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Analysis of $M^{[X_1]}, M^{[X_2]}/G_1, G_2^{(a,b)}/1$ Queue with Priority Services, Server Breakdown, Repair, Modified Bernoulli Vacation, Immediate Feedback

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Abstract

In this investigation, the steady state analysis of two individualistic batch arrival queues with immediate feedback, modified Bernoulli vacation and server breakdown are introduced. Two different categories of customers like priority and ordinary are to be considered. This model propose non-preemptive priority discipline. Ordinary and priority customers arrive as per Poisson processes. The server consistently afford single service for priority customers and the general bulk service for the ordinary customers and the service follows general distribution. The ordinary customers to be served only if the batch size should be greater than or equal to "a", else the server should not start service until "a" customers have accumulated. Meanwhile priority queue is empty; the server becomes idle or go for vacation. If server gets breakdown while the priority customers are being served, they may wait in the head of the queue and get fresh service after repair completion, but in case of ordinary customers they may leave the system. After completion of each priority service, customer may rejoin the system as a feedback customer for receiving regular service because of inappropriate quality of service. Supplementary variable technique and probability generating function are generally used to solve the Laplace transforms of time-dependent probabilities of system states. Finally, some performance measures are evaluated and express the numerical results.

Keywords: Batch arrivals; Bulk service; Priority queues; Breakdown; Immediate feedback; Modified Bernoulli vacation

MSC 2010 No.: 60K25, 68M30, 90B22

1. Introduction

Hypothesis of priority queue has been done in communication strategies for the past two decades. Priority is different from a normal queue because it does not comes under FCFS. It is a special type of queue in which each customers are dealt with priority and served according to its priority. Preemptive and non-preemptive are the two different types of priority service which offered in queueing system. Ordinary customers will be served when arriving priority customers wait until the service is completed. This belongs to non-preemptive priority rule. In case of preemptive priority customers will constantly interrupt the ordinary service. Gao (2015) examined two classes of customers for general retrials with preemptive priority service. Rajadurai et al. (2016) considered working vacations and vacation interruption for $M/G/1$ retrial queue with preemptive priority service. Subramanian et al. (2009) investigated $M/M/1$ retrial queue with negative arrival under non-preemptive priority service. Jeganathan et al. (2013) have discussed non-preemptive priority service for retrials with inventory system. Krishnamoorthy and Divya (2018) investigated $(M,M)/(PH,PH)/1$ non-preemptive priority queue with working interruption. Bhagat (2020) described multi server non-preemptive priority service, which is susceptible breakdown and repairs. Additionally, discretionary priority service is used in which both disciplines has been considered. Cho and Kwan (1993) explored combined preemptive/non-preemptive priority discipline in $M/G/1$ queueing system. Fajardo and Drekic (2016) observed general mixed priority, which depends on server discretion. Ayyappan and Somasundaram (2019) analysed discretionary priority service in $M^{[X_1]}, M^{[X_2]}/G_1, G_2/1$ retrial queue. Haghighi (1998) explained queueing models in industry and business.

The concept of bulk service has gained tremendous significance in queueing system. The server offers services in batches, the batch size also vary from "a" to "b". Bulk queues have wide range of applications in manufacturing systems, elevators, transport vehicles, etc. Neuts (1967) define the general class of bulk queues with Poisson input. Holman and Chaudhry (1981) discussed bulk service and its consequences. The book by Chaudhry and Templeton (1983) explained bulk service queueing systems in detail. Haghighi and Mishev (2013) determined three possible stages for the handling of job applications in a hiring process as a network queuing model. Chakravarthy (1993) explained a finite $MAP/G/1$ queue with group services. Jeyakumar and Senthilnathan (2016) investigated single server batch arrival and general bulk service queueing system with multiple working vacations.

In actual existence, customers may be severed frequently for some specific reason. Until being satisfied the customer will get the service from the server, which is known as immediate feedback. Haghighi (1998) examined parallel multi server queueing system with task-splitting and feedback service. Choi et al. (2000) investigated a non-Markovian queue with multi class of customers as well as feedback service provided by the server. Rakesh Kumar and Soodan (2019) analysed a single server queuing model with correlated reneging, balking and feedback customers. Zadeh and Shahkar (2008) explored Bernoulli feedback and Bernoulli vacation for two phases of queue. Azhagappan and Deepa (2020) have discussed interrupted closedown balking and feedback customers for $M/M/1$ queue.

Due to various reasons like maintenance, working at other queue or simply taking a break, the server would not be available for some period. These periods of time is mentioned as vacation, which is introduced by Keilson and Servi (1986). Krishnakumar and Arivudainambi (2002) investigated Bernoulli vacation schedule for $M/G/1$ retrial queue. Two phases of batch arrival including vacation time which is examined by Choudhury and Madan (2004) under Bernoulli vacation schedule. Jose and Beena (2020) explained multi server retrial inventory system with vacation policy. Ayyappan et al. (2021) analysed single server priority retrial queue with balking, Bernoulli vacation, working breakdown and admission control policy.

In this paper, we discuss two different categories of customers like priority and ordinary with immediate feedback, breakdown, modified Bernoulli vacation, followed by non-preemptive priority discipline. Server provide single service and bulk service for priority and ordinary customers, respectively. When priority queue becomes empty then the server begins service with ordinary customers under general bulk service rule. If the length of the ordinary queue is less than "a" then the server becomes idle or it goes for vacation under modified Bernoulli vacation schedule. Meanwhile the server afford service to priority customers, abruptly the server gets breakdown. On that occasion priority customer wait in the head of the queue and get fresh service after repair completion, but in case of ordinary customers leave the system permanently. In addition, we consider feedback service only for high priority customers.

The manuscript of this work is synchronized as follows. Section 2 contains a brief explanation of our model. The governing equations have been enlisted in Section 3. Steady-State analysis of our model comprises in Section 4. In Section 5, some performance measures has been enrolled. Particular cases has been obtained in Section 6. Section 7 exhibits numerical and graphical result.

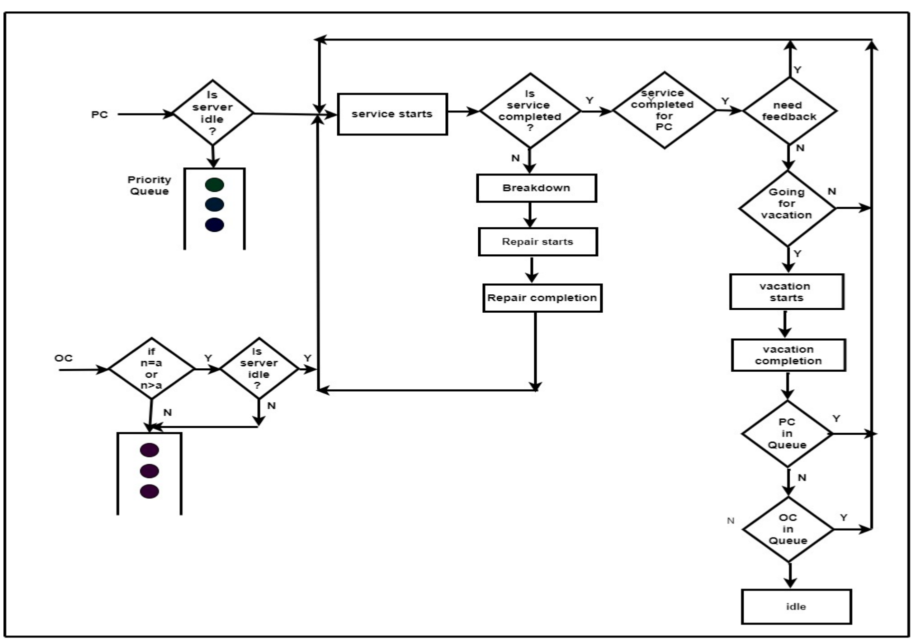

Figure 1. Schematic representation

2. Description of the Model

(1) **Arrival Mechanism:** Two different types of customers arrive in batches with independent Poisson compound process. Let $\lambda_1, \lambda_2 > 0$ be the arrival rate for priority and ordinary customers respectively. Assume that the first order probabilities for priority and ordinary customers $\lambda_1 c_i dt$ $(i = 1, 2, 3, ...)$ and $\lambda_2 c_j dt$ $(j = 1, 2, 3, ...)$ with batch size i and j customers arrive at the system during a short interval of time $(t, t + dt)$. Here, $0 \leq c_i \leq 1$, $\sum_{i=1}^{\infty} c_i = 1$, $0 \leq c_j \leq 1$, $\sum_{j=1}^{\infty} c_j = 1$.

(2) **Service Mechanism:** Priority and ordinary customers ordinate in batches with distinct queues. Server renders single service for priority customers and general bulk service for ordinary customers with service rate $\mu_i(\nu)$, $i = 1, 2$, respectively. Server initiate service for ordinary customers, if the priority queue becomes empty as well as the ordinary customers have queue size minimum "a". Additionally this model follows a non-preemptive priority discipline.

(3) **Breakdown and repair:** If server gets breakdown while priority customers being served, the same priority customers wait in the same queue until repair completion and get new service, but in case of low priority customers leave the system. The breakdown follows exponential distribution with rate α. The breakdown server immediately send off for repair, and repair follows general distribution with rate $\eta(\nu)$.

(4) **Feedback:** Feedback service is being served only for priority customers. After completion of service for each customer in priority queue. The positive customer is annoyed with their service, either they may join the queue and earn re-service with probability r, $0 \leq r \leq 1$ or may leave the system with probability $1 - r$.

(5) **Vacation:** Whenever the server completes each priority service, it leaves the system for vacation with probability θ, otherwise next customer will be served with probability $(1 - \theta)$.

3. Analysis of Queue Size Distribution

Pursuit of this section is to constructing the governing equations. On account of non-Markovian queueing system, supplementary variable and probability generating function have been used to solve this model.

To obtain a bivariate Markov process $\{N_1(t), N_2(t), Y(t), t > 0\}$, make use of the supplementary variables $B_i^0(t)$, $V^0(t)$ and $R^0(t)$. Here, $N_1(t)$, $N_2(t)$ indicates the length of priority and ordinary queue at time t. $B_i^0(t)$ for $i = 1, 2$, $V^0(t)$, $R^0(t)$ indicates elapsed service time for priority and ordinary customers, vacation and repair at time t.

The status of the server is denoted by $Y(t)$, where its values are as follows:

$$Y(t) = \begin{cases} 0, & \text{If the server is idle,} \\ 1, & \text{If the server is busy with priority customers,} \\ 2, & \text{If the server is busy with ordinary customers,} \\ 3, & \text{If the server is on vacation,} \\ 4, & \text{If the server is in repair at time } t. \end{cases}$$

Also, $\mu_1(\nu)$, $\mu_2(\nu)$, $\gamma(\nu)$ and $\eta(\nu)$ are the hazard rate for priority and ordinary customer, vacation and repair:

$$\mu_i(\nu) = \frac{dB_i(\nu)}{1 - B_i(\nu)}, i = 1, 2; \quad \gamma(\nu) = \frac{dV(\nu)}{1 - V(\nu)}; \quad \eta(\nu) = \frac{dR(\nu)}{1 - R(\nu)}.$$

The probability densities are as follows:

$$I_{0,k}(t) = \Pr\{N_1(t) = 0, N_2(t) = k, Y(t) = 0\}, \text{ for } \quad t \geq 0, 0 \leq k \leq a - 1,$$

$$P_{m,n}(\nu, t)d\nu = \Pr\{N_1(t) = m, N_2(t) = n, Y(t) = 1; \nu \leq B_1^0(t) \leq \nu + d\nu\},$$

$$Q_{m,n}(\nu, t)d\nu = \Pr\{N_1(t) = m, N_2(t) = n, Y(t) = 2; \nu \leq B_2^0(t) \leq \nu + d\nu\},$$

$$V_{m,n}(\nu, t)d\nu = \Pr\{N_1(t) = m, N_2(t) = n, Y(t) = 3; \nu \leq V^0(t) \leq \nu + d\nu\},$$

$$R_{m,n}(\nu, t)d\nu = \Pr\{N_1(t) = m, N_2(t) = n, Y(t) = 4; \nu \leq R^0(t) \leq \nu + d\nu\},$$

for $\nu \geq 0,\ t \geq 0,\ m \geq 0$ and $n \geq 0$.

4. Equations Governing the System

Idle state of the server

$$\begin{aligned} \frac{d}{dt} I_{0,n}(t) = &-(\lambda_1 + \lambda_2) I_{0,n}(t) + (1 - \delta_{0n}) \lambda_2 c_k \sum_{k=1}^{n} I_{0,(n-k)}(t) \\ &+ (1 - \theta) \int_0^\infty Q_{0,n}(\nu, t) \mu_2(\nu) d\nu + (1 - \theta)(1 - r) \int_0^\infty P_{0,n}(\nu, t) \mu_1(\nu) d\nu \\ &+ \int_0^\infty R_{0,n}(\nu, t) \eta(\nu) d\nu + \int_0^\infty V_{0,n}(\nu, t) \gamma(\nu) d\nu \\ &\text{for } \ 0 \leq n \leq a - 1. \end{aligned} \tag{1}$$

Server busy with priority queue

$$\begin{aligned}\frac{\partial}{\partial t}P_{m,n}(\nu,t)+\frac{\partial}{\partial \nu}P_{m,n}(\nu,t)=&-(\lambda_1+\lambda_2+\alpha+\mu_1(\nu))P_{m,n}(\nu,t)\\&+\lambda_1(1-\delta_{0m})\sum_{i=1}^{m}c_iP_{m-i,n}(\nu,t)\\&+\lambda_2(1-\delta_{0n})\sum_{j=1}^{n}c_jP_{m,n-j}(\nu,t)\quad \text{for } m,n\geq 0.\end{aligned}\tag{2}$$

Server busy with ordinary queue

$$\begin{aligned}\frac{\partial}{\partial t}Q_{m,n}(\nu,t)+\frac{\partial}{\partial \nu}Q_{m,n}(\nu,t)=&-(\lambda_1+\lambda_2+\alpha+\mu_2(\nu))Q_{m,n}(\nu,t)\\&+\lambda_1(1-\delta_{0m})\sum_{i=1}^{m}c_iQ_{m-i,n}(\nu,t)\\&+\lambda_2(1-\delta_{0n})\sum_{j=1}^{n}c_jQ_{m,n-j}(\nu,t)\quad \text{for } m,n\geq 0.\end{aligned}\tag{3}$$

The server during vacation

$$\begin{aligned}\frac{\partial}{\partial t}V_{m,n}(\nu,t)+\frac{\partial}{\partial \nu}V_{m,n}(\nu,t)=&-(\lambda_1+\lambda_2+\alpha+\gamma(\nu))V_{m,n}(\nu,t)\\&+\lambda_1(1-\delta_{0m})\sum_{i=1}^{m}c_iV_{m-i,n}(\nu,t)\\&+\lambda_2(1-\delta_{0n})\sum_{j=1}^{n}c_jV_{m,n-j}(\nu,t)\quad \text{for } m,n\geq 0.\end{aligned}\tag{4}$$

The server under repair process for priority

$$\begin{aligned}\frac{\partial}{\partial t}R_{m,n}(\nu,t)+\frac{\partial}{\partial \nu}R_{m,n}(\nu,t)=&-(\lambda_1+\lambda_2+\eta(\nu))R_{m,n}(\nu,t)\\&+\lambda_1(1-\delta_{0m})\sum_{i=1}^{m}c_iR_{m-i,n}(\nu,t)\\&+\lambda_2(1-\delta_{0n})\sum_{j=1}^{n}c_jR_{m,n-j}(\nu,t)\quad \text{for } m,n\geq 0.\end{aligned}\tag{5}$$

Define the boundary conditions at $\nu=0$,

$$\begin{aligned}P_{m,n}(0,t)=&\lambda_1c_{m+1}I_{0,n}(t)+(1-\theta)(1-r)\int_0^\infty P_{m+1,n}(\nu,t)\mu_1(\nu)d\nu\\&+r\int_0^\infty P_{m,n}(\nu,t)\mu_1(\nu)d\nu+(1-\theta)\int_0^\infty Q_{m+1,n}(\nu,t)\mu_2(\nu)d\nu\\&+\int_0^\infty V_{m+1,n}(\nu,t)\gamma(\nu)d\nu+\int_0^\infty R_{m+1,n}(\nu,t)\eta(\nu)d\nu,\end{aligned}\tag{6}$$

$$\begin{aligned} Q_{0,0}(0,t) &= \lambda_2 \sum_{n=a}^{b}\sum_{k=0}^{a-1} c_{n-k} I_{0,k}(t) + (1-\theta)(1-r)\sum_{n=a}^{b}\int_0^\infty P_{0,n}(\nu,t)\mu_1(\nu)d\nu \\ &+ \sum_{n=a}^{b}(1-\theta)\int_0^\infty Q_{0,n}(\nu,t)\mu_2(\nu)d\nu + \sum_{n=a}^{b}\int_0^\infty V_{0,n}(\nu,t)\gamma(\nu)d\nu \\ &+ \sum_{n=a}^{b}\int_0^\infty R_{0,n}(\nu,t)\eta(\nu)d\nu, \end{aligned} \tag{7}$$

$$\begin{aligned} Q_{0,n}(0,t) &= \lambda_2 \sum_{n=a}^{b}\sum_{k=0}^{a-1} c_{b+n-k} I_{0,k}(t) + \sum_{n=a}^{b}\int_0^\infty V_{0,b+n}(\nu,t)\gamma(\nu)d\nu \\ &+ \sum_{n=a}^{b}(1-\theta)\int_0^\infty Q_{0,b+n}(\nu,t)\mu_2(\nu)d\nu + \sum_{n=a}^{b}\int_0^\infty R_{0,b+n}(\nu,t)\eta(\nu)d\nu \\ &+ (1-\theta)(1-r)\sum_{n=a}^{b}\int_0^\infty P_{0,b+n}(\nu,t)\mu_1(\nu)d\nu, \end{aligned} \tag{8}$$

$$V_{m,n}(0,t) = \theta(1-r)\int_0^\infty P_{m,n}(\nu,t)\mu_1(\nu)d\nu + \theta\int_0^\infty Q_{m,n}(\nu,t)\mu_2 d\nu, \tag{9}$$

$$R_{m,n}(0,t) = (1-\delta_{0m})\alpha\int_0^\infty P_{m-1,n}(\nu,t)d\nu + \alpha\int_0^\infty Q_{m,n}(\nu,t)d\nu. \tag{10}$$

The initial conditions are,

$$P_{m,n}(0) = R_{m,n}(0) = Q_{m,n}(0) = V_{m,n}(0) = 0,\ \ m,n \geq 0, \quad I_{0,0}(0) = 1,$$

and

$$I_{0,n}(t) = 0,\ \text{where } n \geq a.$$

Now we define the Probability Generating Functions (PGF),

$$\begin{aligned} I(t,z_l) &= \sum_{n=0}^{a-1} z_l^n I_{0,n}(t), \quad A(\nu,t,z_h,z_l) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} z_h^m z_l^n A_{m,n}(\nu,t), \\ A(\nu,t,z_h) &= \sum_{m=0}^{\infty} z_h^m A_m(\nu,t), \quad A(\nu,t,z_l) = \sum_{n=0}^{\infty} z_l^n A_n(\nu,t), \end{aligned} \tag{11}$$

here $A = P,\ Q,\ V,\ R.$

From Equations (1) to (10), by applying Laplace transforms and using Equation (11), we obtain

$$\overline{P}(\nu,s,z_h,z_l) = \overline{P}(0,s,z_h,z_l)e^{-\phi_1(s,z)\nu - \int_0^\nu \mu_1(t)dt}, \tag{12}$$

$$\overline{Q}(\nu,s,z_h,z_l) = \overline{Q}(0,s,z_h,z_l)e^{-\phi_1(s,z)\nu - \int_0^\nu \mu_2(t)dt}, \tag{13}$$

$$\overline{V}(\nu,s,z_h,z_l) = \overline{V}(0,s,z_h,z_l)e^{-\phi_2(s,z)\nu - \int_0^\nu \gamma(t)dt}, \tag{14}$$

$$\overline{R}(\nu,s,z_h,z_l) = \overline{R}(\nu,0,s,z_h,z_l)e^{-\phi_2(s,z)\nu - \int_0^\nu \eta(t)dt} \text{ at time } t, \tag{15}$$

where,

$$\phi_1(s,z) = s + \lambda_1(1 - C(z_h)) + \lambda_2(1 - C(z_l)) + \alpha,$$
$$\phi_2(s,z) = s + \lambda_1(1 - C(z_h)) + \lambda_2(1 - C(z_l)) \text{ at time t,}$$

$$\overline{Q}(0,s,z_l) = \frac{\left\{\begin{array}{l} \lambda_1 C(g(z_l)) \sum_{k=0}^{a-1} \overline{I}_{0,k}(s) + \lambda_2 \sum_{k=0}^{a-1} \sum_{n=1}^{b-k-1} c_n \overline{I}_{0,k}(s)(z_l^b - z_l^{n+k}) \\ + \lambda_2 C(z_l) \sum_{k=0}^{a-1} z_l^k \overline{I}_{0,k}(s) - z_l^b \big[\sum_{k=0}^{a-1} (s + \lambda_1 + \lambda_2) \overline{I}_{0,k}(s) - 1 \big] \\ + \int_0^\infty \sum_{k=0}^{b-1} (z_l^b - z_l^k)[(1-\theta)(1-r)\overline{P}_{0,k}(\mu_1(\nu)) \\ + (1-\theta)\overline{Q}_{0,k}(\mu_2(\nu)) + \overline{V}_{0,k}(\gamma(\nu)) + \overline{R}_{0,k}(\eta(\nu))] d\nu \end{array}\right\}}{\left\{\begin{array}{l} z_l^b - \{(1-\theta)\overline{B}_2(\sigma_1(s,z)) + \theta \overline{B}_2(\sigma_1(s,z))\overline{V}(\sigma_2(s,z)) \\ + \alpha \big[\frac{1 - \overline{B}_2(\sigma_1(s,z))}{\sigma_1(s,z)} \big] \overline{R}(\sigma_2(s,z))\} \end{array}\right\}}, \tag{16}$$

$$\overline{P}(0,s,z_h,z_l) = \frac{\left\{\begin{array}{l} \lambda_1 [C(z_h) - C(g(z_l))] \sum_{k=0}^{a-1} \overline{I}_{0,k}(s) + \{(1-\theta)[\overline{B}_2(\phi_1(s,z)) \\ - \overline{B}_2(\sigma_1(s,z))] + \theta[\overline{B}_2(\phi_1(s,z))\overline{V}(\phi_2(s,z)) \\ - \overline{B}_2(\sigma_1(s,z))\overline{V}(\sigma_2(s,z))] + \alpha \big[\frac{1 - \overline{B}_2(\phi_1(s,z))}{\phi_1(s,z)} \big] \overline{R}(\phi_2(s,z)) \\ - \alpha \big[\frac{1 - \overline{B}_2(\sigma_1(s,z))}{\sigma_1(s,z)} \big] \overline{R}(\sigma_2(s,z))\} \overline{Q}(0,s,z_l) \end{array}\right\}}{\left\{\begin{array}{l} z_h - \{[(1-\theta)(1-r) + z_h r + \theta(1-r)\overline{V}(\phi_2(s,z))]\overline{B}_1(\phi_1(s,z)) \\ + \alpha z_h \big[\frac{1 - \overline{B}_1(\phi_1(s,z))}{\phi_1(s,z)} \big] \overline{R}(\phi_2(s,z))\} \end{array}\right\}}, \tag{17}$$

$$\overline{V}(0,s,z_h,z_l) = \theta(1-r)\overline{P}(0,s,z_h,z_l)\overline{B}_1(\phi_1(s,z)) + \theta \overline{Q}(0,s,z_l)\overline{B}_2(\phi_1(s,z)), \tag{18}$$

$$\overline{R}(0,s,z_h,z_l) = \alpha z_h \overline{P}(0,s,z_h,z_l)\big[\frac{1 - \overline{B}_1(\phi_1(s,z))}{\phi_1(s,z)}\big] + \alpha \overline{Q}(0,s,z_l)\big[\frac{1 - \overline{B}_2(\phi_1(s,z))}{\phi_1(s,z)}\big]. \tag{19}$$

Theorem 4.1.

When the system is in standard service, breakdown, repair and vacation by using the Laplace transforms the Probability Generating Functions (PGF) of the number of customers in the respective queue is given by

$$\overline{P}(s,z_h,z_l)=\overline{P}(0,s,z_h,z_l)\left[\frac{1-\overline{B}_1(\phi_1(s,z))}{\phi_1(s,z)}\right], \tag{20}$$
$$\overline{Q}(s,z_h,z_l)=\overline{Q}(0,s,z_l)\left[\frac{1-\overline{B}_2(\phi_1(s,z))}{\phi_1(s,z)}\right], \tag{21}$$
$$\overline{V}(s,z_h,z_l)=\overline{V}(0,s,z_h,z_l)\left[\frac{1-\overline{V}(\phi_2(s,z))}{\phi_2(s,z)}\right], \tag{22}$$
$$\overline{R}(s,z_h,z_l)=\overline{R}(0,s,z_h,z_l)\left[\frac{1-\overline{R}(\phi_2(s,z))}{\phi_2(s,z)}\right]. \tag{23}$$

Proof:

Integrate the preceeding Equations (12) to (15) with respect to ν and apply the solution of renewal theory,

$$\int_0^\infty \big[1-H(\nu)\big]e^{-s\nu}d\nu=\frac{1-\overline{h(s)}}{s}. \tag{24}$$

Here, the LST of the distribution function H(ν) is denoted as $\overline{h(s)}$, we acquire the results from (20) to (23). Hence, we receive the absolute outcome of the probability generating functions (PGF) for the successive states, $\overline{P}(s,z_h,z_l)$, $\overline{Q}(s,z_h,z_l)$, $\overline{V}(s,z_h,z_l)$, and $\overline{R}(s,z_h,z_l)$. ■

5. Steady State Analysis

Using Tauberian property,

$$\lim_{s\to 0} s\overline{f}(s)=\lim_{t\to\infty} f(t).$$

Despite of the state of the system, the probability generating function (PGF) of the queue size is

$$W_q(z_h,z_l)=\frac{Nr(z_h,z_l)}{Dr(z_h,z_l)}, \tag{25}$$

where

$$Nr(z_h,z_l)=N_2(z)F_1(z)D_1(z)+N_1(z)F_2(z)D_2(z)+I(D_1(z)D_2(z)\phi_1(z)\phi_2(z)),$$
$$Dr(z_h,z_l)=D_1(z)D_2(z)\phi_1(z)\phi_2(z),$$

here,

$$
\begin{aligned}
N_2(z) = {} & \lambda_1[C(z_h) - C(g(z_l))] \sum_{k=0}^{a-1} \overline{I}_{0,k}(z_l^b - [(1-\theta)\overline{B}_2(\sigma_1(z)) + \theta\overline{B}_2(\sigma_1(z))\overline{V}(\sigma_2(z)) \\
& + \alpha \left[\frac{1-\overline{B}_2(\sigma_1(z))}{\sigma_1(z)}\right] \overline{R}(\sigma_1(z))]) + [(1-\theta)[\overline{B}_2(\phi_1(z)) - \overline{B}_2(\sigma_1(z))] \\
& + \theta[\overline{B}_2(\phi_1(z))\overline{V}(\phi_2(z)) - \overline{B}_2(\sigma_1(z))\overline{V}(\sigma_2(z))] \\
& + \alpha \left[\frac{1-\overline{B}_2(\phi_1(z))}{\phi_1(z)}\right] \overline{R}(\phi_2(z)) - \alpha \left[\frac{1-\overline{B}_2(\sigma_1(z))}{\sigma_1(z)}\right] \overline{R}(\sigma_2(z))] \\
& [\lambda_1(C(g(z_l)) - z_l^b) \sum_{k=0}^{a-1} \overline{I}_{0,k} + \lambda_2 \sum_{k=0}^{a-1} \sum_{n=a-1}^{b-k-1} C_n \overline{I}_{0,k}(z_l^b - z_l^{n+k}) \\
& + \lambda_2 \sum_{k=0}^{a-1} (C(z_l)z_l^k - z_l^b)\overline{I}_{0,k} + \sum_{k=0}^{b-1}(z_l^b - z_l^k)W_{0,k}],
\end{aligned}
$$

$$
\begin{aligned}
F_1(z) = {} & (1-\overline{B}_1(\phi_1(z)))(\phi_2(z)) + \theta(1-r)\overline{B}_1(\phi_1(z))(1-\overline{v}(\phi_2(z)))(\phi_1(z)) \\
& + \alpha z_h(1-\overline{B}_1(\phi_1(z)))(1-\overline{R}(\phi_2(z))),
\end{aligned}
$$

$$
\begin{aligned}
D_2(z) = {} & [z_l^b - \{(1-\theta)\overline{B}_2(\sigma_1(z)) + \theta\overline{B}_2(\sigma_1(z))\overline{V}(\sigma_2(z)) + \alpha \left[\frac{1-\overline{B}_2(\sigma_1(z))}{\sigma_1(z)}\right] \overline{R}(\sigma_2(z))\}] \\
& [z_h - \{[(1-\theta)(1-r) + z_h r + \theta(1-r)\overline{V}(\phi_2(z))]\overline{B}_1(\phi_1(z)) \\
& + \alpha z_h \left[\frac{1-\overline{B}_1(\phi_1(z))}{\phi_1(z)}\right] \overline{R}(\phi_2(z))\}],
\end{aligned}
$$

$$
\begin{aligned}
N_1(z) = {} & \lambda_1(C(g(z_l)) - z_l^b) \sum_{k=0}^{a-1} \overline{I}_{0,k} + \lambda_2 \sum_{k=0}^{a-1} \sum_{n=a-1}^{b-k-1} c_n \overline{I}_{0,k}(z_l^b - z_l^{n+k}) \\
& + \lambda_2 \sum_{k=0}^{a-1} (C(z_l)z_l^k - z_2^b)\overline{I}_{0,k} + \sum_{k=0}^{b-1}(z_l^b - z_l^k)W_{0,k},
\end{aligned}
$$

$$
\begin{aligned}
F_2(z) = {} & (1-\overline{B}_2(\phi_1(z)))\phi_2(z) + \theta\overline{B}_2(\phi_1(z))(1-\overline{v}(\phi_2(z)))(\phi_1(z)) \\
& + \alpha(1-\overline{B}_2(\phi_1(z)))(1-\overline{R}(\phi_2(z))),
\end{aligned}
$$

$$
D_1(z) = \left[z_l^b - (1-\theta)\overline{B}_2(\sigma_1(z)) + \theta\overline{B}_2(\sigma_1(z))\overline{V}(\sigma_2(z)) + \alpha \left[\frac{1-\overline{B}_2(\sigma_1(z))}{\sigma_1(z)}\right] \overline{R}(\sigma_2(z))\right],
$$

$$
\begin{aligned}
W_{0,k} = {} & \sum_{k=0}^{a-1}[(1-\theta)(1-r)\int_0^\infty \overline{P}_{0,k}(\nu)\mu_1(\nu)d\nu + (1-\theta)\int_0^\infty \overline{Q}_{0,k}(\nu)\mu_2(\nu)d\nu \\
& + \int_0^\infty \overline{V}_{0,k}(\nu)\gamma(\nu)d\nu + \int_0^\infty \overline{R}_{0,k}(\nu)\eta(\nu)d\nu.
\end{aligned}
$$

6. Stability condition

The probability generating function has to satisfy $W_q(1,1) = 1$ at time t, In order to satisfy this condition, apply L'Hopital's rule and equating the expression consecutively

$$N_2^{''}(1)F_1^{'}(1)D_1^{'}(1) + N_1^{'}(1)F_2^{'}(1)D_2^{''}(1) + I(D_1^{'}(1)D_2^{''}(1)\phi_1(1)\phi_2^{'}(1)) = D_1^{'}(1)D_2{''}(1)\phi_1(1)\phi_2^{'}(1),$$

where,

$$\begin{aligned}
N_2^{''}(1) = {} & 2\lambda_1 E(X)(1 - E(X_1)\sum_{k=0}^{a-1}\{b + (\theta\overline{B}_2(\alpha)E(V) - \left[\frac{1-\overline{B}_2(\alpha)}{\alpha}\right] + (1-\overline{B}_2(\alpha))E(R)) \\
& (\lambda_1 E(X_1) + \lambda_2)E(X)\} - 2[(\theta\overline{B}_2(\alpha)E(V) - (1-\overline{B}_2(\alpha))E(R) - \left[\frac{1-\overline{B}_2(\alpha)}{\alpha}\right]) \\
& (\lambda_1 + \lambda_2)E(X) + (\theta\overline{B}_2(\alpha)E(V) - (1-\overline{B}_2)E(R) - \left[\frac{1-\overline{B}_2(\alpha)}{\alpha}\right]) \\
& (\lambda_1 E(X_1) + \lambda_2)E(X)][\lambda_1\sum_{k=0}^{a-1} I_{0,k}(E(X)E(X_1) - b) \\
& + \lambda_2(E(X) + k + b) + \sum_{k=0}^{b-1}(b-k)W_{0,k} + \lambda_2\sum_{n=1}^{b-k-1} c_n(b-n-k)],
\end{aligned}$$

$$\begin{aligned}
D_2^{''}(1) = {} & 2[b + \left\{\theta\overline{B}_2(\alpha)E(V) + (1-\overline{B}_2(\alpha))E(R) - \left[\frac{1-\overline{B}_2(\alpha)}{\alpha}\right]\right\} \\
& (E(X_1)\lambda_1 + \lambda_2)E(X)][1 + \{r\overline{B}_1(\alpha) + \theta(1-r)\overline{B}_1(\alpha)E(V) + (1-\overline{B}_1(\alpha))E(R) \\
& - \left[\frac{1-\overline{B}_2(\alpha)}{\alpha}\right]\}(\lambda_1 + \lambda_2)E(X)],
\end{aligned}$$

$$\begin{aligned}
N_1^{'}(1) = {} & \lambda_1(E(X)E(X_1) - b)\sum_{k=0}^{a-1} I_{0,k} + \lambda_2\sum_{k=0}^{a-1}\sum_{n=1}^{b-n-k} c_n I_{0,k}(b-n-k) \\
& + \lambda_2\sum_{k=0}^{a-1} I_{0,k}(E(X) + k - b) + \sum_{k=0}^{b-1} W_{0,k},
\end{aligned}$$

$$D_1^{'}(1) = b + [1 - \overline{B}_2(\alpha) + \theta\overline{B}_2(\alpha)E(V) - \left[\frac{1-\overline{B}_2(\alpha)}{\alpha}\right]](\lambda_1 E(X_1) + \lambda_2)E(X),$$

$$F_1^{'}(1) = [(\alpha E(R) - 1)(\overline{B}_1(\alpha)) + \theta(1-r)E(V)(\overline{B}_1(\alpha))][(\lambda_1 + \lambda_2)E(X)],$$

$$F_2^{'}(1) = [(\alpha E(R) - 1)(\overline{B}_2(\alpha)) + \theta\alpha E(V)(\overline{B}_2(\alpha))][(\lambda_1 + \lambda_2)E(X)].$$

since $W_{0,k}$, $(k = 0, 1, 2, ..., b - 1)$ is a probability of "k" customers in the queue. It follows that left hand side of the above expression must be positive. Thus, $W_q(1, 1) = 1$, if $D_1'(z) D_2''(z) \phi_1(z) \phi_2'(z) > 0$. If

$$\rho = \frac{[\left(\frac{1-\overline{B}_2(\alpha)}{\alpha}\right) - (1 - \overline{B}_2(\alpha))E(R) - \theta \overline{B}_2(\alpha)E(V)](\lambda_1 E(X_1) + \lambda_2)E(X)}{b}, \text{ then } \rho < 1, \quad (26)$$

is the condition to be satisfied for existence of steady state of the proposed model. Equation (25) gives the (PGF) of the number of customers involving $b + a$ unknowns. We can express $W_{0,k}$ $k = 0, 1, 2, ..., a - 1$ as in terms of $I_{0,k}$, $k = 0, 1, 2, ..., a - 1$ such that the numerator has $b + 1$ constants. Now Equation (25) has $b + 1$ unknowns. By applying "Rouche's theorem", $[z_l^b - (1-\theta)\overline{B}_2(\sigma_1(z)) + \theta \overline{B}_2(\sigma_1(z))\overline{V}(\sigma_2(z)) + \alpha\left[\frac{1-\overline{B}_2(\sigma_1(z))}{\sigma_1(z)}\right]\overline{R}(\sigma_2(z))]D_1(z)\phi_1(z)\phi_2(z)$ has one zero on the boundary and the remaining "b" roots inside the unit circle. Due to analyticity of $W_q(z_h, z_l)$ the numerator must vanish at these points and gives $b + 1$ equations with $b + 1$ unknowns, which can be solved by numerical technique.

7. The Expected Queue Length

The expected queue length for priority customers queue,

$$L_{q_1} = \frac{d}{dz_h} W_q(z_h, 1)|_{z_h=1}, \quad (27)$$

and the expected queue length for ordinary customers queue,

$$L_{q_2} = \frac{d}{dz_l} W_q(1, z_l)|_{z_l=1}, \quad (28)$$

then

$$L_{q_1} = \frac{Dr''(1)Nr'''(1) - Dr'''(1)Nr''(1)}{3(Dr''(1))^2}, \quad (29)$$

$$L_{q_2} = \frac{Dr'''(1)Nr^{iv}(1) - Dr^{iv}(1)Nr'''(1)}{4(Dr'''(1))^2}, \quad \text{at time } t. \quad (30)$$

8. The Expected Waiting Time

The expected waiting time of priority customers queue is

$$W_{q_1} = \frac{L_{q_1}}{\lambda_1} \text{ at time } t. \quad (31)$$

The expected waiting time for ordinary customers queue is

$$W_{q_2} = \frac{L_{q_2}}{\lambda_2} \text{ at time } t. \quad (32)$$

9. Particular Cases

Case 1: In the absence of priority queue, server renders service to the customers without breakdown and feedback, that is $\lambda_1 = 0$, $\alpha = 0$, $r = 1$. Hence, the proposed model becomes $M^X/G^{(a,b)}/1$ queueing system with Bernoulli vacation,

$$W_q(z) = \frac{\left\{\begin{array}{l} [\lambda_2 \sum_{k=0}^{a-1}\sum_{n=a-1}^{b-k-1} C_n \overline{I}_{0,k}(z_l^b - z_l^{n+k}) + \lambda_2 \sum_{k=0}^{a-1}(C(z_l)z_l^k - z_l^b)\overline{I}_{0,k} + \sum_{k=0}^{b-1}(z_l^b - z_l^k)W_k] \\ \quad [(1-\overline{B}_2(\phi(z)))\phi(z) + \theta\overline{B}_2\phi(z)(1-\overline{V}(\phi(z)))\phi(z)] \\ \quad + I[z_l^b - \{(1-\theta)+\theta\overline{V}\phi(z)\overline{B}(\phi(z))\}]\phi(z) \end{array}\right\}}{[z_l^b - \{(1-\theta)+\theta\overline{V}\phi(z)\overline{B}(\phi(z))\}]\phi(z)},$$

where $\phi(z) = (1 - C(z_l))\lambda_2$.

The preceding result coincides with the result of Ayyappan and Supraja (2018).

Case 2: In the absence of priority queue, server renders service to the customers without bulk service, breakdown and feedback, that is $\lambda_1 = 0$, $\alpha = 0$, $a = b = 1$ $r = 1$ and $C(z_l) = z_l$. Hence, the proposed model becomes $M^X/G/1$ queue with Bernoulli vacation,

$$W_q(z) = \frac{I(1-z_l)[(1-\theta)+\theta\overline{V}(\phi(z))]\overline{B}_2(\phi(z))\phi(z)}{[(1-\theta)+\theta\overline{V}(\phi(z))]\overline{B}_2(\phi(z))\phi(z)}.$$

Tthe above result is exactly matched with the result of Madan (2004).

10. Numerical Results

This segment shows the numerical and graphical behavior of this model. We examined the usual service time, breakdown, repair and vacation time which are distributed exponentially.

Table 1 exhibits that when an arrival rate (λ_1) for priority queue escalates, then the expected queue length (L_{q_1}, L_{q_2}) and the expected waiting time (W_{q_1}, W_{q_2}) also rise at $\lambda_2 = 1$, $\alpha = 5$, $\mu = 20$, $\eta = 10$, $\gamma = 10$, and $\lambda_1 = 5.0$ to 5.5.

Table 2 shows that when the breakdown rate (α) escalates, then the expected queue length (L_{q_1}, L_{q_2}) and expected waiting time (W_{q_1}, W_{q_2}) also rise at $\lambda_1 = 5$, $\lambda_2 = 1$, $\mu = 10$, $\gamma = 10$, and $\alpha = 4$ to 10.

Table 3 indicates that when repair rate (η) escalates, then the expected queue length (L_{q_1}, L_{q_2}) and expected waiting time (W_{q_1}, W_{q_2}) decrease at $\lambda_1 = 3$, $\lambda_2 = 1$, $\mu = 10$, $\gamma = 10$, $\alpha = 2$, and $\eta = 5$ to 10.

Figure 2 through 4 illustrate the performance measures of this model. Figure 2 displays the behaviour of expected length of queue (L_{q_1}, L_{q_2}). When the priority arrival rate (λ_1) rises, the expected length of queue size is also enriched. Figure 3 displays the behavior of queue size (L_{q_1}, L_{q_2})

which depends upon the breakdown rate (α). When the breakdown rate raises the length of queue is also enriched. Figure 4 illustrates the expected queue length (L_{q_1}, L_{q_2}) which depends upon the repair rate (η). When the repair rate enriched, the expected length of queue is decreased.

Table 1. Effect of priority arrival rate (λ_1)

λ_1	L_{q_1}	W_{q_1}	L_{q_2}	W_{q_2}
5.0	20.0304	4.0061	2.4577	2.4577
5.1	20.7271	4.0641	2.5081	2.5081
5.2	21.4369	4.1225	2.5506	2.5506
5.3	22.1583	4.1808	2.5860	2.5860
5.4	22.8917	4.2392	2.6151	2.6151
5.5	23.6366	4.5455	2.6386	2.6386

Table 2. Effect of breakdown rate (α)

α	L_{q_1}	W_{q_1}	L_{q_2}	W_{q_2}
5	3.4568	0.6914	4.2256	4.2256
6	4.2297	0.8459	5.2405	5.2405
7	4.5339	0.9068	6.2635	6.2635
8	4.6715	0.9343	7.1705	7.1705
9	4.7565	0.9513	7.7746	7.7746
10	4.8345	0.9669	7.8003	7.8003

Table 3. Effect of repair rate(η)

η	L_{q_1}	W_{q_1}	L_{q_2}	W_{q_2}
5	8.5332	2.8444	1.0720	1.0720
6	6.8782	2.2927	0.9978	0.9978
7	5.8585	1.9528	0.9278	0.9278
8	5.1693	1.7231	0.8661	0.8661
9	4.6715	1.5572	0.8125	0.8125
10	4.2947	1.4316	0.7660	0.7660

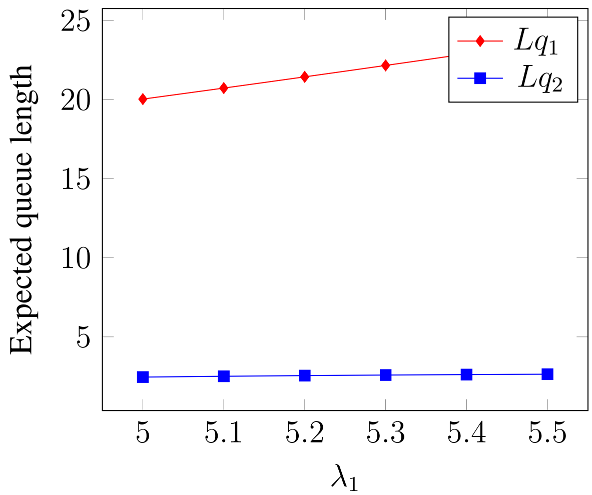

Figure 2. Expected queue length verses priority arrival rate λ_1

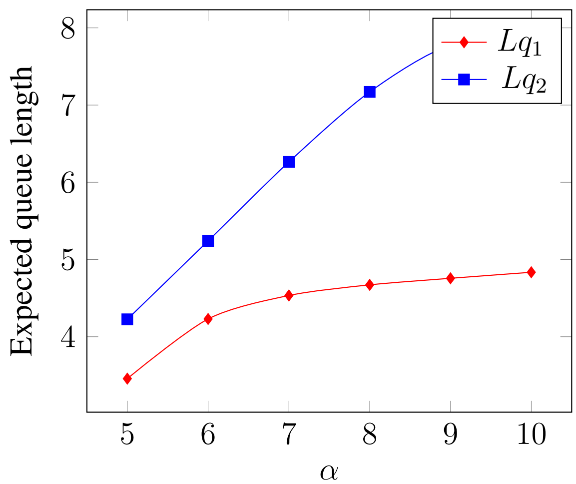

Figure 3. Expected queue length verses breakdown rate α

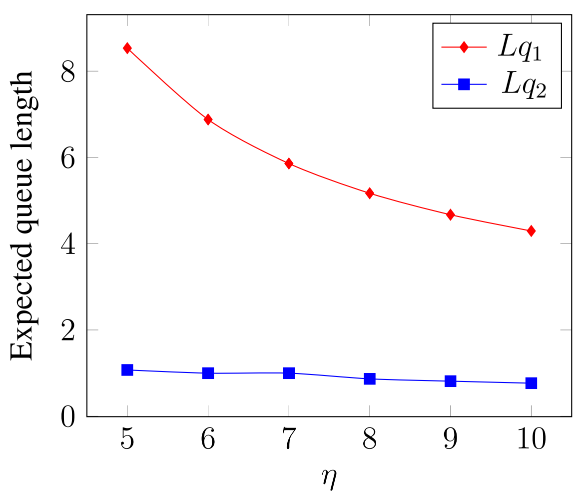

Figure 4. Expected queue length verses repair rate η

11. Conclusion

Two types of batch arrival queues with immediate feedback, modified Bernoulli vacation and server breakdown, are discussed. Server afford single service for priority and bulk service for ordinary queue under non-preemptive priority rule. During priority service the server may break down; after repair completion the priority customers may get fresh service but in case of ordinary customers they last in the system. In the same manner, the unsatisfied priority customers may get fresh service from the server. Some performance measures and also quantitative results are obtained along with graphical representations. Our article is composite and implemented in our day-to-day life, hence various parameters are discussed.

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