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Hermite Wavelets Method for System of Linear Differential Equations

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Abstract

In this research paper, we present an accurate technique for solving the system of linear differential equations. Such equations often arise as a result of modeling in many systems and applications of engineering and science. The proposed scheme is based on Hermite wavelets basis functions and operational matrices of integration. The demonstrated scheme is simple as it converts the problem into algebraic matrix equation. To validate the applicability and efficacy of the developed scheme, some illustrative examples are also considered. The results so obtained with the help of the present proposed numerical technique by using Hermite wavelets are observed to be more accurate and favorable in comparison with those by using Haar wavelets.

Keywords: Hermite wavelets; Haar wavelets; Function approximation; Convergence analysis; Operational matrix of integration; Numerical examples; Differential equations

MSC 2010 Classifications: 65N99, 65H10

1. Introduction

The simultaneous differential equations turn up frequently in quantum mechanics, electric circuit analysis, heat transfer, classical mechanics, and in many other areas of science and engineering. Researchers have tried to solve systems of differential equations by implementing various techniques. But in the last few decades, the popularity of wavelets is being acknowledged due to their implementation in the fields of signal processing, control applications, and numerical analysis. From the previous research, it is assumed that wavelets based numerical methods are more accurate and efficient in comparison to classical techniques. The theory of wavelets is comparatively a new and upcoming methodology that is being tremendously used in present-day mathematical research. This theory is continuously developing due to the increasing and rigorous implementation of wavelet methods as a significant tool in various disciplines of engineering, science, and applied mathematics. Wavelets can be defined as mathematical functions that have been considerably used for the representation and segmentation of waves in digital signal processing, compression of images, for analyzing time and frequency of waves, algorithms for smooth and quick implementations, and in various domains of applied and pure mathematics. This research paper aims to prove the applicability of a technique based on Hermite wavelets as an effective tool to solve the simultaneous differential equations.

Hermite Wavelets Method (HWM) is successfully implemented to obtain accurate solutions of various linear as well as non-linear boundary value problems of fifth and sixth order in Iqbal and Mohyud-din (2013). The same methodology can also be extended to find solutions of other linear as well as nonlinear diversified physical problems of complex nature. In Berwal et al. (2013), a solution of simultaneous linear differential equations is obtained with Haar wavelets, whereas in Entezari (2019), fractional partial differential equations have been solved with the help of Bernstein wavelets operational matrices of integration. Hermite wavelet-based method is devised to obtain solutions of Jaulent–Miodek equation in Gupta and Ray (2015), whereas in Khakrangin et al. (2021), Haar wavelet operational matrix of the fractional order integration has been efficiently used to get solution of the fuzzy fractional differential equations. In Khashem (2019), the Hermite wavelet-based numerical approach is presented to find the estimated solution of Bratu's problem and in Kumbinarasaiah (2017), delay differential equations have been solved with the aid of a method based on Hermite wavelets. The Hermite wavelet and operational matrix of integration are utilized for the solution of integro-differential equations in Kumbinarasaiah and Mundewadi (2021), whereas in Lepik (2008), initial and boundary value problems have been discussed to demonstrate the applicability of Haar wavelets for solving differential equations of order higher than two. In Oruc (2018), the hyperbolic telegraph equation in two-dimensional space has been solved with the help of wavelets and in Saeed and Rehman (2014), Hermite wavelet method has been applied to find solution of fractional delay differential equations after converting them to fractional nondelay differential equations.

Haar wavelet collocation method has been implemented to solve singular initial value problems and compared favorably with the exact and existing numerical methods in Shiralashetti et al. (2016) and solutions of nonlinear singular initial value problems with the help of Hermite wavelets are introduced in Shiralashetti and Srinivasa (2018). Modified Hermite wavelets based numerical method has been established for solving singular initial and boundary value problems by expanding the unknown function as a series of Hermite wavelets with unknown coefficients in Shiralashetti and Srinivasa (2019) and with the help of a series of derivatives in Shiralashetti and Hanaji (2020). In Singh and Kumar (2016), nonlinear Volterra integral equations of the first kind have been solved by using Haar wavelets and in Singh and Kumar (2018), a collocation method based on Haar wavelets has been presented for solving fourth order nonlinear Kuramoto-Sivashinsky equation. In Singh (2019), a useful numerical scheme has been developed to solve the generalized Burger's type equations by converting them to nonlinear ordinary differential equations and then solving the system of linear equations thus obtained by implementing Haar wavelet-based collocation method. Hermite wavelets have been favorably utilized for the evaluation of Numerical Integration in Singh and Kaur (2021), whereas in Waleeda and Haleema (2021), Haar wavelet collocation points and operational matrices of integration have been used to find solutions of linear ordinary differential equations with variable coefficients. The literature reveals that so far, Hermite wavelets have never been utilized for solving system of linear differential equations. In this paper, we have introduced an efficient technique based on the basis functions of Hermite wavelets, for solving system of linear differential equations of first and second order. The highest order derivative occurring in the established system of differential equations is expressed as a series of Hermite wavelets basis functions and wavelet coefficients. After integration and application of initial conditions, the Hermite wavelet solution of the given system of first or second order is secured as explained in the exhibited method. The numerical results so obtained validate the efficiency and accuracy of the proposed scheme.

The proposed schemes to solve systems of linear differential equations of the first and second order are simple, easy to implement, and the time involved in the process of the execution of the demonstrated methods is also very less. In addition, the applicability of the rendered technique in the situation when exact solution is not regular enough, contributes to the novelty of the paper. An increase in the degree of the approximating polynomial leads to a higher rate of convergence. So the desired convergence is achieved due to the fact that in case of Hermite wavelets, the approximation is done through a polynomial of degree greater than one. It is pertinent to mention that in the case of absence of exact solution, a favorable comparison with the numerical solution derived from classical numerical methods, decides the relevance and suitability of the proposed techniques.

This paper has been organized as follows. Hermite wavelets based method to solve system of linear differential equations of first order have been presented in Section 2 and for that of second order in Section 3. In Section 4, the theorem on convergence analysis has been given, and in Section 5 some numerical examples have been given to prove the better accuracy of results with the proposed methods as compared to Haar wavelets based technique. In Section 6, stability analysis has been discussed.

2. Proposed Hermite Wavelets Technique for System of Linear Differential Equations of First Order

Let us consider the following first-order linear equations

$$
\begin{aligned}
y'_1(t) &= f_1(t, y_1, y_2, \dots, y_n),\\
y'_2(t) &= f_2(t, y_1, y_2, \dots, y_n),\\
y'_3(t) &= f_3(t, y_1, y_2, \dots, y_n),\\
&\dots\\
y'_n(t) &= f_n(t, y_1, y_2, \dots, y_n),
\end{aligned}
$$

subject to the initial conditions $y_1(0) = s_1,\ y_2(0) = s_2, \dots, y_n(0) = s_n$.

We approximate the function $y'_n(t)$ using Hermite wavelet basis as follows:

$$y_n'(t) = \sum_{p=1}^{\infty}\sum_{q=0}^{\infty} A_{p,q}\, \Phi_{p,q}(t), \tag{1}$$

where $A_{p,q}$ are the wavelet coefficients and

$$\Phi_{p,q}(t) = \begin{cases} \dfrac{2^{\frac{k+1}{2}}}{\sqrt{\pi}} H_q(2^k t - 2p + 1), & \dfrac{p-1}{2^{k-1}} \le t < \dfrac{p}{2^{k-1}} \\ 0, & \text{elsewhere,} \end{cases}$$

and $q = 0, 1, \cdots, M-1$, $H_q(t)$ denotes the Hermite polynomial of degree q.

By truncating the infinite series of $y_n'(t)$ in (1), we assume that

$$y_n'(t) = \sum_{p=1}^{2^{k-1}}\sum_{q=0}^{M-1} A_{p,q}\, \Phi_{p,q}(t) = A^T\Phi(t), \tag{2}$$

where A and Φ are matrices of order $2^{k-1}M \times 1$ and are defined as:

$$A^T = \left[A_{1,0}, \dots, A_{1,M-1}, \dots, A_{2^{k-1},0}, \dots, A_{2^{k-1},M-1}\right],$$

and

$$\Phi = [\Phi_{1,0}, \dots, \Phi_{1,M-1}, \dots, \Phi_{2^{k-1},0}, \dots, \Phi_{2^{k-1},M-1}]^T,$$

where T means the transpose of a matrix.

Integrating both sides of (2) with respect to t with limits from 0 to t, we obtain

$$y_n(t) = y_n(0) + \sum_{p=1}^{2^{k-1}}\sum_{q=0}^{M-1} \left(A_{p,q}\right)_n \int_0^t \Phi_{p,q}(t)dt, \qquad n = 1, 2, 3, \dots \;.$$

Applying the initial conditions, we obtain:

$$y_n(t) = s_n + \sum_{p=1}^{2^{k-1}}\sum_{q=0}^{M-1} (A_{p,q})_n\, P_{p,q}(t), \qquad n = 1, 2, 3, \dots \tag{3}$$

where

$$P_{p,q}(t) = \int_0^t \Phi_{p,q}(t)dt.$$

Discretizing by choosing the collocation points as:

$$t_l = \frac{l - 0.5}{(M-1).\,2^{k-1}}, \quad l = 1,2,3, \dots, ((M-1).\,2^{k-1}).$$

From (2) and (3), we obtain

$$y_n'(t_l) = \sum_{p=1}^{2^{k-1}} \sum_{q=0}^{M-1} A_{p,q}\, \Phi_{p,q}(t_l),$$

and

$$y_n(t_l) = s_n + \sum_{p=1}^{2^{k-1}} \sum_{q=0}^{M-1} (A_{p,q})_n\, P_{p,q}(t_l)\,, \qquad n = 1, 2, 3, \ldots\ .$$

Substituting the values of $y_n'(t_l)$ and $y_n(t_l)$ into the discretized system of differential equations of the first order, we obtain the system of linear equations which can be solved with the help of Gauss elimination method or Gauss Jordan method. The solution of the linear algebraic equations gives the values of wavelet coefficients $(A_{p,q})_n$. By putting these values of the wavelet coefficients in (3), we obtain the Hermite wavelet solution of the given system of linear differential equations of first order.

3. Proposed Hermite Wavelets Technique for System of Linear Differential Equations of Second Order

We consider the following system of second order linear equations

$$y''_1(t) = f_1(t, y_1, y_2, \ldots, y_n, y'_1, y'_2, \ldots, y'_n),$$
$$y''_2(t) = f_2(t, y_1, y_2, \ldots, y_n, y'_1, y'_2, \ldots, y'_n),$$
$$y''_3(t) = f_3(t, y_1, y_2, \ldots, y_n, y'_1, y'_2, \ldots, y'_n),$$
$$\cdots$$
$$y''_n(t) = f_n(t, y_1, y_2, \ldots, y_n, y'_1, y'_2, \ldots, y'_n),$$

subject to the initial conditions

$$y_1(0) = s_1,\ y_2(0) = s_2, \ldots\ , y_n(0) = s_n,$$

$$y'_1(0) = e_1,\ y'_2(0) = e_2, \ldots, y'_n(0) = e_n.$$

As explained in previous section of differential equations of first order, we assume that

$$y_n''(t) = \sum_{p=1}^{2^{k-1}} \sum_{q=0}^{M-1} A_{p,q}\Phi_{p,q}(t) = A^T\, \Phi(t). \tag{4}$$

Integrating both sides of the above equation (4) two times with respect to t, from 0 to t and applying the initial conditions, we obtain:

$$y'_n(t) = y'_n(0) + \sum_{p=1}^{2^{k-1}} \sum_{q=0}^{M-1} (A_{p,q})_n \int_0^t \Phi_{p,q}(t)dt\,, \qquad n = 1, 2, 3, \ldots\ ,$$

or

$$y'_n(t) = e_n + \sum_{p=1}^{2^{k-1}} \sum_{q=0}^{M-1} (A_{p,q})_n\ P_{p,q}(t) \qquad n = 1, 2, 3, \ldots\ , \tag{5}$$

where

$$P_{p,q}(t) = \int_0^t \Phi_{p,q}(t)dt,$$

and

$$y_n(t) = s_n + te_n + \sum_{p=1}^{2^{k-1}} \sum_{q=0}^{M-1} (A_{p,q})_n \int_0^t \int_0^t \Phi_{p,q}(t) dt\, dt, \qquad n = 1, 2, 3, \dots ,$$

or

$$y_n(t) = s_n + te_n + \sum_{p=1}^{2^{k-1}} \sum_{q=0}^{M-1} (A_{p,q})_n Q_{p,q}(t) , \qquad n = 1, 2, 3, \dots , \tag{6}$$

where

$$Q_{p,q}(t) = \int_0^t \int_0^t \Phi_{p,q}(t) dt\, dt.$$

Discretizing by using the collocation points as:

$$t_l = \frac{l - 0.5}{(M-1).2^{k-1}} , \quad l = 1,2,3, \dots, ((M-1).2^{k-1}).$$

From (4), (5) and (6), we obtain

$$y''_n(t_l) = \sum_{p=1}^{2^{k-1}} \sum_{q=0}^{M-1} A_{p,q} \Phi_{p,q}(t_l),$$

$$y'_n(t_l) = e_n + \sum_{p=1}^{2^{k-1}} \sum_{q=0}^{M-1} (A_{p,q})_n \; P_{p,q}(t_l) \qquad n = 1, 2, 3, \dots ,$$

and

$$y_n(t_l) = s_n + (t_l)e_n + \sum_{p=1}^{2^{k-1}} \sum_{q=0}^{M-1} (A_{p,q})_n Q_{p,q}(t_l) , \qquad n = 1, 2, 3, \dots .$$

Discretize the given system of equations by using collocation points t_l. Substituting the values of $y''_n(t_l)$, $y'_n(t_l)$ and $y_n(t_l)$ into the discretized system of differential equations of second order, we obtain a system of linear algebraic equations which can be solved with the help of Gauss elimination method or Gauss Jordan method. The solution of this system of linear equations gives the values of wavelet coefficients $(A_{p,q})_n$. By putting these values of wavelet coefficients in the expression for $y_n(t)$ in (6), we obtain the Hermite wavelet solution of the given system of linear differential equations of second order.

4. Convergence Analysis

For the convergence analysis, we discuss the following theorem:

Theorem 4.1.

Consider a continuous function $u(t)$ in $H^2[0, 1)$ defined on $[0, 1)$ be bounded, then the Hermite wavelet expansion of $u(t)$ converges to it.

Proof:

As $u(t)$ is bounded real valued function on $[0, 1)$. The values of Hermite wavelet coefficients of continuous functions $u(t)$ are defined as:

$$A_{n,m} = \int_0^1 u(t)\psi_{n,m} dt = \int_\Lambda u(t) \frac{2^{\frac{k+1}{2}}}{\sqrt{\pi}} h_m(2^k t - 2n + 1) dt,$$

over the interval defined as $\Lambda = \left[\frac{n-1}{2^{k=1}}, \frac{n}{2^{k-1}}\right)$.

Letting $2^k t - 2n + 1 = \theta$. Therefore,

$$A_{n,m} = \frac{2^{\frac{k+1}{2}}}{\sqrt{\pi}} \int_{-1}^{1} u\left(\frac{\theta - 1 + 2n}{2^k}\right) h_m(\theta) 2^{-k} d\theta,$$

$$A_{n,m} = \frac{2^{\frac{-k+1}{2}}}{\sqrt{\pi}} \int_{-1}^{1} u\left(\frac{\theta - 1 + 2n}{2^k}\right) h_m(\theta) d\theta.$$

Applying mean value theorem, we obtain:

$$A_{n,m} = \frac{2^{\frac{-k+1}{2}}}{\sqrt{\pi}} u\left(\frac{\Delta - 1 + 2n}{2^k}\right) \int_{-1}^{1} h_m(\theta) d\theta, \text{ for some } \Delta \in (-1,1).$$

Taking $\int_{-1}^{1} h_m(\theta) d\theta = \eta$, we obtain:

$$|A_{n,m}| = \left|\frac{2^{\frac{-k+1}{2}}}{\sqrt{\pi}}\right| u \left|\frac{\Delta - 1 + 2n}{2^k}\right| \eta,$$

as given that u is bounded. Therefore, the infinite series $\sum_{n,m=0}^{\infty} A_{n,m}$ is absolutely convergent. Hence, the Hermite series expansion of $u(t)$ converges uniformly. ■

5. Numerical Experiments

We perform some numerical examples to prove the efficiency of the proposed numerical scheme based on Hermite wavelets. The accuracy of the numerical results is obtained by using the following relation

$$Absolute\ Error = |y_{Exact} - y_{Approximate}|.$$

To establish the efficiency of the proposed numerical scheme based on Hermite wavelets basis functions, a comparison study is also presented in this research by using Haar wavelets.

Example 5.1.

Let us take the following system of linear differential equations of first order (Berwal et al. (2013))

$$y_1'(t) = y_3(t) - \cos t, \quad y_1(0) = 1\ ,$$
$$y_2'(t) = y_3(t) - e^t, \quad y_2(0) = 0\ ,$$
$$y_3'(t) = y_1(t) - y_2(t), \quad y_3(0) = 2\,.$$

The exact solution of this system of differential equations is:

$$y_1(t) = e^t, \quad y_2(t) = \sin t, \quad y_3(t) = e^t + \cos t.$$

We assume that

$$y_1'(t) = \sum_{p=1}^{2^{k-1}} \sum_{q=0}^{M-1} (A_{p,q})_1\, \Phi_{p,q}(t) = A_1^T \Phi, \tag{7}$$

$$y_2'(t) = \sum_{p=1}^{2^{k-1}} \sum_{q=0}^{M-1} \left(A_{p,q}\right)_2\, \Phi_{p,q}(t) = A_2^T \Phi, \tag{8}$$

and

$$y_3'(t) = \sum_{p=1}^{2^{k-1}} \sum_{q=0}^{M-1} (A_{p,q})_3\, \Phi_{p,q}(t) = A_3^T \Phi. \tag{9}$$

Integrating the above equations (7), (8) and (9) with respect to t and with limits from 0 to t, we obtain:

$$y_1(t) = y_1(0) + \sum_{p=1}^{2^{k-1}} \sum_{q=0}^{M-1} (A_{p,q})_1\, P_{p,q}(t) = 1 + A_1^T P, \tag{10}$$

$$y_2(t) = y_2(0) + \sum_{p=1}^{2^{k-1}} \sum_{q=0}^{M-1} (A_{p,q})_2\, P_{p,q}(t) = A_2^T P, \tag{11}$$

and

$$y_3(t) = y_3(0) + \sum_{p=1}^{2^{k-1}} \sum_{q=0}^{M-1} (A_{p,q})_3\, P_{p,q}(t) = 2 + A_3^T P. \tag{12}$$

Substituting the values from the equations (7) - (12) in the given system of differential equations, we obtain:

$$A_1^T \Phi - A_3^T P - 2 + \cos t = 0, \tag{13}$$

$$A_2^T \Phi - A_3^T P - 2 + e^t = 0, \tag{14}$$

and

$$A_3^T \Phi - A_1^T P - 1 + A_2^T P = 0. \tag{15}$$

Rewriting the equations (13) and (14), we get,

$$A_1^T = A_3^T P \Phi^{-1} - U \Phi^{-1}, \tag{16}$$

$$A_2^T = A_3^T P \Phi^{-1} - V \Phi^{-1}, \tag{17}$$

where U and V are distinct values of $(2-\cos t)$ and $(2-e^t)$ respectively, each of order $1\times(2^{k-1}\times M-1)$. Substituting the values of A_1^T and A_2^T in (15), we obtain:

$$A_3^T = U\Phi^{-1}P\Phi^{-1} - V\Phi^{-1}P\Phi^{-1} + \Phi^{-1}.$$

Substituting this value of A_3^T in (16) and (17), we obtain:

$$A_1^T = U\Phi^{-1}P\Phi^{-1}P\Phi^{-1} - V\Phi^{-1}P\Phi^{-1}P\Phi^{-1} + \Phi^{-1}P\Phi^{-1} + (2-\cos t)\Phi^{-1},$$

and

$$A_2^T = U\Phi^{-1}P\Phi^{-1}P\Phi^{-1} - V\Phi^{-1}P\Phi^{-1}P\Phi^{-1} + \Phi^{-1}P\Phi^{-1} + (2-e^t)\Phi^{-1}.$$

Putting the values of the so obtained wavelet coefficients in (10) - (12), we get the required wavelets solution.

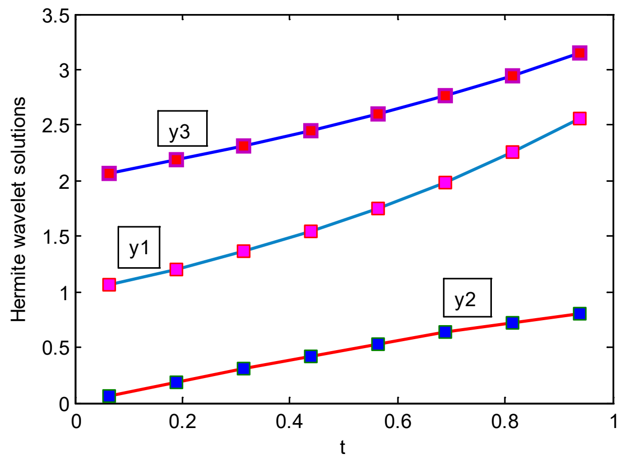

Figure 1: Comparison of exact and approximate solution of Example 5.1 for $k=1, M=8$

Table 1: Absolute errors (Hermite wavelets) of Example 5.1 for $k=1$ and $M=8$

$t/16$	y_1	y_2	y_3
1	4.4501e-010	2.6766e-010	3.1900e-010
3	3.6564e-010	2.3674e-010	2.5527e-010
5	4.2361e-010	2.8429e-010	2.8923e-010
7	4.4753e-010	3.1405e-010	2.9716e-010
9	4.9133e-010	3.5535e-010	3.1776e-010
11	5.1557e-010	3.8726e-010	3.2291e-010
13	5.7582e-010	4.3812e-010	3.5204e-010
15	4.8983e-010	4.1764e-010	2.7674e-010

Table 2: Absolute errors (Haar wavelets) of Example 5.1 for $J = 2$

$t/16$	y_1	y_2	y_3
1	2.0500e-003	6.7749e-005	2.1759e-004
3	2.2924e-003	1.7128e-004	7.0872e-004
5	2.6264e-003	2.0407e-004	1.2889e-003
7	3.0665e-003	1.5218e-004	1.9635e-003
9	3.6288e-003	1.1183e-006	2.7380e-003
11	4.3303e-003	2.6421e-004	3.6184e-003
13	5.1894e-003	6.5952e-004	4.6112e-003
15	6.2259e-003	1.2012e-003	5.7239e-003

Figure 1 exhibits the parity of the exact solution and the Hermite wavelets solutions for Example 5.1 for$k = 1, M = 8$. Table 1 presents the absolute errors for Example 5.1 by using the Hermite wavelet method and taking $k = 1,\ M = 8$. Table 2 gives the absolute errors of Example 5.1 by using the Haar wavelet method as discussed in Berwal et al. (2013) and taking $J = 2$. Example 5.1 shows that the Hermite wavelets based collocation method gives better results in comparison to the method based on Haar wavelets as presented in Berwal et al. (2013).

Example 5.2.

In a forest that has a variety of trees, choose a few of them which are very old and are about to ie or fall and become humus in the subsequent years. Let the variables ω_1, ω_2, ω_3, and t be described by

$$\omega_1(t) = \text{Bio-mass which decayed into humus},$$
$$\omega_2(t) = \text{Bio-mass of the dead trees},$$
$$\omega_3(t) = \text{Biomass of the living trees},$$
$$t = \text{Time in the number of decades}.$$

Let us assume that initially there is no dead tree and no humus is present at $t = 0$. If the initial number of units of biomass of the living trees is denoted by ω_0, the typical biological model is given by (Berwal et al. (2013))

$$\omega_1'(t) = -\omega_1(t) + 3\omega_2(t), \quad \omega_1(0) = 0,$$
$$\omega_2'(t) = -3\omega_2(t) + 5\omega_3(t), \quad \omega_2(0) = 0,$$
$$\omega_3'(t) = -5\omega_3(t), \quad w_3(0) = \omega_0.$$

The exact solution of this system of differential equations is given by:

$$\omega_1(t) = \frac{15}{8}\omega_0(e^{-5t} - 2e^{-3t} + e^{-t}),$$
$$\omega_2(t) = \frac{5}{2}\omega_0(-e^{-5t} + e^{-3t}),$$

and

$$\omega_3(t) = \omega_0 e^{-5t}.$$

We assume that

$$\omega_1'(t) = \sum_{p=1}^{2^{k-1}} \sum_{q=0}^{M-1} \left(A_{p,q}\right)_1 \Phi_{p,q}(t) = A_1^T \Phi, \tag{18}$$

$$\omega_2'(t) = \sum_{p=1}^{2^{k-1}} \sum_{q=0}^{M-1} (A_{p,q})_2 \Phi_{p,q}(t) = A_2^T \Phi, \tag{19}$$

and

$$\omega_3'(t) = \sum_{p=1}^{2^{k-1}} \sum_{q=0}^{M-1} (A_{p,q})_3 \Phi_{p,q}(t) = A_3^T \Phi. \tag{20}$$

Integrating the above three equations (18) - (20) with respect to t from 0 to t, we get:

$$\omega_1(t) = \omega_1(0) + \sum_{p=1}^{2^{k-1}} \sum_{q=0}^{M-1} (A_{p,q})_1 P_{p,q}(t) = A_1^T P, \tag{21}$$

$$\omega_2(t) = \omega_2(0) + \sum_{p=1}^{2^{k-1}} \sum_{q=0}^{M-1} (A_{p,q})_2 P_{p,q}(t) = A_2^T P, \tag{22}$$

and

$$\omega_3(t) = \omega_3(0) + \sum_{p=1}^{2^{k-1}} \sum_{q=0}^{M-1} (A_{p,q})_3 P_{p,q}(t) = \omega_0 + A_3^T P. \tag{23}$$

Substituting the values from the above equations (21) - (23) in the given system, we get:

$$A_1^T \Phi - A_1^T P - 3A_2^T \Phi = 0, \tag{24}$$

$$A_2^T \Phi + 3A_2^T P - 5\omega_0 - 5A_3^T P = 0, \tag{25}$$

and

$$A_3^T \Phi + 5A_3^T P + 5\omega_0 = 0. \tag{26}$$

Solving the above three equations (24) - (26), we obtain:

$$A_1^T = \frac{3A_2^T \Phi}{(\Phi - P)},$$

$$A_2^T = \frac{5\omega_0 + 5A_3^T P}{(\Phi + 3P)},$$

and

$$A_3^T = \frac{-5\omega_0}{\Phi + 5P}.$$

Substituting these wavelet coefficients in (21) - (23), we get the values of $\omega_1(t)$, $\omega_2(t)$, $\omega_3(t)$ and hence the required solution.

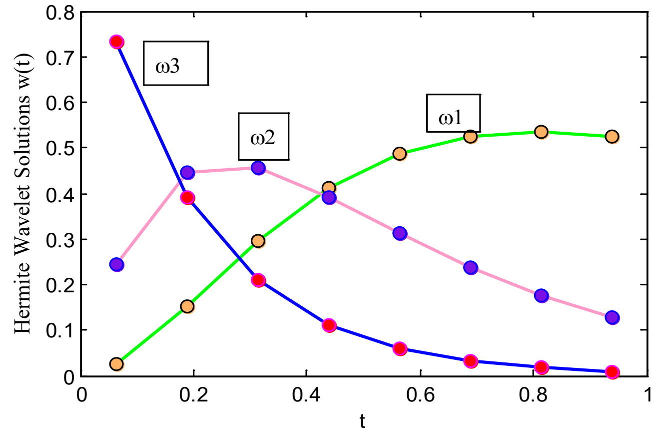

Figure 2: Comparison of exact and Hermite wavelets solutions of Example 5.2 for $k = 1,\ M = 8$

Table 3: Absolute errors (Hermite Wavelets) of Example 5.2 for $k = 1$ and $M = 8$

$t/16$	θ_1	θ_2	θ_3
1	6.9365e-005	9.5154e-005	3.9129e-005
3	2.0588e-005	2.8636e-005	1.1928e-005
5	1.4262e-005	1.9972e-005	8.3713e-006
7	5.8316e-006	8.3690e-006	3.5851e-006
9	4.2291e-006	6.0961e-006	2.6214e-006
11	5.2473e-007	9.4938e-007	4.7970e-007
13	3.9793e-006	5.6244e-006	2.3773e-006
15	1.5856e-005	2.1617e-005	8.8368e-006

Table 4: Absolute errors (Haar Wavelets) of Example 5.2 for $J = 2$

$t/16$	θ_1	θ_2	θ_3
1	1.1063e-002	4.3032e-002	3.0289e-002
3	7.2766e-003	2.7322e-003	7.4873e-003
5	8.7569e-003	7.9591e-003	5.6269e-004
7	5.5781e-003	8.2070e-003	2.6952e-003
9	2.1493e-003	5.6751e-003	2.6966e-003
11	2.9179e-004	3.0972e-003	2.1003e-003
13	1.6712e-003	1.2187e-003	1.4683e-003
15	2.2525e-003	6.5114e-005	9.6613e-004

Figure 2 depicts the contrast of exact solution with solution by Hermite wavelets for Example 5.2 with $k = 1, M = 8$, and $\omega_0 = 1$. Table 3 gives the absolute errors for Example 5.2 by implementation of the Hermite wavelet method and taking $k = 1, M = 8$, and $\omega_0 = 1$. Table 4 presents the absolute errors for Example 5.2 with the Haar wavelets method as presented in Berwal et al. (2013) and taking $J = 2$ and $\omega_0 = 1$. Results show that Hermite wavelets based collocation method gives better results in comparison to Haar wavelets based collocation method as discussed in Berwal et al. (2013).

Example 5.3.

A mechanical system with two degrees of freedom satisfies the equations

$$2x''(t) + 3y'(t) = 4,$$
$$2y''(t) - 3x'(t) = 0,$$

subject to the initial conditions

$$x(0) = x'(0) = y(0) = y'(0) = 0.$$

The exact solution of this system is given by:

$$x(t) = \frac{8}{9}\left(1 - \cos\frac{3t}{2}\right), \quad y(t) = \frac{4}{3}t - \frac{8}{9}\sin\frac{3t}{2}.$$

We assume that

$$x''(t) = \sum_{p=1}^{2^{k-1}} \sum_{q=0}^{M-1} \left(A_{p,q}\right)_1 \Phi_{p,q}(t) = A_1^T \Phi, \tag{27}$$

and

$$y''(t) = \sum_{p=1}^{2^{k-1}} \sum_{q=0}^{M-1} \left(A_{p,q}\right)_2 \Phi_{p,q}(t) = A_2^T \Phi. \tag{28}$$

Integrating the above two equations (27) and (28) with respect to t, from 0 to t, we attain:

$$x'(t) = x'(0) + \sum_{p=1}^{2^{k-1}} \sum_{q=0}^{M-1} \left(A_{p,q}\right)_1 P_{p,q}(t) = A_1^T P, \tag{29}$$

and

$$y'(t) = y'(0) + \sum_{p=1}^{2^{k-1}} \sum_{q=0}^{M-1} \left(A_{p,q}\right)_2 P_{p,q}(t) = A_2^T P. \tag{30}$$

Integrating these two equations (29) and (30) with respect to t, from 0 to t and applying initial conditions, we obtain:

$$x(t) = x(0) + tx'(0) + \sum_{p=1}^{2^{k-1}} \sum_{q=0}^{M-1} \left(A_{p,q}\right)_1 Q_{p,q}(t) = A_1^T Q, \tag{31}$$

and

$$y(t) = y(0) + ty'(0) + \sum_{p=1}^{2^{k-1}} \sum_{q=0}^{M-1} \left(A_{p,q}\right)_2 Q_{p,q}(t) = A_2^T Q. \tag{32}$$

Substituting all the values from (27) - (30) in the given system of equations, we obtain:

$$2A_1^T \Phi + 3A_2^T P = 4,$$
$$2A_2^T \Phi - 3A_1^T P = 0.$$

Solving these two equations, we obtain the wavelet coefficients A_1 and A_2. By putting the values of the wavelet coefficients in (31) and (32), we get values of $x(t)$ and $\mathrm{y}(t)$ and hence the required Hermite wavelet solution.

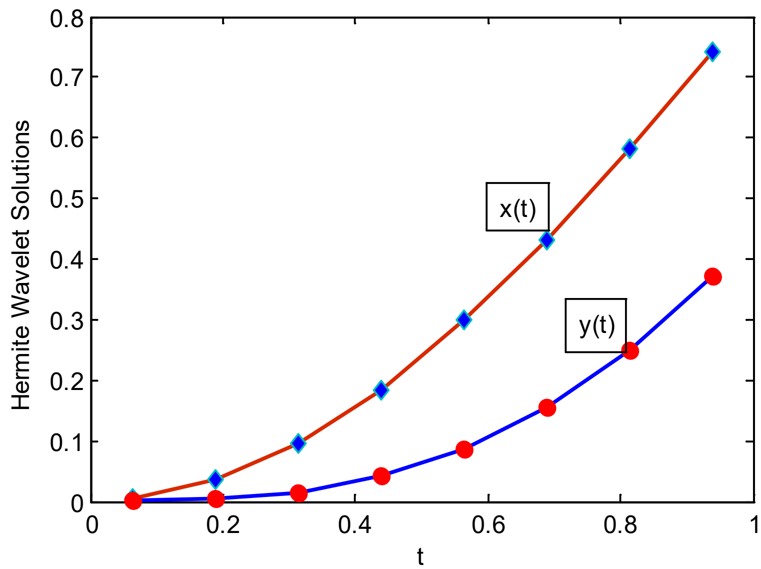

Figure 3: Comparison of exact and Hermite wavelets solutions of Example 5.3 for $k = 1,\ M = 8$

Table 5: Absolute errors (Hermite Wavelets) of Example 5.3 for $k = 1, M = 8$

$t/16$	$x(t)$	$y(t)$
1	4.2577e-010	4.2705e-010
3	1.2665e-009	1.5540e-009
5	1.8050e-009	2.7162e-009
7	2.1289e-009	3.9751e-009
9	2.2107e-009	5.2684e-009
11	2.0458e-009	6.5520e-009
13	1.6591e-009	7.7953e-009
15	9.5411e-010	8.8553e-009

Table 6: Absolute errors (Haar Wavelets) of Example 5.3 for $J = 2$

$t/16$	$x(t)$	$y(t)$
1	3.1173e-005	2.4100e-004
3	2.1011e-004	6.9768e-004
5	5.5544e-004	1.0799e-003
7	1.0427e-003	1.3427e-003
9	1.6369e-003	1.4485e-003
11	2.2943e-003	1.3686e-003
13	2.9646e-003	1.0859e-003
15	3.5939e-003	5.9570e-j004

Figure 3 depicts the comparability of the exact solution with Hermite wavelet solutions for Example 5.3 when $k = 1, M = 8$. Table 5 presents the absolute error of Example 5.3 using the Hermite wavelets method and taking $k = 1, M = 8$. Table 6 gives the absolute error for Example 5.3 with the Haar wavelets method as discussed in Berwal et al. (2013), by taking $J = 2$.

All the numerical experiments and graphical representations were computed in double precision with some MATLAB codes on a personal computer System Vostro 1400 Processor.

6. Stability Analysis

There are many ways to describe the idea of stability analysis. Clearly a computation is stable if it does not "blow up". In this section, we discuss the concept of stability of the numerical computations using the procedure based on Patra & Ray (2014). The stability criterion in this analysis is related with 1% of the relative error, where

$$\text{Relative error} = \left|\frac{u_{approximate} - u_{exact}}{u_{approximate}}\right| \times 100 \leq 1\% .$$

The values of relative errors for Example 5.1, Example 5.2 and Example 5.3 are less than 1%. Therefore, the proposed method is stable.

7. Conclusion

The computational and graphical representation of numerical results of the above illustrative examples exhibits that Hermite wavelets provide an efficient numerical technique to find solutions of systems of linear differential equations. It is also concluded that solutions obtained from Hermite wavelets are much better as compared to those derived by using Haar wavelets (as discussed in Berwal et al. (2013)). For future scope, the proposed method can be extended to attain the approximate solutions of systems of nonlinear differential equations as well as two and three-dimensional systems of differential equations. The presented work can prove to be very beneficial for the advancement of research leading to the expansion of subjects of numerical analysis and differential equations.

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