (4) is a unstable focus and the origin is a saddle point.

$iv)$ If the parameters $\beta,\ \nu,\ \gamma,\ \alpha$ and N_0 satisfy one of the conditions (10) to (13) and the condition (16) with $V(\alpha) > 0$, then the nontrivial equilibrium point of the system (4) is a unstable node and the origin is a saddle point .

$v)$ If the parameters $\beta,\ \nu,\ \gamma,\ \alpha$ and N_0 satisfy one of the conditions (14) to (15), then the nontrivial equilibrium point of the system (4) is a saddle point and the origin is a focus or a node or a center (see the Theorem 3.1).

Proof:

The Jacobian matrix of the system (4) at $Z_0 = (0\ ,\ 0)$ is

$$M_0(\alpha) = \begin{pmatrix} \alpha - \gamma - \beta y_* & -\gamma - \nu \\ \beta y_* & \alpha \end{pmatrix}.$$

The trace and determinant of $M_0(\alpha)$ are, respectively,

$$T_0(\alpha) = 2\alpha - \gamma - \beta y_* \quad \text{and} \quad D_0(\alpha) = \big(\alpha - \alpha_b\big)^2 - \frac{u(\beta)}{4}.$$

The Jacobian matrix of system (4) at Z_1 is

$$M(\alpha) = \begin{pmatrix} -\frac{\gamma+\nu}{A_1(\alpha)} & (\alpha - \gamma - \beta y_*)A_1(\alpha) \\ \\ \frac{\alpha}{A_1(\alpha)} & \beta y_* A_1(\alpha) \end{pmatrix}, \text{ for all } \alpha \in (0, \gamma) \cup (\gamma + \nu, +\infty).$$

The trace and determinant of $M(\alpha)$ are, respectively,

$$T(\alpha) = \frac{\beta y_*}{A_1(\alpha)}\Big(A_1(\alpha)^2 - A_0^2\Big) \text{ and } D(\alpha) = -D_0(\alpha).$$

Let $\Delta(\alpha) = T(\alpha)^2 - 4D(\alpha)$ be the discriminant of characteristic equation of $M(\alpha)$.

We have $\Delta(\alpha) = V(\alpha)$. By studing the signs of $T(\alpha)$ and of $D(\alpha)$, we obtain

- $T(\alpha) < 0$ if the parameters fulfill one of the conditions (7) to (9).
- $T(\alpha) > 0$ is positive if the parameters fulfill one of conditions (10) to (13).
- $D(\alpha) < 0$ and $D_0(\alpha) > 0$, if the parameters fulfill one of conditions (14) and (15).
- $D(\alpha) > 0$ and $D_0(\alpha) < 0$, if the parameters satisfy the condition (16). Hence, the proof of theorem 3.1 is completed. ∎

4. Bifurcation analysis

4.1. Hopf bifurcation

Let us recall that there is no regular method to study the limit cycles of the systems in the plane. Perhaps, one of the most important approaches, together with the Poincaré-Bendixson theory, is the Poincare-Andronov-Hopf bifurcation (Françoise (2005); Rudiger Seydel (2010); Albert (2019); Kielhöfer (2004)), which is the only genuinely two dimensional bifurcation (i.e., it cannot be observed in systems of dimension 1), which can occur in generic two dimensional autonomous systems depending on one parameter (co-dimension 1 bifurcation). In this section, we give the conditions for the existence of Hopf bifurcation in a neighborhood of the equilibrium points of the system (4).

4.1.1. *Hopf bifurcation of system (4) at the point of* $Z_0 = (0\ ,\ 0)$

In this subsection, we study the Hopf bifurcation of the system (4) at $Z_0 = (0, 0)$; where α is the bifurcation parameter.

Theorem 4.1.

If $\beta_1 < \beta < \beta_2$, then the system (4) admits a Hopf bifurcation at $\big(Z_0 = (0\ ,\ 0)\ ,\ \alpha_b\big)$, where $\alpha_b = \frac{\gamma+\beta y_*}{2}$, $\beta_1 = \frac{X_1(\gamma+\nu)+\nu}{N_0}$, $\beta_2 = \frac{X_2(\gamma+\nu)+\nu}{N_0}$, $X_1 = \frac{\gamma+2\nu-\sqrt{\nu(\nu+2\gamma)}}{\gamma}$ and $X_2 = \frac{\gamma+2\nu+\sqrt{\nu(\nu+2\gamma)}}{\gamma}$.

Proof:

Let $M_0(\alpha)$ be the Jacobean matrix of (4) at Z_0. Then,

$$M_0(\alpha) = \begin{pmatrix} \alpha-\gamma-\beta y_* & -\gamma-\nu \\ \\ \beta y_* & \alpha \end{pmatrix}.$$

The trace and determinant of $M_0(\alpha)$ are, respectively,

$$T_0(\alpha) = 2\alpha-\gamma-\beta y_* \quad \text{and} \quad D_0(\alpha) = \big(\alpha-\alpha_b\big)^2 - \frac{u(\beta)}{4}.$$

Let $\Delta(\alpha) = T_0(\alpha)^2 - 4D_0(\alpha)$ be the discriminant of the characteristic equation of $M_0(\alpha)$.

We have

$$\Delta(\alpha) = u(\beta) = \gamma^2(\beta-\beta_1)(\beta-\beta_2).$$

If $\beta_1 < \beta < \beta_2$, then $u(\beta) < 0$. So, $M_0(\alpha)$ admits two conjugate complex eigenvalues

$$W(\alpha) = h(\alpha) + i\omega(\alpha) \text{ and } \overline{W}(\alpha) = h(\alpha) - i\omega(\alpha),$$

where $h(\alpha) = \frac{T(\alpha)}{2}$ and $\omega(\alpha) = \frac{\sqrt{-u(\beta)}}{2}$.

Likewise, if $\alpha = \alpha_b = \frac{\gamma+\beta y^*}{2}$ and $\beta_1 < \beta < \beta_2$, then

$$h\big(\alpha_b\big) = 0, \quad \frac{dh(\alpha)}{d\alpha}\,|_{\alpha=\alpha_b} = 1 > 0 \text{ and } \omega\big(\alpha_b\big) > 0.$$

Moreover, the only eigenvalues of $M\big(\alpha_b\big)$ are

$$W\big(\alpha_b\big) = i\omega\big(\alpha_b\big) \quad \text{and} \quad \overline{W}\big(\alpha_b\big) = -i\omega\big(\alpha_b\big).$$

Hence, according to the Poincaré-Andronov-Hopf theorem (Françoise (2005); Rudiger Seydel (2010); Albert (2019); Kielhöfer (2004)), the conclusion of Theorem 4.1 follows. ∎

4.1.2. *Numerical simulation*

To illustrate the Theorem 4.1, we take $\gamma = \frac{4}{5}$, $\nu = \frac{9}{10}$, $\beta = \frac{7}{10}$, and $N_0 = 10$. Then,

$$x^* = \frac{9}{7},\ y^* = \frac{488}{119} \text{ and } \mathcal{A}_2 = (0,\ +\infty)^4 \times (1.7,\ +\infty)\ .$$

Therefore, we have,

$$\alpha_b = \frac{1092}{595} \quad \text{and } \beta_1 \approx 0.12 < \beta = 0.7 < \beta_2 \approx 1.1\ .$$

Then, the system (4) admits a Hopf bifurcation at $\big(Z_0\ ,\ \alpha_b\big)$ inside of

$$Q := \left\{ (z_1, z_2) \in \left[-\frac{9}{7}, \frac{61}{7} \right] \times \left[-\frac{488}{119}, \frac{702}{119} \right], -\frac{641}{119} \le z_1 + z_2 \le \frac{549}{119} \right\}.$$

For the simulations, we denote the initial conditions by $(z_1^0,\ z_2^0)$ (see Figures 1, 2, 3, 4, 5, 6, 7, 10 and 11 in the appendix).

Remark 4.1.

We remark from our simulation that there are three critical values $\alpha_{c_1} < \alpha_{c_2} < \alpha_{c_3}$ in $(\alpha_b,\ +\infty)$ such that:

• for all $\alpha \in \big[\alpha_b,\ \alpha_{c_1}\big)$, there exists a stable limit cycle (see Figures 2, 3, 4 and 5). This means that the disease appears periodically in the population. We can therefore predict and control the infection.

• for all $\alpha \in \big[\alpha_{c_1},\ \alpha_{c_2}\big] \cup \big[\alpha_{c_3},\ +\infty)$, we observe the disappearance of the limit cycle with the explosion of the rate of infections (see Figures 6, 7, 10 and 11). This means that the infection becomes uncontrollable (pandemic) after a certain period.

• for all $\alpha \in \big(\alpha_{c_2},\ \alpha_{c_3}\big)$, the limit cycle disappears with a considerable decrease in the rate of infection and an explosion in the rate of cure (see Figures 8 and 9). This means that the infection becomes controllable and can be eradicated after a certain period.

4.1.3. *Hopf bifurcation at Z_1*

In this subsection, we study the Hopf bifurcation of the system (4) at Z_1. So, we take $(\beta,\ \nu,\ \gamma,\ \alpha,\ N_0) \in \mathcal{A}_1$.

Theorem 4.2.

If

$$\frac{\nu}{N_0} < \beta < \frac{\gamma+\nu}{N_0}, \quad \alpha_1 < \alpha_0 < \alpha_2, \quad \alpha \in (0,\gamma) \cup (\gamma+\nu) \ \text{ and } \ V(\alpha) < 0, \tag{17}$$

then the system (4) admits a Hopf bifurcation in a neigborhood of $(Z_1,\ \alpha_0)$, where

$$\alpha_0 = \gamma + \frac{\nu}{1 - A_0}.$$

Proof:

The Jacobian matrix of system (4) at Z_1 is

$$M(\alpha) = \begin{pmatrix} -\frac{\gamma+\nu}{A_1(\alpha)} & (\alpha - \gamma - \beta y_*)A_1(\alpha) \\ \\ \frac{\alpha}{A_1(\alpha)} & \beta y_* A_1(\alpha) \end{pmatrix}, \quad \text{for all } \ \alpha \in (0,\gamma) \cup (\gamma+\nu, +\infty).$$

The trace and determinant are, respectively,

$$T(\alpha) = \frac{\beta y_*}{A_1(\alpha)}\Big(A_1(\alpha)^2 - A_0^2\Big) \quad \text{and} \quad D(\alpha) = -D_0(\alpha).$$

Let $\Delta(\alpha) = T(\alpha)^2 - 4D(\alpha)$ be the discriminant of characteristic equation of $M(\alpha)$. We have $\Delta(\alpha) = V(\alpha)$. According to (17), we have $V(\alpha) < 0$. Then, the matrix $M(\alpha)$ admits two conjugate complex eigenvalues

$$W(\alpha) = h(\alpha) + i\omega(\alpha) \text{ and } \overline{W}(\alpha) = h(\alpha) - i\omega(\alpha),$$

where

$$h(\alpha) = \frac{T(\alpha)}{2} \text{ and } \omega(\alpha) = \frac{\sqrt{-\Delta(\alpha)}}{2}.$$

Moreover,

$$h\big(\alpha_0\big) = 0 \quad \text{and} \quad \frac{dh(\alpha)}{d\alpha} \,|_{\alpha=\alpha_0} = \frac{\beta y_*}{2}\Big(1 + \frac{A_0^2}{A_1(\alpha_0)^2}\Big)A_1'(\alpha_0) > 0.$$

Since $\alpha_1 < \alpha_0 < \alpha_2$, we have $\omega\big(\alpha_0\big) > 0$. Likewise, if $\alpha = \alpha_0$, the only eigenvalues of $M\big(\alpha_0\big)$ are

$$W(\alpha_0\big) = i\omega(\alpha_0\big) \text{ and } \overline{W}(\alpha_0\big) = -i\omega(\alpha_0\big).$$

Hence, according to the Poincaré-Andronov-Hopf theorem (Françoise (2005); Rudiger Seydel (2010); Albert (2019); Kielhöfer (2004)), the conclusion of Theorem 4.2 follows. ■

4.2. Bogdanov-Takens bifurcation

A Bogdanov-Takens bifurcation (Jean-Baptiste and Claude (1993)) is an example of a codimension-2 bifurcation, which is far more complex than codimension-1 bifurcations, and it describes very rich dynamics of the given system. The basic idea of what happens with a Bogdanov-Takens bifurcation is that we have two codimension-1 bifurcation curves that collide at a single point. The point where the two bifurcation curves collide is where the Bogdanov-Takens bifurcation happens. For later use, we set,

$$BT := \Big\{(\beta,\ \nu,\ \gamma,\ \alpha,\ N_0) \in (0\ ,\ +\infty)^5,\ \ \beta = \beta_0,\ \ \alpha = \alpha_0\Big\},$$

a cusp bifurcation surface of codimension 2 for system (4) (i.e., Bogdanov-Takens bifurcation surface), where

$$\beta_0 \in \big\{\beta_1\ ;\ \beta_2\big\},\ \alpha_0 = \frac{\gamma + \beta_0 y^*}{2},\ \beta_1 = \frac{X_1(\gamma+\nu)+\nu}{N_0},\ \ \beta_2 = \frac{X_2(\gamma+\nu)+\nu}{N_0},$$
$$X_1 = \frac{\gamma + 2\nu - \sqrt{\nu(\nu+2\gamma)}}{\gamma} \quad \text{and} \quad X_2 = \frac{\gamma + 2\nu + \sqrt{\nu(\nu+2\gamma)}}{\gamma}.$$

Theorem 4.3.

If $(\beta, \nu, \gamma, \alpha, N_0) \in BT,$ then the equilibrium $Z_0 = (0, 0)$ of system (4) is a cusp of codimension two, i.e., it is a Bogdanov-Takens singularity.

Proof:

Let $(\beta, \nu, \gamma, \alpha, N_0) \in BT$. Then, the system (4) becomes

$$\begin{cases} \dot{z}_1 = -\alpha_0 z_1 - (\gamma+\nu) z_2 - \beta_0 z_1 z_2, \\ \dot{z}_2 = \beta_0 y^* z_1 + \alpha_0 z_2 + \beta_0 z_1 z_2. \end{cases} \tag{18}$$

In order to find the canonical normal form of the cusp, we take the smooth invertible transformations

$$u = z_1 \quad \text{and} \quad v = -\alpha_0 z_1 - (\gamma+\nu) z_2.$$

We can rewrite system (18) as follows,

$$\begin{cases} \dot{u} &= v - \frac{\alpha_0\beta_0}{\gamma+\nu} u^2 + o(\|(u,\ v)\|^2), \\ \dot{v} &= (\alpha_0^2 - \beta_0(\gamma+\nu)y^*)u + \alpha_0\beta_0 u^2 + o(\|(u,\ v)\|^2). \end{cases} \tag{19}$$

Making the change of variables by $X = u$ and $Y = v - \frac{\alpha_0\beta_0}{\gamma+\nu} u^2 + o(\|(u,\ v)\|^2)$, then the system (19) become

$$\begin{cases} \dot{X} &= Y, \\ \dot{Y} &= \mu_1 X + \mu_2 X^2 - 2\mu_0 XY + o(\|(X\ ,\ Y)\|^2), \end{cases} \tag{20}$$

where $\mu_0 = \frac{\alpha_0\beta_0}{\gamma+\nu}$, $\mu_1 = \alpha_0^2 - \beta_0(\gamma+\nu)y^*$ and $\mu_2 = \alpha_0\beta_0$.

Making the final change of variables by $u = \frac{4\mu_0^2}{\mu_2} X$, $v = \frac{8\mu_0^3}{\mu_2^2} Y$, and $\tau = \frac{\mu_2}{2\mu_0} t$ (we still denote $u,\ v,\ \tau$ by $X,\ Y,\ t$, respectively), we obtain

$$\begin{cases} \dot{X} &= Y, \\ \dot{Y} &= \frac{4\mu_0^2\mu_1}{\mu_2^2} X + X^2 - XY + o(\|(X,\ Y)\|^2). \end{cases} \tag{21}$$

From the result by Bogdanov-Takens theorem in (Jean-Baptiste and Claude (1993)), we conclude that system (21) undergoes Bogdanov-Takens bifurcation. ■

4.2.1. *Simulation*

For the simulation, we take $\gamma = \frac{4}{5}$, $\nu = \frac{9}{10}$ and $N = 10$. So we obtain

$$\beta_0 \in \Big\{\beta_1 = 0.32375,\ \beta_2 = 0.96125\Big\} \quad \text{and} \quad \alpha_0 \in \Big\{\alpha_{01} = 1.0638235,\ \alpha_{02} = 2.3709664\Big\}$$

and we take the initial conditions $(z_1^0, z_2^0) \in \Big\{(2,9)\ ,\ (3,7)\ ,\ (5,7)\ ,\ (4,9)\ ,\ (7,5)\Big\}$ (see figures 12, 13, and 14 in the appendix).

Remark 4.2.

If there exists $(\gamma,\ \beta,\ \nu,\ N_0) \in (0, +\infty)^4$ such that

$$\beta < \frac{\gamma+\nu}{N_0} \quad \text{and} \quad \Big(\gamma + \frac{\nu}{1-A_0} = \alpha_1 \ \text{ or } \ \gamma + \frac{\nu}{1-A_0} = \alpha_2\Big),$$

then the system (4) can admit a Bogdonov-Takens bifurcation at Z_1, where α_1 and α_2 are defined in Subsection 3.2.1.

5. Auto-oscillation

In this section, we prove a theorem for auto-oscillations of (4) on the following nonempty closed and convex set of $\mathbb{R}^2$

$$Q_k := \Big\{(z_1,\ z_2) \in \big[-\delta,\ N_0 - x_*\ \big] \times \big[-\delta,\ N_0 - ky_*\ \big],\quad -2\delta \leq z_1 + z_2 \leq N_0 - (ky_* + x_*)\Big\},$$

where $\delta := \frac{\overline{\delta}}{q},\ \overline{\delta} = \min\Big\{x_*,\ y_*\Big\},\ q \geq 2,$ and $k < \frac{N_0}{y_*}$ are some constant real numbers.

Let $Z \in Q_k$. From now on, we consider the following system

$$\dot{Z} = \tau_Z f(Z), \tag{22}$$

where f is the vector field defined by (4) and τ_Z is the metric projection on the closed and convex tangent cone to Q_k at Z (Appell et al. (2017); Lobanova and Sadovskii (2007)). Let

$$\alpha_* = \frac{\gamma(\beta N_0 + \gamma)}{\gamma + \nu},\ \ \alpha_3 = \gamma + \beta\Big(N_0 + y_*(1-k)\Big),\ \ \alpha_m = \max\left\{\frac{\alpha_*}{2};\ \alpha_3;\ \beta\delta\right\},$$
$$\delta_0 = \frac{1}{2}\left(y_* - \frac{\gamma}{\beta}\right),\ \delta_1 = x_* - \delta_0,\ \ \overline{\delta}_0 = \min\Big\{\delta_0,\ \delta_1\Big\},\ \ q_0 = \max\Big\{2,\ \frac{\overline{\delta}}{\delta_0},\ \frac{\overline{\delta}}{\delta_1},\ \frac{2\beta\overline{\delta}}{\alpha_*}\Big\},$$
$$k_0 = 1 + \frac{\beta(\gamma + \beta N_0)(\gamma + 2\nu)}{2\gamma(\beta N_0 - \nu)},\ \beta_0 = \min\left\{\frac{\gamma^2 + 2\nu^2 + 4\gamma\nu}{N_0\gamma},\ \ \frac{2\nu}{\gamma + 2\nu}\right\}.$$

Theorem 5.1.

$$\text{Let } Q_k \text{ be a closed and convex set of } \mathbb{R}^2 \text{ of which interior is nonempty, and} \tag{23}$$

$$f : Q_k \to \mathbb{R}^2 \text{ be a locally lipschitz function .} \tag{24}$$

Suppose that there exist some positive real numbers $\beta,\ \nu,\ \gamma,\ \alpha,\ N_0,\ q$ and k such that

$$\frac{(\gamma + 2\nu)^2}{2\nu} < N_0,\ \ \frac{\gamma + 2\nu}{N_0} < \beta < \beta_0,\ \ q_0 < q,\ \ k_0 < k < \frac{N_0}{y_*}\ \text{ and }\ \alpha_m < \alpha, \tag{25}$$

and

$$\forall Z \in \partial Q_k,\ \ \exists\, u \in T_Z,\ \ \langle u\ ,\ f(Z)\rangle > 0. \tag{26}$$

If the conditions (23) - (26) hold, then the system (22) admits a unique closed trajectory Γ of which orbit is a globally stable limit cycle on Q_k.

Proof:

We will just verify if the hypothesis of Lobanova-Sadovskii theorem (Lobanova and Sadovskii (2007)) are satisfied under the conditions of Theorem 4.1. We check that $O(0\ ,\ 0) \in \mathring{Q}_k,\ \ Q_k$ is a closed and convex set. Moreover, f is locally Lipschitz on Q_k because its components f_1 and f_2 defined by

$$f_1(Z) = (\alpha - \gamma - \beta y_*)z_1 - (\gamma + \nu)z_2 - \beta z_1 z_2 \quad \text{and} \quad f_2(Z) = \beta y_* z_1 + \alpha z_2 + \beta z_1 z_2,$$

are polynomial functions.

Next, we prove that there exist a real positive definite matrix B and an application

$$\mu : (0\,,\, +\infty) \to (0\,,\, +\infty) \text{ such that } \quad \text{for all } Z \in Q_k, \quad \langle BZ,\ f(Z)\rangle \geq \mu(\|Z\|).$$

Let $B := \begin{pmatrix} \beta y_* & 0 \\ 0 & \gamma + \nu \end{pmatrix}$ and $Z \in Q_k$. Then,

$$\langle BZ,\ f(Z)\rangle = \beta y_* \left(\alpha - \gamma - \beta y\right) z_1^2 + (\gamma + \nu)\left(\alpha + \beta z_1\right) z_2^2.$$

Since $y < N_0 + y_*(1-k)$, we have $-\beta y > -\beta\Big(N_0 + y_*(1-k)\Big)$.

Moreover, we have $z_1 \geq -\delta$. Therefore, we obtain

$$\begin{aligned}\langle BZ,\ f(Z)\rangle &\geqslant \beta y_*\Big(\alpha - \gamma - \beta\Big(N_0 + y_*(1-k)\Big)\Big) z_1^2 + (\gamma + \nu)\left(\alpha - \beta\delta\right) z_2^2 \\ &\geqslant \beta y_* \left(\alpha - \alpha_3\right) z_1^2 + (\gamma + \nu)\left(\alpha - \beta\delta\right) z_2^2.\end{aligned}$$

According to (25), we have $\alpha > \max\left\{\ \beta\delta\ ;\ \alpha_3\ ;\ \frac{\alpha_*}{2}\ \right\}$. So, we can take

$$\eta := \min\Big\{\beta y_* \left(\alpha - \alpha_3\right)\ ;\ (\gamma + \nu)\left(\alpha - \beta\delta\right)\Big\} > 0,$$

and $\mu(r) = \eta r^2$, for all $r > 0$. Then, for all $Z \in Q_k$, $\langle BZ,\ f(Z)\rangle \geq \mu(\|Z\|)$.

Now, let us find $r_0 > 0$ such that for all $Z \in Q_k$, $\langle JZ,\ f(Z)\rangle \geq r_0\|Z\|^2$.

Let $Z \in Q_k$. Then, we have

$$\begin{aligned}\langle JZ,\ f(Z)\rangle &= \beta y z_1^2 + (\gamma + \beta x)\, z_2^2 + (\gamma + \beta y^*) z_1 z_2 \\ &\geq \left(\beta y - \frac{1}{2}(\gamma + \beta y^*)\right) z_1^2 + \left(\beta x + \frac{1}{2}(\gamma - \beta y^*)\right) z_2^2 \\ &\geq \beta\left(z_2 + y^* - \frac{1}{2}\left(\frac{\gamma}{\beta} + y^*\right)\right) z_1^2 + \beta\left(z_1 + x^* + \frac{1}{2}\left(\frac{\gamma}{\beta} - y^*\right)\right) z_2^2 \\ &\geq \beta\left(\frac{1}{2}\left(y^* - \frac{\gamma}{\beta}\right) - \delta\right) z_1^2 + \beta\left(x^* + \frac{1}{2}\left(\frac{\gamma}{\beta} - y^*\right) - \delta\right) z_2^2 \\ \langle JZ,\ f(Z)\rangle &\geq \beta\left(\delta_0 - \delta\right) z_1^2 + \beta\left(\delta_1 - \delta\right) z_2^2.\end{aligned}$$

Moreover, according to (25), we have $\delta_0 - \delta > 0$ and $\delta_1 - \delta > 0$. So, we can take

$$r_0 := \beta \min\Big\{\delta_0 - \delta\ ;\ \delta_1 - \delta\Big\} > 0.$$

Then, for all $Z \in Q_k$, $\quad \langle JZ,\ f(Z)\rangle \geq r_0\|Z\|^2$.

Now, we prove that for all $Z \in Q_k$ and $Z \neq O, \quad f(Z) \notin N_Z$.

Case 1: If $Z \in \mathring{Q}_k$, then $N_Z = \{O\}$. Thus, $f(Z) \in N_Z$ if and only if $f(Z) = O$.

Since, there exists $r_0 > 0$ such that for all $Z \in Q_k$, $\langle JZ,\ f(Z)\rangle \geq r_0\|Z\|^2$, then,

$$f(Z) \in N_Z \text{ implies } Z = O.$$

Thus, for all $Z \in \mathring{Q}_k \setminus \{O\}$, $f(Z) \notin N_Z$.

Case 2: If $Z \in \partial Q_k$, according to (26), there exists $u \in T_Z$, such that $\langle u,\ f(Z)\rangle > 0$.

So, for all $Z \in \partial Q_k$, $f(Z) \notin N_Z$. Thus, for all $Z \in Q_k \setminus \{O\}$, $f(Z) \notin N_Z$.

Hence, according to the Lobanova-Sadovskii theorem (Lobanova and Sadovskii (2007)), the proof of Theorem 5.1 is completed. ∎

5.1. Application and simulation

To apply Theorem 5.1, we take $\big(\beta, \nu, \gamma, \alpha, N_0\big) \in \mathcal{A}_2$ such that

$$\gamma = \frac{2}{25}, \quad \nu = \frac{9}{100}, \quad \beta = \frac{7}{100}, \quad N_0 = 9 \text{ and } \frac{\alpha_*}{2} = \frac{71}{425} < \alpha.$$

Then, we obtain $x_* = \frac{9}{7}$, $y_* = \frac{432}{119}$, and $\delta = \frac{x_*}{2} = \frac{9}{14}$. Moreover, the values under consideration verify the condition (25), i.e.,

$$\frac{(\gamma + 2\nu)^2}{2\nu} = \frac{338}{225} < N_0, \quad \frac{\gamma + 2\nu}{N_0} = \frac{13}{225} < \beta < \beta_0 \approx 0.0713889,$$

$$q_0 < q = 2, \text{ and } k_0 < k = 2 < \frac{N_0}{y_*} = \frac{1071}{432}.$$

We obtain

$$Q_k := \left\{(z_1, z_2) \in \mathbb{R}^2,\ -\frac{9}{14} \leq z_1 \leq \frac{72}{7},\ -\frac{9}{14} \leq z_2 \leq \frac{207}{119}, \quad -\frac{9}{7} \leq z_1 + z_2 \leq \frac{54}{119}\right\},$$

which is a closed and convex set of $\mathbb{R}^2$ of which interior is nonempty.

For all $Z = (z_1,\ z_2) \in Q_k$, we have

$$\begin{cases} \dot{z}_1 = f_1(Z) = (\alpha - \gamma - \beta y_*)z_1 - 0.17z_2 - 0.07z_1z_2, \\ \dot{z}_2 = f_2(Z) = 0.2541176z_1 + \alpha z_2 + 0.07z_1z_2. \end{cases} \tag{27}$$

The vector field $f = (f_1, f_2)$ is a locally Lipschitz map on Q_k and $O_{\mathbb{R}^2} \in Q_k$.

Let $Z \in \partial Q_k$. Then, we distinguish two cases.

Case 1: Z is a corner point. In this case the tangent cone at Z is the angular domain. For example if $Z = \big(-\frac{9}{14}\ ,\ -\frac{9}{14}\big),$ then we have

$$T_Z = \Big\{(z_1\ ,\ z_2) \in \mathbb{R}^2,\ \ z_1 \geq -\frac{9}{14}\ \ \text{and}\ \ z_2 \geq -\frac{9}{14}\ \Big\},$$

and

$$N_Z = \Big\{(z_1\ ,\ z_2) \in \mathbb{R}^2,\ \ z_1 \leq -\frac{9}{14}\ \ \text{and}\ \ z_2 \leq -\frac{9}{14}\ \Big\}.$$

Moreover, for $\alpha \approx 0.18034,$ we have $u = (1,\ 0\ ) \in T_Z$ and $\langle u,\ f(Z)\rangle \approx 0.25 > 0.$

Case 2: Z is not a corner point. In this case the tangent cone at Z is the half plane. For example if $\big(-\frac{9}{14}\ ,\ 0\big),$ then we have

$$T_Z = \Big\{(z_1\ ,\ z_2) \in \mathbb{R}^2,\ -\frac{9}{14} \leq z_1\ \ \text{and}\ \ z_2 = 0\ \Big\},$$

and

$$N_Z = \Big\{(z_1\ ,\ z_2) \in \mathbb{R}^2,\ z_1 \leq -\frac{9}{14}\ \ \text{and}\ \ z_2 = 0\Big\}.$$

Moreover, for $\alpha \approx 0.18034,$ we have $u = (1,\ 0\ ) \in T_Z$ and $\langle u,\ f(Z)\rangle \approx 0.25 > 0.$

Then, for all $Z \in \partial Q_k,$ there exist $u \in T_Z$ such that $\langle u\ , f(Z)\rangle > 0.$

So, for these values of the parameters, the system (22) admits a unique closed trajectory of which orbit is stable. For the simulations, we denote the initial conditions by $(z_1^0,\ z_2^0)$ (see Figures 15, 16, and 17 in the appendix).

Remark 5.1.

On Figure 15 and Figure 16 below, we have simulated respectively the phase portrait and the chronic of the system (22) on Q under the conditions of the Theorem 5.1 and we observe the existence of a unique limit cycle which is globally stable. This means that when the SIR system is perturbed by external factors (such as climate change), the infection (the disease) persists over time (a long term). We remark that there exists a $\alpha_c \in (\alpha_m,\ +\infty)$ such that for all $\alpha \in (\alpha_m,\ \alpha_c),$ there exists a unique stable limit cycle (see Figure 15 and Figure 16). This means that the disease appears periodically in the population. We can therefore predict and control the infection. But for all $\alpha \geq \alpha_c,$ we observe the disappearance of the limit cycle and the decreases of the rate of infections (see Figure 17). This means that the infection becomes controllable and can be eradicated after a certain period.

6. Conclusion

In this work, we have studied the dynamic behavior of the perturbed epidemiological model SIR on a nonempty, closed and convex set. Indeed, we have studied firstly the existence and the number of the equilibrium points. Moreover, we have studied the nature of these equilibrium points. Also, we have analyzed the existence of the Hopf and Bogdanov-Takens bifurcations in a neighborhood of some equilibrium points. Finally, we have demonstrated under certain conditions on the parameters, the existence and the uniqueness of a globally stable limit cycle. The results obtained in this paper with the perturbed SIR epidemiological model are not obtained with the classical SIR model. Therefore, taking into account the external phenomena that can influence an epidemic would allow a better understanding of the true behavior and evolution of this epidemic.

Acknowledgment:

We thank all reviewers for their valuable comments. The first author is grateful to the Deutscher Akademischer Austauschdienst German Academic Exchange Service (DAAD) for its fellowship. The other authors acknowledge the grant of The World Academy of Sciences (TWAS) for the advancement for science in developing countries.

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Appendix

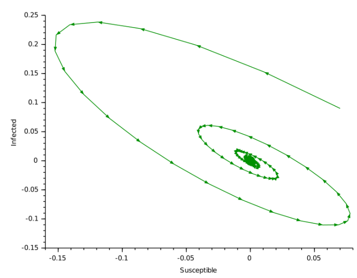

Figure 1. Phase portrait of the differential system (4) for $\alpha = 1.58 < \alpha_b$ and $(z_1^0,\ z_2^0) = (0.07, 0.09)$.

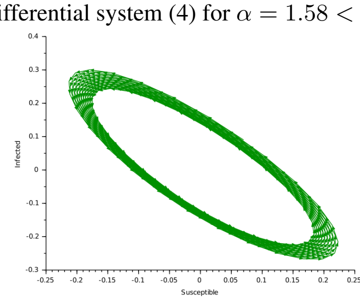

Figure 2. Phase portrait of the differential system (4) for $\alpha = \alpha_b = \frac{1092}{595}$ and $(z_1^0\ ,\ z_2^0) = (0.07\ ,\ 0.09)$.

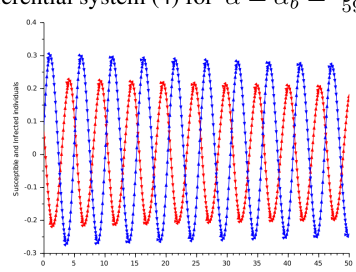

Figure 3. Chronic of the differential system (4) for $\alpha = \alpha_b = \frac{1092}{595}$ and $(z_1^0\ ,\ z_2^0) = (0.07\ ,\ 0.09)$.

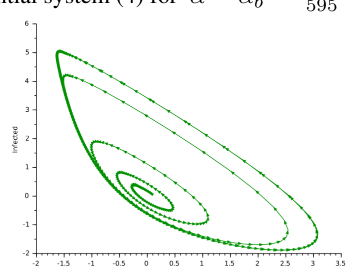

Figure 4. Phase portrait of the differential system (4) for $\alpha \in (\alpha_b\ ,\ \alpha_{c_1})$ with $\alpha_{c_1} = 2.01$, $\alpha_b = \frac{1092}{595}$ and $(z_1^0,\ z_2^0) = (0.07\ ,\ 0.09)$.

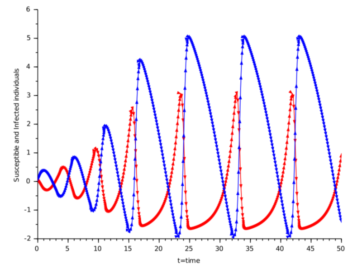

Figure 5. Chronic of the differential system (4) for $\alpha \in \left(\alpha_b \ , \ \alpha_{c_1}\right)$ with $\alpha_{c_1} = 2.01$, $\alpha_b = \frac{1092}{595}$ and $(z_1^0,\ z_2^0) = (0.07\ ,\ 0.09)$.

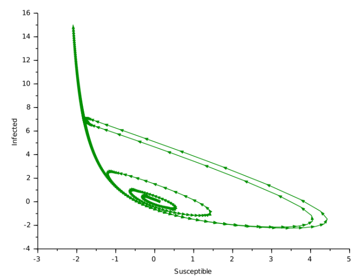

Figure 6. Phase portrait of the differential system (4) for $\alpha \in [\alpha_{c_1} \ , \ \alpha_{c_2}]$ with $\alpha_{c_1} = 2.01$, $\alpha_{c_2} = 2.02$, and $(z_1^0, z_2^0) = (0.07, 0.09)$.

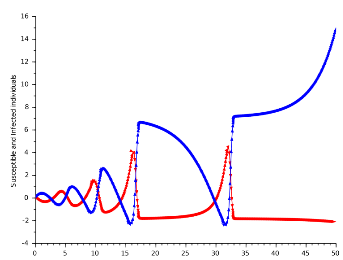

Figure 7. Chronic of the differential system (4) for $\alpha \in [\alpha_{c_1} \ , \ \alpha_{c_2}]$ with $\alpha_{c_1} = 2.01$, $\alpha_{c_2} = 2.02$, and $(z_1^0, z_2^0) = (0.07, 0.09)$.

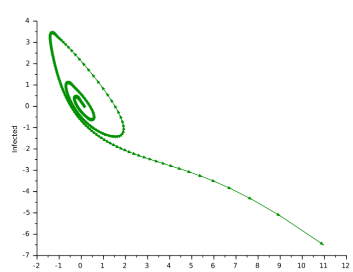

Figure 8. Phase portrait of the differential system (4) for $\alpha \in (\alpha_{c_2} \ , \alpha_{c_3})$ with $\alpha_{c_2} = 2.02, \alpha_{c_3} = 3.575$, and $(z_1^0,\ z_2^0) = (0.07\ ,\ 0.09)$.

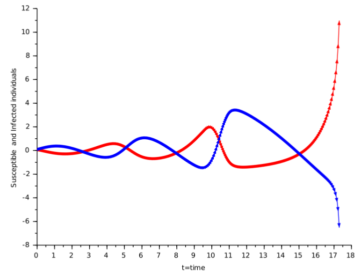

Figure 9. Chronic of the differential system (4) for $\alpha \in (\alpha_{c_2}, \alpha_{c_3})$ with $\alpha_{c_2} = 2.02, \alpha_{c_3} = 3.575$ and $(z_1^0, z_2^0) = (0.07, 0.09)$.

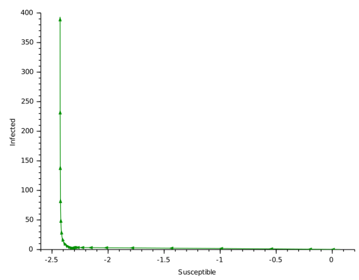

Figure 10. Phase portrait of the differential system (4) for $\alpha \geq \alpha_{c_3} = 3.575$ and $(z_1^0, z_2^0) = (0.07, 0.09)$.

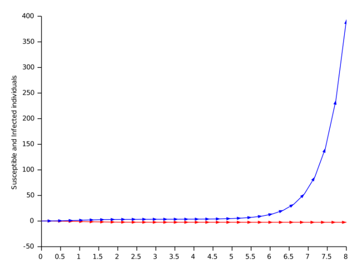

Figure 11. Chronic of the differential system (4) for $\alpha \geq \alpha_{c_3} = 3.575$ and $(z_1^0, z_2^0) = (0.07, 0.09)$.

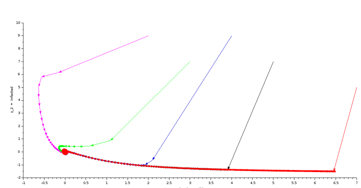

Figure 12. Phase portrait of the differential system (18) with $\beta_0 \in \{\beta_1 + 0.99, \beta_2 + 0.99\}$ and $\alpha_0 \in \{\alpha_{01} + 0.99, \alpha_{02} + 0.99\}$.

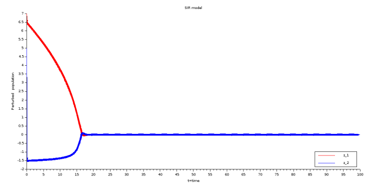

Figure 13. Chronic of the differential system (18) with $\beta_0 \in \{\beta_1 + 0.99, \quad \beta_2 + 0.99\}$ and $\alpha_0 \in \{\alpha_{01} + 0.99, \;\; \alpha_{02} + 0.99\}$.

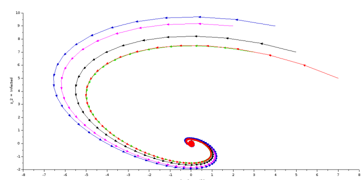

Figure 14. Phase portrait of the differential system (18) with $\beta_0 \in \{\beta_1 - 0.99, \;\beta_2 - 0.99\}$ and $\alpha_0 \in \{\alpha_{01} - 0.99, \;\alpha_{02} - 0.99\}$.

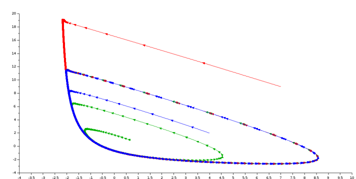

Figure 15. Phase portrait of the differential system (22) for $\alpha \approx 0.18034$ and $(z_1^0, z_2^0) \in \{(0.7, 0.9),\ (7, 9),\ (4, 2)\}$.

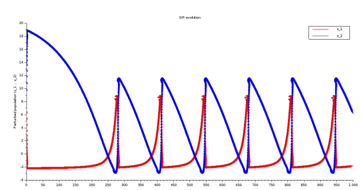

Figure 16. Chronic of the differential system (22) for $\alpha \approx 0.18034$ and $(z_1^0,\ z_2^0) = (7,\ 9)$.

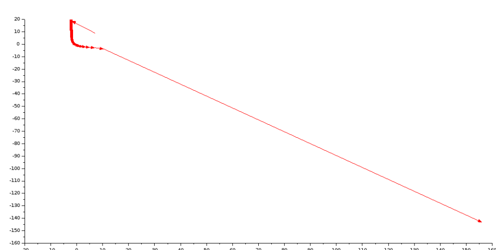

Figure 17. Phase portrait of (22) for $(z_1^0,\ z_2^0) = (7,\ 9)$ and $\alpha \geqslant \alpha_c = 0.1804$.