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G. Ayyappan
Pondicherry Engineering College

S. Sankeetha
Pondicherry Engineering College

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MAP/PH(1), PH(2)/2 Queue with Multiple Vacation, Optional Service, Consultations and Interruptions

[1]G. Ayyappan and [2]S. Sankeetha

Department of Mathematics
Pondicherry Engineering College
Puducherry, India
[1]ayyappanpec@hotmail.com; [2]sangeetha.sivarajp@gmail.com

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Abstract

Two types of services are explored in this paper: regular server and main server, both of which provide both regular and optional services. Customers arrive using the Markovian Arrival Process (MAP), and service time is allocated based on phase type. The regular server uses the main server as a resource. Customers' service at the primary server is disrupted as a result. When the queue size is empty, the main server can take several vacations. This system has been represented as a QBD Process that investigates steady state with the use of matrix analytic techniques, employing finite-dimensional block matrices. Our model's waiting time distribution has been examined in more detail during the busy times. The system's key parameters are assessed, and a few graphs and numerical representations are constructed.

Keywords: Markovian Arrival Process; Phase type service; Vacation; Optional service; Interruptions

MSC 2010 No.: 60K25, 68M30, 90B22

1. Introduction

By introducing phase type distribution and the adaptable Markovian Point Process, Neuts (1979) has made an incredible contribution to stochastic processes. Both discrete and continuous temporal distributions of phase type were studied. MAP is a useful appliance in the point process, which includes the Markov Modulated Poisson Process, Poisson Process and phase type renewal process. By introducing the matrix analytic approach, he enhanced his contribution. Remarkable cases was investigated in MAP, and looked into detailed descriptions of the Markovian Arrival Process by Chakravarthy (2010). In the books of Neuts (1981, 1989) as well as in the book Latouche and Ramaswami (1999), the matrix analytic approach was thoroughly studied.

Choudhary and Tadj (2009) investigated steady state behaviour in M/M/1 queue model with an additional phase of service. Madan (2000) introduced an optional service that was served in the second phase. Medhi (2002) developed the Pollackzek-Khinchin formula for a Probability Generating Function. Chaudhary and Tadj (2009) introduced interruption as an optional service. Chakravarthy (2013) researched the second phase of a multi-server queue with an optional service after phase type distribution in service and MAP in arrival. Ammar (2014) analysed the two server Markovian model with impatient customer. External arrival tasks from outside the system were explored by Haghighi and Mishev (2016a), as well as internal arrival tasks through rapid feedback or a splitting process, both of which occurred with the concept of delay.

Chakravarthy and Karatza (2013) proposed a two-parallel server queuing model with exponential service and Markovian arrival. The stability condition and waiting time distribution of the system were investigated using the matrix analytic method. The retrial queueing system with multiserver, exponential service time distribution and Markovian arrival process was investigated by Artalejo and Chakravarthy (2006). Additionally, the maximum orbit size distribution function was discovered and numerically demonstrated. Klimenok et al. (2018) looked into a queueing model with many servers, batch Markovian arrival, phase type service, and backup servers. They demonstrated the need for their stable system by calculating the LST for distribution of elapsed time. Ke et al. (2011) investigated a multi-server model using a modified Bernoulli vacation. They came up with a method for computing invariant probability and a cost model for identifying the ideal number of servers at the lowest total average cost.

Vacation is a period of time during which the server is unavailable for service. This vacation can be taken by the server for a variety of reasons, including the need to complete other tasks such as data verification, filing records, and so on while no customers are being served in the system. In the queuing model, Levy and Yechiali (1975) introduced the concept of vacation. Bernoulli scheduled vacation was unleashed by Keilson and Servi (1986). Levy and Yechiali (1976) investigated multi-server queue with an exponential vacation distribution rate. This vacation is classified as a working vacation by Servi and Finn (2002), Doshi (1986), Takagi (1991), and Tian, Li and Zhang (2009) presented an excellent survey of the vacation model. Tiang and Zhang (2006) also done their contribution on vacation queueing model.

Do (2010) developed and researched the notion of working vacation in a retail queue. Ayyappan and Gowthami (2019) investigated two server queueing models with Bernoulli vacation following exponential distribution, where inter arrival time follows MAP and service obeys phase type distribution. Zhang and Hou (2011) looked at the MAP/G/1 queue with working vacations and vacation interrupts. They discovered queue length distribution, as well as system size distribution during the prior to the arrival period and Laplace-Stieltjes transform (LST) of waiting time, utilizing additional variable approach in conjunction with matrix-analytic method and censoring methodology. Single-Server Poisson Queuing System with Delayed Feedback was researched by Haghighi et al. (2011). Banik et al. (2007) published a set of numerical results with working vacations on the GI/M/N queue. A single server service queueing system with delayed feedback and splitting was studied by Haghighi and Mishev (2016b). Chen et al. (2009) modelled the GI/M/1 queue using Phase-type working vacations and vacation interruptions, resulting in a Phase-type vacation time distribution. Sreenivasan et al. (2013) expanded Li and Tian's (2007) work on arrivals and service following MAP and phase type respectively, as well as the N-policy vacation.

2. Model Description

Customers arrive as per the Markovian Arrival with the parameters D_0 and D_1 of dimensions m in this classical system, which has two servers, one main and one regular, each with an infinite capacity queue. Customers who join the queue are served instantly if one of the servers is available; otherwise, they must wait in line. Both the regular server and the main server provides regular and optional service for the customers which follows phase type distribution $(\alpha_1, t_1), (\alpha_2, t_2), (\beta_1, s_1), (\beta_2, s_2)$, respectively, of order n. The vector $T_1^0, T_2^0, S_1^0, S_2^0$ is given by $T_1^0 = -T_1 e, T_2^0 = -T_2 e, S_1^0 = -S_1 e, S_2^0 = -S_2 e$ where e is the column vector of 1's of proper order. After relishing the regular service provided by either regular server or main server, the customer can opt for optional service on the basis of the needs of the customer with the probability $p_i (i = 1, 2)$ or the customer leaves the system with the probability $q_i (i = 1, 2)$ where $p_i + q_i = 1$.

Apart from delivering regular and optional services, the main server also provides consultation to the normal server when it is needed. With the parameter θ, consultation follows an exponential distribution. When there is a lack of assurance while delivering service, the ordinary server seeks consultation. Customers served by the main server are interrupted when consultation is provided. During the interruption, the customer served by the primary server must wait until the consultation is concluded. It will become unethical if the customer served by the main customer is regularly interrupted. As a result, only M interruptions are permitted, limiting the count of consultations for the regular server. Allow K to be the maximum count of consultations supplied by the main server, ensuring that only K times the customer will be interrupted when the regular server is providing regular service. If the count of interruptions exceeds M or K, the regular server must wait until the main server's service to the customer is completed. Only when the normal server is delivering regular service does it require consultation. On the other hand the regular server is certain while providing optional service. The regular server is idle, when no customer is in the system. But multiple vacation can be taken by the main server whenever there is nil customer in the system. Multiple vacation follows exponential distribution with the parameter η.

To find a matrix-geometric type solution, this model is investigated as a QBD process. Refer to Neuts (1981) and Latouche and Ramaswami (1999) for a detailed study of Matrix Analytic Methods. The state space of QBD under considered model is defined and the form of infinitesimal generator is also investigated using the following notations.

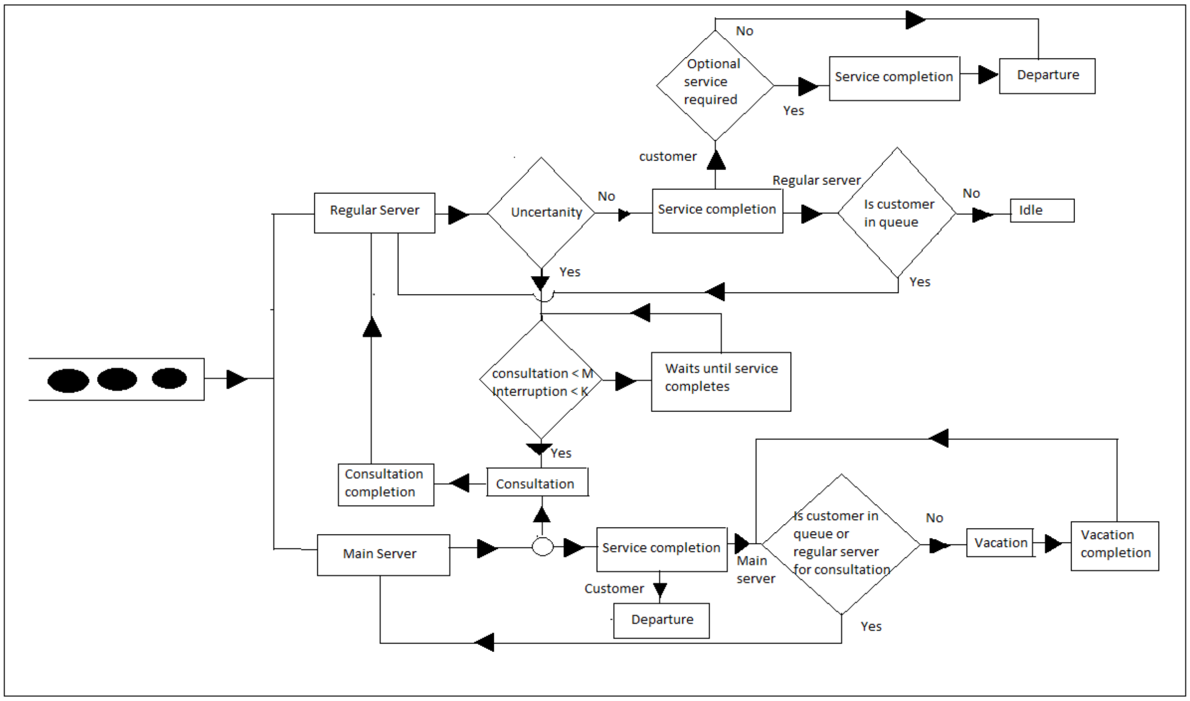

Figure 1. Model Representation in Schematic Form

- I_j is the identity matrix of dimension j.
- e represents the column vector of appropriate dimension with 1 as its entries.
- e_1 is of dimension $n_1 m[(M+1)(2Nn_2+1)]$ with first $NMmn_1n_2$ elements are 1 and the remaining elements are 0.
- e_2 is of dimension $n_1 m[(M+1)(2Nn_2+1)]$ with 1 from $NMmn_1n_2+1$ to $2NMmn_1n_2$ and the remaining elements are 0.
- e_3 is of dimension $n_1 m[(M+1)(2Nn_2+1)]$ with 1 from $2NMmn_1n_2$ to $Mmn_1(2Nn_2+1)$ and the remaining elements are 0.
- e_4 is of dimension $n_1 m[(M+1)(2Nn_2+1)]$ with 1 from $Mmn_1(2Nn_2+1)+1$ to $mn_1[Nn_2(2M+1)+M]$ and the remaining elements are 0.
- e_5 is of dimension $n_1 m[(M+1)(2Nn_2+1)]$ with 1 from $mn_1[Nn_2(2M+1)+M]+1$ to $mn_1[2Nn_2(M+1)+M]$ and the remaining elements are 0.
- e_6 is of dimension $n_1 m[(M+1)(2Nn_2+1)]$ with 1 from $mn_1[2Nn_2(M+1)+M]+1$ to $mn_1(M+1)(2Nn_2+1)$ and the remaining elements are 0.
- e_7 is of dimension $2m[n_2+n_1(M+1)]$ with 1 from $2m(n_2+Mn_1)+1$ to $m[2n_2+n_1(2M+1)]$ and the remaining elements are 0.
- e_8 is of dimension $2m[n_2+n_1(M+1)]$ with 1 from $m(2n_2+Mn_1)+1$ to $2m(n_2+Mn_1)$ and the remaining elements are 0.
- $N(t)$ is the system's total number of consumers.
- $S_1(t)$ be the regular servers.

$$S_1(t) = \begin{cases} 0, & \text{if regular server is idle,} \\ 1, & \text{if regular server is offering regular service,} \\ 2, & \text{if regular server is offering optional service.} \end{cases}$$

- $S_2(t)$ be the main server status.

$$S_2(t) = \begin{cases} 0, & \text{if main server is offering consultation,} \\ 1, & \text{if main server is offering regular service,} \\ 2, & \text{if main server is offering optional service,} \\ 3, & \text{if main server is availing vacation.} \end{cases}$$

- $B_1(t)$ is the count of consultations received by the regular server up to M times throughout a customer's service.
- $B_2(t)$ is count of interruptions completed by the customer at the main server while the server was busy with regular service.
- $B_3(t)$ is the count of interruptions completed by the customer the main server while the server was busy with optional service.
- $C_1(t)$ is the regular server's service phase.
- $C_2(t)$ is the main server's service phase.
- $M(t)$ is the Markovian Arrival Process phase.

Let $\{(N(t), S_1(t), S_2(t), B_1(t), B_2(t), B_3(t), C_1(t), C_2(t), M(t)); t \geq 0\}$ be a Markov Process with the state space $\Omega = l(0) \cup l(1) \cup l(i)$, where

$$\begin{aligned}
l(0) &= \{(0, 0, l) : 1 \leq l \leq m\}, \\
l(1) &= \{(1, 0, 1, k_2, l) : 1 \leq k_2 \leq n_2, 1 \leq l \leq m\} \cup \{(1, 0, 2, k_2, l) : 1 \leq k_2 \leq n_2, 1 \leq l \leq m\} \\
&\quad \cup \{(1, 1, 0, j_1, k_1, l) : 0 \leq j_1 \leq M, 1 \leq k_1 \leq n_1, 1 \leq l \leq m\}, \\
l(i) &= \{(i, 1, 1, j_1, j_2, k_1, k_2, l) : i \geq 1, 0 \leq j_1 \leq M, 0 \leq j_2 \leq N, 1 \leq k_1 \leq n_1, 1 \leq k_2 \leq n_2, 1 \leq l \leq m\} \\
&\quad \cup \{(i, 1, 2, j_1, j_3, k_1, k_2, l) : i \geq 1, 0 \leq j_1 \leq M, 0 \leq j_3 \leq N, 1 \leq k_1 \leq n_1, 1 \leq k_2 \leq n_2, 1 \leq l \leq m\} \\
&\quad \cup \{(i, 2, 1, j_2, k_1, k_2, l) : i \geq 1, 0 \leq j_2 \leq N, 1 \leq k_1 \leq n_1, 1 \leq k_2 \leq n_2, 1 \leq l \leq m\} \\
&\quad \cup \{(i, 2, 2, j_3, k_1, k_2, l) : i \geq 1, 0 \leq j_3 \leq N, 1 \leq k_1 \leq n_1, 1 \leq k_2 \leq n_2, 1 \leq l \leq m\} \\
&\quad \cup \{(i, 2, 3, k_1, l) : i \geq 2, 1 \leq k_1 \leq n_1, 1 \leq l \leq m\} \\
&\quad \cup \{(i, 1, 3, j_1, k_1, l) : i \geq 2, 0 \leq j_1 \leq M, 1 \leq k_1 \leq n_1, 1 \leq l \leq m\}.
\end{aligned}$$

The Quasi birth-death (QBD) process of infinitesimal generator matrix is given by:

$$Q = \begin{bmatrix} B_{00} & B_{01} & 0 & 0 & 0 & 0 & 0 \cdots \\ B_{10} & B_{11} & B_{12} & 0 & 0 & 0 & 0 \cdots \\ 0 & B_{21} & A_1 & A_0 & 0 & 0 & 0 \cdots \\ 0 & 0 & A_2 & A_1 & A_0 & 0 & 0 \cdots \\ \cdots & \cdots & \cdots & \ddots & \ddots & \ddots & \cdots \end{bmatrix},$$

where each block matrix of Q is as follows:

$$B_{00} = \begin{bmatrix} D_0 \end{bmatrix},$$

$$B_{01} = \begin{bmatrix} 0 \; 0 \; 0 \; D_1 \; 0 \end{bmatrix},$$

$$B_{10} = \begin{bmatrix} q_2 S_1^0 \otimes I_m \\ S_2^0 \otimes I_m \\ q_1 T_1^0 \otimes I_m \otimes e_M \\ q_1 T_1^0 \otimes I_m \otimes e_M \\ T_2^0 \otimes I_M \end{bmatrix},$$

$$B_{11} = \begin{bmatrix} S_1 \oplus D_o & p_2 S_1^0 \beta_1 \otimes I_m & 0 & 0 & 0 \\ 0 & S_2 \oplus D_0 & 0 & 0 & 0 \\ 0 & 0 & b_{(3,3)}^{(1,1)} & 0 & p_1 T_1^0 \alpha_1 \otimes I_m \otimes e_M \\ 0 & 0 & \eta \otimes I_{mn_1} & b_{(4,4)}^{(1,1)} & p_1 T_1^0 \alpha_1 \otimes I_m \otimes e_M \\ 0 & 0 & 0 & 0 & S_2 \oplus D_0 \end{bmatrix},$$

$$b_{(33)}^{(11)} = \begin{bmatrix} T_1 \oplus (D_0 + \theta I_m) & \theta I_{mn_1} & 0 \\ 0 & T_1 \oplus (D_0 + \theta I_m) & \theta I_{mn_1} \\ 0 & 0 & T_1 \oplus D_o \end{bmatrix},$$

$$b_{(44)}^{(11)} = \begin{bmatrix} T_1 \oplus (D_0 + \eta I_m) & 0 & 0 \\ 0 & T_1 \oplus D_0 & 0 \\ 0 & 0 & T_1 \oplus D_0 \end{bmatrix},$$

$$B_{12} = \begin{bmatrix} I_m \otimes D_1 & 0 & 0 & 0 \; 0 & 0 \\ 0 & I_m \otimes D_1 & 0 & 0 \; 0 & 0 \\ b_{(31)}^{(12)} & 0 & 0 & 0 \; 0 & 0 \\ 0 & 0 & I_{Mn_1} \otimes D_1 & 0 \; 0 & 0 \\ 0 & 0 & 0 & 0 \; 0 & I_m \otimes D_1 \end{bmatrix},$$

$$b_{(31)}^{(12)} = \begin{bmatrix} I_m \otimes D_1 & 0 & 0 \\ 0 & I_m \otimes D_1 & 0 \\ 0 & 0 & I_m \otimes D_1 \end{bmatrix},$$

$$B_{21} = \begin{bmatrix} q_1 T_1^0 \otimes I_{mn_2} \otimes e_{MN} & 0 & I_M \otimes (q_2 S_1^0 \otimes I_m) \otimes e_N & 0 & 0 \\ 0 & q_1 T_1^0 \otimes I_{mn_2} \otimes e_{MN} & I_M \otimes (S_2^0 \otimes I_m) \otimes e_N & 0 & 0 \\ 0 & 0 & 0 & q_1 T_1^0 \otimes I_{Mm} & 0 \\ (T_2 \otimes I_{mn_2}) \otimes e_N & 0 & 0 & 0 & (q_2 S^1 \otimes I_{mn_1}) \otimes e_N \\ 0 & (T_2 \otimes I_{mn_2}) \otimes e_N & 0 & 0 & (S_2 \otimes I_{mn_1}) \otimes e_N \\ 0 & 0 & 0 & T_2 \otimes I_m & 0 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} a_{(11)}^1 & (p_1 S_1^0 \beta_1 \otimes I_{mn_1 n_2}) \otimes e_N & 0 & a_{(14)}^1 & 0 & 0 \\ 0 & a_{(22)}^1 & 0 & 0 & a_{(25)}^1 & 0 \\ \eta \otimes I_{mn_1} & 0 & a_{(33)}^1 & 0 & 0 & (p_1 T_1^0 \alpha_1 \otimes Im) \otimes e_M \\ 0 & 0 & 0 & a_{(44)}^1 & (p_2 S_1^0 \beta_1 \otimes I_{Nmn_1}) & 0 \\ 0 & 0 & 0 & 0 & a_{(55)}^1 & 0 \\ 0 & 0 & 0 & \eta I_{mn_2} & 0 & T_2 \oplus (D_0 - \eta I_m) \end{bmatrix},$$

$$a_{(11)}^1 = a_{(22)}^2 = \begin{bmatrix} [(T_1 + S_1) \oplus (D_0 - \theta I_m)] \otimes I_{mn_2} & 0 & \theta I_{mn_1 n_2} \\ 0 & [(T_1 + S_1) \oplus D_0] \otimes I_{mn_2} & 0 \\ 0 & 0 & [(T_1 + S_1) \oplus D_0] \otimes I_{mn_1} \end{bmatrix},$$

$$a_{(33)}^1 = \begin{bmatrix} T_1 \oplus (D_0 - \eta I_m) & 0 & 0 \\ 0 & T_1 \oplus D_0 & 0 \\ 0 & 0 & T_1 \oplus D_0 \end{bmatrix},$$

$$a_{(44)}^1 = \begin{bmatrix} [(T_2 + S_1) \oplus (D_0 - \theta I_m)] \otimes I_m & \theta I_{mn_1 n_2} \\ 0 & [(T_2 + S_1) \oplus D_0] \otimes I_m \end{bmatrix},$$

$$a_{(55)}^1 = \begin{bmatrix} [(T_2 + S_2) \oplus (D_0 - \theta I_m)] \otimes I_m & \theta I_{mn_1 n_2} \\ 0 & [(T_2 + S_2) \oplus D_0] \otimes I_m \end{bmatrix},$$

$$a_{(14)}^1 = a_{(25)}^1 = \begin{bmatrix} p_1 T_1^0 \alpha_1 \otimes I_{mn_2} \otimes e_M & 0 \\ 0 & p_1 T_1^0 \alpha_1 \otimes I_{mn_2} \otimes e_M \end{bmatrix},$$

$$A_2 = \begin{bmatrix} a^2_{(11)} & q_2S_1^0\beta_1 \otimes I_{mn_1n_2} \otimes e_N & 0 & 0 & 0\,0 \\ S_2^0\beta_2 \otimes I_{mn_1n_2} \otimes e_N & a^2_{(22)} & 0 & 0 & 0\,0 \\ 0 & 0 & q_1T_1^0\alpha_1 \otimes I_m \otimes e_M & 0 & 0\,0 \\ T_2^0\alpha_2 \otimes I_{mn_2} & 0 & 0 & q_2S_1^0\beta_1 \otimes I_{mn_1n_2} & 0\,0 \\ 0 & [T_2^0\alpha_2 \otimes I_m] \otimes e_N & 0 & S_2^0\alpha_2 \otimes I_{mn_1n_2} & 0\,0 \\ 0 & 0 & T_2^0\alpha_2 \otimes I_m & 0 & 0\,0 \end{bmatrix},$$

$$a^2_{(11)} = a^2_{(22)} = \begin{bmatrix} q_1T_1^0\alpha_1 \otimes I_{mn_2} \otimes e_M & 0 \\ 0 & q_1T_1^0\alpha_1 \otimes I_{mn_2} \otimes e_M \end{bmatrix},$$

$$A_0 = \begin{bmatrix} I_{MNmn_1n_2} \otimes D_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & I_{MNmn_1n_2} \otimes D_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_{Mn_1} \otimes D_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{Nn_1n_2} \otimes D_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{Nn_1n_2} \otimes D_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{n_1} \otimes D_1 \end{bmatrix}.$$

3. Analysis of the Stability Condition in the System

Let us define $A = A_0 + A_1 + A_2$ of order $n_1m[(M+1)(2Nn_2+1)]$ is an irreducible infinitesimal generator matrix. A adhere to $\Psi A = 0$; $\Psi e = 1$, where $\Psi = (\Psi_0, \Psi_1, \Psi_2, \Psi_3, \Psi_4, \Psi_5)$ is an invariant probability vector.

- $\Psi_0(q_1T_1^0\alpha_1 \otimes I_{mn_2} \otimes e_{M-1} + [(T_1+S_1) \oplus (D_0 - \theta I_m)] \otimes I_{mn_2} + I_{Nmn_1n_2} \otimes D_1)(q_1T_1^0\alpha_1 \otimes I_{mn_2} + [(T_1+S_1) \oplus D_0] \otimes I_{mn_2} + I_{Nmn_1n_2} \otimes D_1)^2 + \Psi_1(S_2^0\beta_2 \otimes I_{mn_1n_2} \otimes e_N) + \Psi_2(\eta \otimes I_{mn_1}) + \Psi_3(T_2^0\alpha_2 \otimes I_{mn_2}) = 0.$
- $\Psi_0(q_2S_1^0\beta_1 \otimes I_{mn_1n_2} \otimes e_N + p_2S_1^0\beta_1 \otimes I_{mn_1n_2} \otimes e_N) + \Psi_1(q_1T_1^0\alpha_1 \otimes I_{mn_2} \otimes e_{M-1} + [(T_1+S_1) \oplus (D_0 - \theta I_m)] \otimes I_{mn_2} + I_{Nmn_1n_2} \otimes D_1)(q_1T_1^0\alpha_1 \otimes I_{mn_2} + [(T_1+S_1) \oplus D_0] \otimes I_{mn_2} + I_{Nmn_1n_2} \otimes D_1)(q_1T_1^0\alpha_1 \otimes I_{mn_2} + I_{Nmn_1n_2} \otimes D_1) + \Psi_4T_2^0\alpha_2 \otimes I_m \otimes e_N = 0.$
- $\Psi_2(q_1T_1^0\alpha_1 \otimes I_m + [T_1 \oplus (D_0 - \eta I_m)] + I_{n_1} \otimes D_1)(T_1 \oplus D_0 + I_{n_1} \otimes D_1)^2 + \Psi_5(T_2^0\alpha_2 \otimes I_m) = 0.$
- $\Psi_0(p_1T_1^0\alpha_1 \otimes I_{mn_2} \otimes e_M)^2 + \Psi_3(q_2S_1^0\beta_1 \otimes I_{mn_1} + [(T_1+S_1) \oplus (D_0 - \theta I_m)] \otimes I_m + I_{n_1n_2} \otimes D_1)(q_2S_1^0\beta_1 \otimes I_{mn_1} + [(T_2+S_1) \oplus D_0] \otimes I_m + I_{n_1n_2} \otimes D_1) + \Psi_4(S_2^0\beta_1 \otimes I_{mn_1n_2} + \Psi_5(\eta I_{mn_2}) = 0.$
- $\Psi_1(p_1T_1^0\alpha_1 \otimes I_{mn_2} \otimes e_M)^2 + \Psi_3(p_2S_1^0\beta_1 \otimes I_{Nmn_1}) + \Psi_4([(T_2+S_2) \oplus (D_0 - \theta I_m)] \otimes I_m + I_{n_1n_2} \otimes D_1)([(T_2+S_2) \oplus D_0] \otimes I_m + I_{n_1n_2} \otimes D_1) = 0.$
- $\Psi_3(p_1T_1^0\alpha_1 \otimes I_m \otimes e_M) + \Psi_5([T_2 \oplus (D_0 - \eta I_m)] + I_{n_1} \otimes D_1) = 0.$

The system attains stability when $\Psi A_0 e < \Psi A_2 e$,

$\Psi_0(I_{MNmn_1n_2} \otimes D_1) + \Psi_1(I_{MNmn_1n_2} \otimes D_1) + \Psi_2(I_{Mn_1} \otimes D_1) + \Psi_3(I_{Nn_1n_2} \otimes D_1) + \Psi_4(I_{Nn_1n_2} \otimes D_1) + \Psi_5(I_{n_1} \otimes D_1) < \Psi_0((q_1T_1^0\alpha_1 \otimes I_{mn_2} \otimes e_M)^2 + (q_2S_1^0\beta_1 \otimes I_{mn_1n_2} \otimes e_N)) + \Psi_1((S_2^0\beta_2 \otimes$

$I_{mn_1n_2} \otimes e_N) + (q_1T_1^0\alpha_1 \otimes I_{mn_2} \otimes e_M))^2 + \Psi_2(q_1T_1^0\alpha_1 \otimes I_m \otimes e_M) + \Psi_3(T_2^0\alpha_2 \otimes I_{mn_2} + q_2S_1^0\beta_1 \otimes I_{mn_1n_2}) + \Psi_4([T_2^0\alpha_2 \otimes I_m] \otimes e_N + S_2^0\alpha_2 \otimes I_{mn_1n_2}) + \Psi_5(T_2^0\alpha_2 \otimes I_m)$.

4. The Invariant Probability Vector

As this model is designed as a Quasi birth-death process (QBD), then its steady state distribution under stability condition has a martix-geometric solution. For infinitesimal generator Q, unique solution to $XQ = 0$ and $Xe = 1$ is transition probability vector. This vector X can be partitioned according to the server status into $(X_0, X_1, X_2....)$ where each X_i is the corresponding row vector. The dimension of X_0 is m, X_1 is $2m[n_2 + n_1(M+1)]$ and the remaining probability vectors $X_2, X_3, X_4, ...$ are of equal dimension $n_1m[(M+1)(2Nn_2+1)]$. The matrix geometric structure of the invariant probability vector under stability criterion is as follows:

$$X_i = X_1R^{i-1}, i = 2, 3, 4, ...$$

The lowest positive solution to the quadratic equation is R, where R represents the rate matrix

$$R^2A_2 + RA_1 + A_0 = 0,$$

and the boundary states X_0, X_1 and X_2 are acquired by solving the following equations,

$$\begin{aligned} X_0B_{00} + X_1B_{10} &= 0, \\ X_0B_{01} + X_1B_{11} + X_2B_{21} &= 0, \\ X_1B_{12} + X_2(A_1 + RA_2) &= 0, \end{aligned}$$

subject to the normalizing condition

$$X_0e + X_1e + X_2(I-R)^{-1}e = 1.$$

The Logarithmic Reduction Algorithm can be used to quickly calculate the rate matrix R. This algorithm was developed by Lautouche and Ramaswamy (1999) which facilitate to find the value of R effortlessly.

Step 1: $H \leftarrow (-A_1)^{-1}A_0, L \leftarrow (-A_1)^{-1}A_2, G = L$ and $T = H$.
Step 2:
$U = HL + LH$,
$M = H^2$,
$H = (I-U)^{-1}M$,
$M = L^2$,
$L = (I-U)^{-1}M$,
$G = G + TL$,
$T = TH$, continue Step 1 until $\|e - Ge\|_\infty < \epsilon$.
Step 3: $R = -A_0(A_1 + A_0G)^{-1}$.

We partition the vectors X_1 and X_i for further usage as $X_1 = (u_{01}, u_{02}, u_{13}, u_{23})$ and $X_i = (v_{i11}, v_{i12}, v_{i13}, v_{i21}, v_{i22}, v_{i23})$, $i \geq 2$, with their dimensions and descriptions specified in Tables 1 and 2.

Table 1. Vector Notation vs Dimension.

Vector Notation	Dimension
$u_{j_1 j_2}(j_1 = 0, j_2 = 1, 2)$	mn_2
$u_{j_1 j_2}(j_1 = 1, j_2 = 3)$	Mmn_1
$u_{j_1 j_2}(j_1 = 2, j_2 = 3)$	mn_1
$v_{j_1 j_2}(j_1 = 1, j_2 = 1, 2)$	$MNmn_1n_2$
$v_{j_1 j_2}(j_1 = 1, j_2 = 3)$	Mmn_1
$v_{j_1 j_2}(j_1 = 2, j_2 = 1, 2)$	Nmn_1n_2
$v_{j_1 j_2}(j_1 = 2, j_2 = 3)$	mn_1

Table 2. Vector Notation vs Description.

Vector Notation	Description
u_{01}	Regular server is idle when the main server is providing regular service
u_{02}	Regular server is idle when the main server is providing optional service
u_{13}, v_{13}	Regular server is offering regular service when the main server is availing vacation
u_{23}, v_{23}	Regular server is offering optional service when the main server is availing vacation
v_{11}	Both the regular and main server is providing regular service
v_{12}	Regular server is offering regular service whereas the primary server is providing optional service
v_{21}	Regular server is offering optional service whereas the main server is providing regular service
v_{22}	Both the regular and main server is offering optional service

5. Analysis of Busy Span

The phrase "busy period" refers to the time between a customer's entrance on a system with zero customers and the time when the system's size hits null for the first time. As a result, the busy period and the time spent transitioning from level 1 to 0 are similar.

To explain the transition from level i to $i-1$, $i \geq 2$, Latouche (1978) coined the term fundamental period in context of a QBD process. It is necessary to deal with $i = 0, 1$ independently for the boundary states. It is also worth noting that for each level i, there are $[n_1 m(M+1)(2Nn_2+1)]$ states, where $i \geq 2$.

Notations for busy period on boundary level state

- $G_{jj'}(k,s)$: The probability that the QBD moves by making k left transition to the level $i-1$ and also by entering (i,j') with the condition that it begins from the state (i,j) at time $t=0$.
- $\bar{G}(z,s)$: The matrix$(G_{jj'}(z,s))$.
- The transition matrix $\bar{G}_{jj'}(z,s) = \sum_0^\infty z^k \int_0^\infty e^{-sx} dG_{jj'}(k,x) : |z| \leq 1, Re(s) \geq 0$.
- $\bar{G}(z,s)$: The matrix $(G_{jj'}(z,s))$, satisfies the equation,

$$\bar{G}(z,s) = z\left[sI - A_1\right]^{-1} A_2 + \left[sI - A_1\right]^{-1} A_0 \bar{G}^2(z,s).$$

- $G = G_{jj'} = \bar{G}(0,1)$ is the initial passage time without the boundary states.
- $\bar{G}^{(2,1)}_{(jj')}(k,x)$ is the probability that moves from level 1 to 2 at time $t=0$.
- $\bar{G}^{(1,0)}_{(jj')}(k,x)$ is the probability that moves from level 0 to 1 at time $t=0$.
- $\bar{G}^{(0,0)}_{(jj')}(k,x)$ is the first probability returning to level 0.
- $\Im_{1j}$ is the expected first transit time from the levels i to $i-1$, of the process in state (i,j) at time $t=0$.
- $\bar{\Im}_1$ is the column vector $\Im_{1j}$ as its entries.
- $\Im_{2j}$ is the average number of customer served during the first time between the levels u and $u-1$, starting in state (i,j) at time $t=0$.
- $\bar{\Im}_2$ is the column vector $\Im_{2j}$ as its entries.
- $\bar{\Im}_1^{(2,1)}$ is the mean time of the first segment from the level 2 to 1.
- $\bar{\Im}_2^{(2,1)}$ is the average number of completed service during the first traverse from the level 2 to 1.
- $\bar{\Im}_1^{(1,0)}$ is the mean time of the first segment from the level 1 to 0.
- $\bar{\Im}_2^{(1,0)}$ is the average number of services completed during the first traverse from the level 1 to 0.
- $\bar{\Im}_1^{(0,0)}$ is the average first return time to level 0.
- $\bar{\Im}_2^{(0,0)}$ is the average number of services completed during the first return to the level 0.

One can easily calculate that the matrix $\bar{G}(z,s)$ satisfies

$$\bar{G}(z,s) = z\left[sI - A_1\right]^{-1} A_2 + \left[sI - A_1\right]^{-1} A_0 \bar{G}^2(z,s).$$

For the boundary states, that is, 2, 1 and 0, the equations below are satisfied according to $\bar{G}^{(2,1)}(z,s), \bar{G}^{(1,0)}(z,s)$ and $\bar{G}^{(0,0)}(z,s)$, respectively.

$$\begin{aligned}
\bar{G}^{(2,1)}(z,s) &= z\left[sI - A_1\right]^{-1} B_{21} + \left[sI - A_1\right]^{-1} A_0 \bar{G}(z,s)\bar{G}^{(2,1)}(z,s),\\
\bar{G}^{(1,0)}(z,s) &= z\left[sI - B_{11}\right]^{-1} B_{10} + \left[sI - B_{11}\right]^{-1} B_{12} \bar{G}^{(2,1)}(z,s)\bar{G}^{(1,0)}(z,s),\\
\bar{G}^{(0,0)}(z,s) &= z\left[sI - B_{00}\right]^{-1} B_{01} \bar{G}^{(1,0)}(z,s).
\end{aligned}$$

The moments are as follows:

$$\begin{aligned}
\Im_1 &= -\frac{\partial}{\partial s}\bar{G}(z,s)|_{s=0,z=1} = -\left[A_0(G+1)+A_1\right]^{-1} e,\\
\Im_2 &= \frac{\partial}{\partial z}\bar{G}(z,s)|_{s=0,z=1} = -\left[A_0(G+1)+A_1\right]^{-1} A_2 e,\\
\Im_1^{(2,1)} &= -\frac{\partial}{\partial s}\bar{G}^{(2,1)}(z,s)|_{s=0,z=1} = -\left[A_0(G+1)+A_1\right]^{-1}\left[A_0\Im_1+e\right],\\
\Im_2^{(2,1)} &= \frac{\partial}{\partial z}\bar{G}^{(2,1)}(z,s)|_{s=0,z=1} = -\left[A_0(G+1)+A_1\right]^{-1}\left[B_{21}e+A_0\Im_2\right],\\
\Im_1^{(1,0)} &= -\frac{\partial}{\partial s}\bar{G}^{(1,0)}(z,s)|_{s=0,z=1} = -\left[B_{11}+B_{12}\bar{G}^{(2,1)}(1,0)\right]^{-1}\left[B_{12}\Im_1^{(2,1)}\right]+e,\\
\Im_2^{(1,0)} &= \frac{\partial}{\partial z}\bar{G}^{(1,0)}(z,s)|_{s=0,z=1} = -\left[B_{11}+B_{12}\bar{G}^{(2,1)}(1,0)\right]^{-1}\left[B_{10}e+B_12\Im_2^{(2,1)}\right],\\
\Im_1^{(0,0)} &= -\frac{\partial}{\partial s}\bar{G}^{(0,0)}(z,s)|_{s=0,z=1} = -B_{00}^{-1}\left[e+B_{01}\Im_1^{(1,0)}\right],
\end{aligned}$$

$$\Im_2^{(0,0)} = \frac{\partial}{\partial z}\bar{G}^{(0,0)}(z,s)|_{s=0,z=1} = -B_{00}^{-1}B_{01}\Im_2^{(1,0)}.$$

6. Analysis of Waiting Time Distribution

The detention time distribution for an arriving consumer in the waiting line is calculated in this section using first transit time investigation. The distribution function of detention time of an entering tagged client in the backof the line is $W(t)$, where $t \geq 0$ is the waiting time of an incoming marked customer in the waiting line. Because each new customer must either stay for the end of the server's vacation time or for the end of the service period, we observe that $W(0+) = 0$ in multiserver queueing with Bernoulli vacation. Consider the state space $(*) \cup \{\bar{1}, \bar{2}, ...\}$ of an absorbing continuous-time Markov chain. After entering the absorption state $(*)$, entering tagged consumers will start receiving service, where

$$\begin{aligned}
(*) &= (0,0) \cup \{(1,1,0,j_1,k_1) : 0 \leq j_1 \leq M, 1 \leq k_1 \leq n\},\\
\bar{1} &= \{(1,0,1,k_2) : 1 \leq k_2 \leq n\} \cup \{(1,0,2,k_2) : 1 \leq k_2 \leq n\}.
\end{aligned}$$

For $i \geq 2$,

$$\begin{aligned}
\bar{i} = &\{(i,1,1,j_1,j_2,k_1,k_2) : i \geq 1, 0 \leq j_1 \leq M, 0 \leq j_2 \leq K, 1 \leq k_1, k_2 \leq n\}\\
&\cup \{(i,1,2,j_1,j_3,k_1,k_2) : i \geq 1, 0 \leq j_1 \leq M, 0 \leq j_3 \leq K, 1 \leq k_1, k_2 \leq n\}\\
&\cup \{(i,2,1,j_2,k_1,k_2) : i \geq 1, 0 \leq j_2 \leq K, 1 \leq k_1, k_2 \leq n\}\\
&\cup \{(i,2,2,j_3,k_1,k_2) : i \geq 1, 0 \leq j_3 \leq K, 1 \leq k_1, k_2 \leq n\}\\
&\cup \{(i,2,3,k_1) : i \geq 2, 1 \leq k_1 \leq n\} \cup \{(i,1,3,j_1,k_1) : i \geq 2, 0 \leq j_1 \leq M, 1 \leq k_1 \leq n\}.
\end{aligned}$$

The transition matrix for this absorbing Markov chain is as follows:

$$\bar{Q} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0\cdots \\ H_1 & F_1 & 0 & 0 & 0\cdots \\ H_2 & F_{21} & F & 0 & 0\cdots \\ 0 & 0 & F_2 & F & 0\cdots \\ \cdots\cdots & \cdots & \ddots & \ddots & \cdots \end{bmatrix},$$

where each block matrix is as follows:

$$H_1 = \begin{bmatrix} q_2 S_1^0 \otimes e_{n_2} & 0 \\ S_2^0 \otimes e_{n_2} & 0 \\ q_1 T_1^0 \otimes e_{n_1 n_2} & \eta \otimes I_{n_1} \\ T_2^0 \otimes e_{n_2} & 0 \end{bmatrix},$$

$$F_1 = \begin{bmatrix} S_1 & p_2 S_1^0 \beta_1 & 0 & 0 & \\ 0 & S_2 & 0 & 0 & \\ 0 & 0 & f_{(33)}^{(11)} & p_1 T_1^0 \alpha_1 \otimes e_M & \\ 0 & 0 & 0 & 0 & T_2 \end{bmatrix}, \qquad f_{(33)}^{(11)} = \begin{bmatrix} T_1 - \eta I_{n_1} & 0 & 0 \\ 0 & T_1 & 0 \\ 0 & 0 & T_1 \end{bmatrix},$$

$$H_2 = \begin{bmatrix} 0 & 0 \\ 0 & q_1 T_1^0 \otimes e_{n_2} \otimes e_{MN} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$F_{21} = \begin{bmatrix} q_1 T_1^0 \otimes I_{n_2} \otimes e_{MN} & 0 & 0 & 0 \\ 0 & q_1 T_1^0 \otimes I_{n_2} \otimes e_{MN} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q_2 S_1 \otimes e_N \\ 0 & T_2 \otimes I_{n_2} \otimes e_N & 0 & S_2 \otimes e_N \\ 0 & 0 & T_2 & 0 \end{bmatrix},$$

$$F = \begin{bmatrix} f_{(1,1)} & [P_1 S_1^0 \beta_1 \otimes I_{n_1 n_2}] \otimes I_N & 0 & f_{(1,4)} & 0 & 0 \\ 0 & f_{(2,2)} & 0 & 0 & f_{(2,5)} & 0 \\ 0 & 0 & (T_1 - \eta I_{n_1}) \otimes I_M & 0 & 0 & p_1 T_1^0 \alpha_1 \otimes e_M \\ 0 & 0 & 0 & (T_2 + S_2) \otimes I_{N n_1} & p_2 S_1^0 \beta_1 \otimes I_{N n_1} & 0 \\ 0 & 0 & 0 & 0 & (T_2 + S_2) \otimes I_{N n_1} & 0 \\ 0 & 0 & 0 & \eta I_{n_2} & 0 & T_2 - \eta I_{n_1} \end{bmatrix},$$

$$f_{(11)} = f_{(22)} = \begin{bmatrix} [(T_1+S_1)\oplus(-\theta)]\otimes I_{n_2} & 0 & \theta I_{n_1n_2} \\ 0 & (T_1+S_1)\otimes I_{n_1n_2} & 0 \\ 0 & 0 & (T_1+S_1)\otimes I_{n_1} \end{bmatrix},$$

$$f_{(14)} = f_{(24)} = \begin{bmatrix} p_1T_1^0\alpha_1\otimes I_{n_2}\otimes e_M & 0 \\ 0 & p_1T_1^0\alpha_1\otimes I_{n_2}\otimes e_M \end{bmatrix},$$

$$F_2 = \begin{bmatrix} f^2_{(11)} & q_2S_1^0\beta_1\otimes I_{n_1n_2}\otimes e_N & 0 & 0 & 0\ 0 \\ S_2^0\beta_2\otimes I_{n_1n_2}\otimes e_N & f^2_{(22)} & 0 & 0 & 0\ 0 \\ 0 & 0 & q_1T_1^0\alpha_1\otimes e_M & 0 & 0\ 0 \\ T_2^0\alpha_2\otimes I_{n_2} & 0 & 0 & q_2S_1^0\beta_1\otimes I_{n_1n_2} & 0\ 0 \\ 0 & T_2^0\alpha_2\otimes e_N & 0 & S_2^0\alpha_2 & 0\ 0 \\ 0 & 0 & T_2^0\alpha_2 & 0 & 0\ 0 \end{bmatrix},$$

$$f^2_{(11)} = f^2_{(22)} = \begin{bmatrix} q_1T_1^0\alpha_1\otimes I_{n_2}\otimes e_M & 0 \\ 0 & q_1T_1^0\alpha_1\otimes I_{n_2}\otimes e_M \end{bmatrix}.$$

For calculating the entering tagged customer $(W(t), t \geq 0)$ waiting time distribution, let's start by calculating the invariant probability vector of the system size at the arrival epoch, which is represented by $z(0) = (z_1(0), z_2(0), ...)$. The vector $z_1(0)$ with various service phases is further partitioned as $z_1(0) = (z_{101}, z_{102}, z_{113}, z_{123})$. As the customer arrives according to Marovian Arrival Process, the system size probability vector at the arrival time, in the steady state is as follows:

$$\begin{aligned} z_{1j_1j_2} &= v_{j_1j_2}\left[I_{n_2}\otimes\frac{D_1e_m}{\lambda}\right], j_1 = 0; j_2 = 1, 2, \\ z_{1j_1j_2} &= v_{j_1j_2}\left[I_{n_1}\otimes\frac{D_1e_m}{\lambda}\right], j_1 = 2; j_2 = 3, \\ z_{1j_1j_2} &= v_{j_1j_2}\left[I_{Mn_1}\otimes\frac{D_1e_m}{\lambda}\right], j_1 = 1; j_2 = 3, \\ z_i(0) &= x_i\left[I_{[n_1(M+1)(2Nn_2+1)]}\otimes\frac{D_1e_m}{\lambda}\right], \text{for } i \geq 2, \end{aligned}$$

where the fundamental arrival rate is denoted by λ. Define $z(t) = (z_*(t), z_1(t), z_2(t), ...)$ where $z_i(t), i \geq 2$ is a vector of order $1\times n_1(M+1)(2Nn_2+1)$ and $z_1(t)$ is a vector of order $1\times 2[n_2 + n_1(M+1)]$. The differential equation $z'(t) = z(t)\bar{Q}$ where $t \geq 0$ is modified as

$$\begin{aligned} z'_*(t) &= \sum_{i=1}^{2} z_i(t)H_i, \\ z'_1(t) &= z_1(t)F_1 + z_2(t)F_{21}, \\ z'_i(t) &= z_i(t)F + z_{i+1}(t)F_2, i \geq 2, \end{aligned}$$

the row vector $\omega(s)$ specifies the Laplace-Stieltjes transform (LST) of the initial transit time to level 2 with $i-2$ customers is the product of their individual LST for each service time of those

$i-2$ customers where $i \geq 2$ as in Neuts et al. (1990),

$$\begin{aligned} \omega(s) &= \sum_{i=2}^{\infty} z_i(0) \left[\int_0^{\infty} e^{-sx} e^{Fx} F_2 dx \right]^{i-2} \\ &= \sum_{i=2}^{\infty} z_i(0)[(sI-F)]^{-1} F_2]^{i-2}, \text{ for } s \geq 0. \end{aligned} \tag{1}$$

Let the Laplace-Stieltjes transform of the absorbing time to the state $(*)$ be represented by $\phi(i,s)$ if the process begins at level $i = 1, 2$,

$$\phi(1,s) = \int_0^{\infty} e^{-sx} e^{F_1 x} H_1 dx = (sI - F_1)^{-1} H_1, \tag{2}$$

Similarly,

$$\phi(2,s) = (sI-F)^{-1} H_{21} \phi(1,s) + (sI-F)^{-1} H_2. \tag{3}$$

As a result, we can clearly see that the Laplace - Stieltjes transform for the waiting time distribution is

$$\bar{W}(s) = z_1(0)\phi(1,s) + \omega(s)\phi(2,s). \tag{4}$$

6.1. Expected waiting time

The expected waiting time is

$$E(W) = -\bar{W}'(0) = -z_0(0)\phi'(1,0) - w'(0)e_{n_1(M+1)(2Nn_2+1)} - w(0)\phi'(2,0). \tag{5}$$

The first element in the preceding equation denotes the average time to access the absorbing state (*) if the system is in the state $i = 0$. On differentiating (2) and (3), and setting $s = 0$, we get,

$$\phi'(1,0) = (-1)[-F_1]^{-2} H_1, \tag{6}$$

$$\phi'(2,0) = (-1)[-F]^{-2} H_{21}\phi(1,0) + [-F]^{-1} H_{21}\phi'(1,0) - [-F]^{-2} H_2. \tag{7}$$

By using Equations (6) and (7) along with the primary conditions $z(t) = (z_1(0), z_2(0), ...)$, it's simple to estimate the first terms of (5). From (1) we have

$$w(s) = \sum_{i=2}^{\infty} z_i(0) V^{i-2}, \tag{8}$$

where the stochastic matrix $V = [-F]^{-1} F_2$. We have

$$w(0)e_{n_1(M+1)(2Nn_2+1)} = 1 - z_1(0). \tag{9}$$

Along with the primary conditions $z(t) = (z_1(0), z_2(0), ...)$, using (7) and (8), the last term of Equation (5) can be evaluated. Differentiating (1) and substituting $s = 0$, we get,

$$w'(0) = (-1) \sum_{i=1}^{\infty} z_{i+1}(0) \sum_{j=0}^{i-1} V^j [-F]^{-1} V^{i-j}. \tag{10}$$

By the condition V is stochastic, we have

$$(-1)w'(0)e_{n_1(M+1)(2Nn_2+1)} = \sum_{i=1}^{\infty} z_{i+2}(0) \sum_{j=0}^{i-2} V^j[-F]^{-1}e_{n_1(M+1)(2Nn_2+1)}. \tag{11}$$

Defining an irreducible matrix V_2 satisfying two conditions such that $1 - V + V_2$ is non singular and the generalized inverse is of the form $(I - K_1)$. Then, the matrix $V_2 = v_0 e_{n_1(M+1)(2Nn_2+1)}$ where v_0 is stationary probability vector of U such that $v_0 V = v_0$ and $v_0 e_{(M_2(2n+r)+(nM_1+1))} = 1$. Moreover V_2 satisfies the property $VV_2 = V_2 V = V_2$. Then we have,

$$\sum_{j=0}^{i-1} V^j(I - V + V_2) = 1 - V^i + iV_2, \text{for } \ i \geq 2. \tag{12}$$

Substituting (12) in (11) and simplifying, we get the following:

$$\begin{aligned}(-1)w'(0)e_{n_1(M+1)(2Nn_2+1)} = \{&x_1(I-R)^{-1}\left\{I_{n_1(M+1)(2Nn_2+1)} \otimes \frac{D_1 e_n}{\lambda}\right\} \\ &- w(0) + x_1 R(I-R)^{-2}\left\{I_{n_1(M+1)(2Nn_2+1)} \otimes \frac{D_1 e_n}{\lambda}\right\}\} \\ &\times [I - V + V_2]^{-1}[-F]^{-1}e_{n_1(M+1)(2Nn_2+1).}\end{aligned} \tag{13}$$

Thus, we have found all the terms of (5) which helps further to evaluate the expected waiting time.

7. Performance Measures

This section examines our model in delivering service quality at steady-state. A few performance measures are suggested to assess the feature of system measurements.

- Probability of the regular server is idle
  $P_I = x_{003} + \sum_{i=1}^{\infty} \sum_{s=1}^{2} x_{i0s}$.
- Probability of the main server is on vacation
  $P_V = \sum_{i=1}^{\infty} \sum_{r=1}^{2} x_{ir3}$.
- Probability of the regular server offering regular service
  $P_{RSR} = x_{110} + \sum_{i=2}^{\infty} \sum_{s=1}^{3} x_{i1s}$.
- Probability of the regular server offering optional service
  $P_{RSO} = x_{123} + \sum_{i=2}^{\infty} \sum_{s=1}^{3} x_{i2s}$.
- Probability of the main server offering regular service
  $P_{MSR} = x_{101} + \sum_{i=2}^{\infty} \sum_{r=1}^{2} x_{ir1}$.
- Probability of the main server offering optional service
  $P_{MSO} = x_{123} + \sum_{i=2}^{\infty} \sum_{r=1}^{3} x_{ir2}$.

- Probability of the main server is on vacation while the regular server offering regular service
  $P_{RSRV} = \sum_{i=2}^{\infty} x_{i13}$.
- Probability of the main server is on vacation while the regular server offering optional service
  $P_{RSOV} = \sum_{i=1}^{\infty} x_{i23}$.
- Probability of the main server offers consultation for regular server while offering regular service
  $P_{RSRC} = \sum_{i=1}^{\infty} x_{i10} + \sum_{i=2}^{\infty} \sum_{s=1}^{2} x_{i1s}$.
- Probability that both the server provides regular service
  $P_{RSRMSR} = \sum_{i=2}^{\infty} x_{i11}$.
- Probability of the regular server providing regular service while the main server offers optional service
  $P_{RSRMSO} = \sum_{i=2}^{\infty} x_{i12}$.
- Probability that both the server provides optional service
  $P_{RSOMSO} = \sum_{i=2}^{\infty} x_{i22}$.
- Probability of the regular server providing optional service while the main server offers regular service
  $P_{RSOMSR} = \sum_{i=2}^{\infty} x_{i21}$.
- Probablity that the system is empty
  $P_{emp} = X_0$.
- Expected system size when both the server offers regular service

$$\begin{aligned} E_{RSRMSR} &= \sum_{i=2}^{\infty} i x_{i11} e \\ &= X_2[(1-R)^{-2} + (1-R)^{-1}]e_1. \end{aligned}$$

- Expected system size when the regular server offering regular service while the main server offers optional service

$$\begin{aligned} E_{RSRMSO} &= \sum_{i=2}^{\infty} i x_{i12} e \\ &= X_2[(1-R)^{-2} + (1-R)^{-1}]e_2. \end{aligned}$$

- Expected system size when the regular server offering optional service while the main server offers regular service

$$\begin{aligned} E_{RSOMSR} &= \sum_{i=2}^{\infty} x_{i21} e \\ &= X_2[(1-R)^{-2} + (1-R)^{-1}]e_4. \end{aligned}$$

- Expected system size when both the server offering optional service

$$\begin{aligned} E_{RSOMSO} &= \sum_{i=2}^{\infty} i x_{i22} e \\ &= X_2[(1-R)^{-2} + (1-R)^{-1}]e_5. \end{aligned}$$

- Expected system size when the main server is on vacation and the regular server offering optional service

$$\begin{aligned} E_{RSOV} &= \sum_{i=1}^{\infty} i x_{i23} e \\ &= X_1 e_7 + X_2[(1-R)^{-2} + (1-R)^{-1}] e_6. \end{aligned}$$

- Expected system size when the main server is on vacation and the regular server offering regular service

$$\begin{aligned} E_{RSRV} &= \sum_{i=2}^{\infty} i x_{i13} e \\ &= X_1 e_8 + X_2[(1-R)^{-2} + (1-R)^{-1}] e_3. \end{aligned}$$

- Expected system size

$$\begin{aligned} E_{system} &= \sum_{i=1}^{\infty} i X_i e \\ &= X_1 e_{2m[n_2+n_1(M+1)]} + X_2[(1-R)^{-2} + (1-R)^{-1}] e_{n_1 m[(M+1)(2Nn_2+1)]}. \end{aligned}$$

8. Numerical Results

The qualitative behaviour of this model will be understood in this section with the help of a few illustrations, both numerically and visually, by adjusting various model factors like the arrival process and service time distribution. Three sets of values from the literature provided by Chakravarthy (2010) are taken as input for both the arrival procedure, with mean value one and the service time distribution.

Erlang of order 2 (ERL-A)

$$D_0 = \begin{bmatrix} -2 & 2 \\ 0 & -2 \end{bmatrix}; D_1 = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}.$$

Exponential (Exp-A)

$$D_0 = \begin{bmatrix} -1 \end{bmatrix}; D_1 = \begin{bmatrix} 1 \end{bmatrix}.$$

Hyperexponential (HYP-EXP-A)

$$D_0 = \begin{bmatrix} -1.90 & 0 \\ 0 & -0.19 \end{bmatrix}; D_1 = \begin{bmatrix} 1.710 & 0.190 \\ 0.171 & 0.019 \end{bmatrix}.$$

Consider three phase type distributions for the service process as follows,
Erlang of order 2 (ERL-S)

$$\alpha = (1, 0); T = \begin{bmatrix} -2 & 2 \\ 0 & -2 \end{bmatrix}.$$

Exponential (Exp-S)

$$\alpha = (1); T = \begin{bmatrix}-1\end{bmatrix}.$$

Hyperexponential (HYP-EXP-S)

$$\alpha_1 = (0.3, 0.7); T_1 = \begin{bmatrix}-9 & 3\\ 2 & -8\end{bmatrix}.$$

$$\alpha_2 = (0.4, 0.6); T_2 = \begin{bmatrix}-12 & 6\\ 5 & -10\end{bmatrix}.$$

$$\beta_1 = (0.4, 0.6); T_3 = \begin{bmatrix}-6 & 4\\ 3 & -4\end{bmatrix}.$$

$$\beta_2 = (0.5, 0.5); R = \begin{bmatrix}-12 & 3\\ 3 & -12\end{bmatrix}.$$

Illustration 1.

We explore the impact of the vacation rate (η) versus the predicted system size (Expected System Size) in Tables 3-5 . To satisfy the stability criteria, set $\lambda = 1, p_1 = 0.6, q_1 = 0.4, p_2 = 0.6, q_2 = 0.4$ and the following interpretations are extracted.

- For separate acceptable groupings of service and arrival times, as (η), the vacation rate increases, predicted system size decreases. Because raising the vacation rate reduces the length of vacation time, resulting in greater server availability for providing the service, resulting in a smaller expected system size.
- In Hyper-exponential-A, the expected size of the system decreases rapidly, gradually for Exponential-A and moderately for an Erlang-A correlating the above values for distinct combination of arrival and service times.

Illustration 2.

Tables 6 through 8 represent the fundamental rate (λ) verses expected waiting time (E(W)) for all conceivable options of arrival and service time by fixing $\lambda = 1, \eta = 15, p_1 = 0.6, q_1 = 0.4, p_2 = 0.6, q_2 = 0.4$ so that the stability criteria is satisfied.

- As the fundamental entry rate (λ) maximizes expected waiting time maximizes for distinct suitable groupings of service and arrival times can be interpreted. Usually since, with a fixed service rate, expanding the arrival rate leads to the collection of customer in the system which is direct proposition to the expected waiting time of the system. Moreover, when the customer opts for optional service then consumption of time for service of that particular customer increases. This will also leads to the increase of waiting time of the customers in the queue.

- The expected waiting time measure of the system increases highly on Hyper-exponential-A, averagely on Exponential-A and pretty gradual for an Erlang-A.

Illustration 3.

The effect of regular server offering regular service, that is, service rate (μ_1) verses expected system size is explored in Figures 2 through 10. In the way of fulfilling the stability condition, setting $\lambda = 10, \eta = 15, p_1 = 0.6, q_1 = 0.4, p_2 = 0.6, q_2 = 0.4$, the following are observations that are made.

- For various conceivable options of arrival and service time the regular server offers regular services verses expected system size are interpreted. With the fixed arrival rate and an escalation in regular service by regular server rate will leads to the decrease in the expected system size.
- The expected system size of the system falls highly on Hyper-exponential-A, declines slowly on Exponential-A and sluggishly on Erlang-A.

Illustration 4.

The impact of a regular server providing optional service, that is, service rate (μ_2) versus the expected system size is investigated in Figures 11 through 19. The following observations are performed in order to satisfy the stability criterion, with $\lambda = 10, \eta = 15, p_1 = 0.6, q_1 = 0.4, p_2 = 0.6, q_2 = 0.4$.

- The regular server provides optional services vs expected system size for various probable arrival and service time configurations. With a fixed arrival rate and an increase in optional service by a regular server, the expected system size will be reduced due to the server's increased availability.
- On Hyper-exponential-A, the expected system size decreases rapidly, average on Exponential-A, and slow on Erlang-A.

Illustration 5.

The impact of vacation rate (η) of the main server and service rate (μ_1), that is, the regular server offering regular service versus the expected system size is investigated in Figures 20-28. The following observations are made in order to satisfy the stability criteria by setting $\lambda = 5, p_1 = 0.6, q_1 = 0.4, p_2 = 0.6, q_2 = 0.4$.

- The vacation rate (η) of the main server and service rate (μ_1), that is, the regular server offering regular service versus the expected system size for various conceivable arrival and service time configurations. With a fixed arrival rate and an increase in rate of regular service by a regular server and increase in vacation rate (η), the expected system size will be decrease. As the main server is on vacation, the function of service falls on the regular service as a result their will be an increase in the length of the queue. The piled up customer is direct proportional to the hike in the expected system size of the system.

- Expected system size grows largely on Hyper-exponential-A, average on Exponential-A, and upticks on Erlang-A.

Illustration 6.

Figures 29 though 37 exhibit the impact of the main server's vacation rate (eta) on the service rate (μ_1), that is, regular server providing regular service vs the expected waiting time. By choosing, $\lambda = 5, p_1 = 0.6, q_1 = 0.4, p_2 = 0.6, q_2 = 0.4$, the following observations are performed in order to satisfy the stability criteria.

- For various potential arrival and service time configurations, the main server's service rate (μ_1) and vacation rate (η), that is, the regular server giving regular service versus the expected waiting time. As the duration of vacation of the main server decreases the opportunity of getting service by the customer increases which has a direct impact on reducing waiting time of each customer causing a declining queue. The number of customers in line is directly proportional to the decrease in the system size. This scenario causes the decline in expected waiting time of the customer.
- The expected system size is falls off large on Hyper-exponential-A, average on Exponential-A, and slowly on Erlang-A.

Table 3. Vacation rate (η) vs Expected System Size - Exponential-S

η	Exponential	Erlang	Hyperexponential
2	0.225526066	0.184140293	0.239950236
3	0.224516392	0.183992277	0.239129666
4	0.223558310	0.183849618	0.238346875
5	0.222647892	0.183712049	0.237599309
6	0.2217816131	0.183579322	0.236884653
7	0.220956298	0.183451205	0.236200777
8	0.220169079	0.183327479	0.235545742
9	0.219417355	0.183207939	0.234917768
10	0.220169079	0.183092391	0.234315222
11	0.219417355	0.182980653	0.233736602
12	0.218698766	0.182872553	0.233180522
13	0.218011159	0.182767936	0.232645702
14	0.217352569	0.182666629	0.232130967
15	0.216721198	0.182568507	0.231635199
16	0.216115396	0.182473426	0.231157402

Table 4. Vacation rate (η) vs Expected System Size - Erlang-S

η	Exponential	Erlang	Hyperexponential
2	0.101634059	0.093136107	0.198327823
3	0.101540909	0.093105956	0.197772059
4	0.101450149	0.093076529	0.197249744
5	0.101361685	0.093047792	0.196757627
6	0.101275428	0.093019722	0.196292874
7	0.101191295	0.092992294	0.195853064
8	0.101109205	0.092965487	0.195436058
9	0.101029083	0.092939280	0.195039949
10	0.100950856	0.092913652	0.194663097
11	0.100874457	0.092888584	0.194304011
12	0.100799820	0.092864058	0.193961368
13	0.100726882	0.092840055	0.193633982
14	0.100655586	0.092816560	0.193320784
15	0.100585875	0.092793556	0.193020808
16	0.100517695	0.092771027	0.192733179

Table 5. Vacation rate (η) vs Expected System Size - Hyper-exponential-S

η	Exponential	Erlang	Hyperexponential
2	0.336611408	0.201634059	0.443210789
3	0.336551881	0.201540909	0.436713192
4	0.336521826	0.201450149	0.430677596
5	0.336503758	0.201361685	0.425057777
6	0.336491576	0.201275428	0.419813363
7	0.336482898	0.201191295	0.414908944
8	0.336476378	0.201109205	0.410313349
9	0.336471302	0.201029083	0.405999037
10	0.336467236	0.200950856	0.401941587
11	0.336463908	0.200874457	0.398119279
12	0.336461132	0.200799827	0.394512732
13	0.336458782	0.200726882	0.391104606
14	0.336456767	0.200655586	0.387879335
15	0.336455029	0.200585875	0.384822919
16	0.336453497	0.200517695	0.381922727

Table 6. Fundamental rate (λ) vs Expected Waiting Time - Exponential-S

λ	Exponential	Erlang	Hyperexponential
1	0.032775803	0.012760774	0.495598116
2	0.063333154	0.015591816	1.000476639
3	0.091715248	0.021732759	1.511909029
4	0.117962216	0.032612905	2.027060449
5	0.142111289	0.045201760	2.543005025
6	0.164196958	0.059329986	3.056744090
7	0.184251145	0.059374012	3.565224683
8	0.202303364	0.074850420	4.065357339
9	0.218380890	0.091632242	4.554031648
10	0.232508922	0.10955669	5.028127357
11	0.244710757	0.128513922	5.484517402
12	0.255007951	0.148400759	5.92005688
13	0.263420491	0.16911908	6.331547174
14	0.269966961	0.190574732	6.715654487
15	0.274664703	0.212676828	7.068739278

Table 7. Fundamental rate (λ) vs Expected Waiting Time - Erlang-S

λ	Exponential	Erlang	Hyperexponential
1	0.014679687	0.001444143	0.30196654
2	0.028587963	0.001522725	1.030018797
3	0.041734555	0.001790359	1.975443882
4	0.054129079	0.002149539	2.984112295
5	0.065781038	0.002515812	3.943804405
6	0.076699832	0.002883876	4.778473708
7	0.086894767	0.003248601	4.945885278
8	0.096375054	0.003255991	5.395378078
9	0.105149823	0.003605771	5.443230505
10	0.113228122	0.003951304	5.778545372
11	0.120618928	0.004281623	5.918489991
12	0.127331149	0.004593564	6.074430438
13	0.133373632	0.004639879	6.203610364
14	0.138755167	0.005501025	6.259920081
15	0.143484496	0.008167498	6.310801747

Table 8. Fundamental rate (λ) vs Expected Waiting Time - Hyper-exponential-S

λ	Exponential	Erlang	Hyperexponential
1	0.073288307	0.228484131	5.149883888
2	0.183342768	0.228809406	5.544319914
3	0.252609093	0.229129487	5.830209226
4	0.263909794	0.229443674	6.018092157
5	0.309303104	0.229751201	6.100175356
6	0.359799767	0.230051223	6.118800395
7	0.405744246	0.230342816	6.142871738
8	0.412731463	0.230624962	6.556154522
9	0.447220617	0.230896541	6.637494167
10	0.483485527	0.231156316	6.873502779
11	0.493152554	0.231402923	7.164176741
12	0.513075028	0.231634854	7.426306367
13	0.531071254	0.231850437	7.656327036
14	0.533638682	0.232047817	7.849426352
15	0.541533591	0.232224933	7.999695832

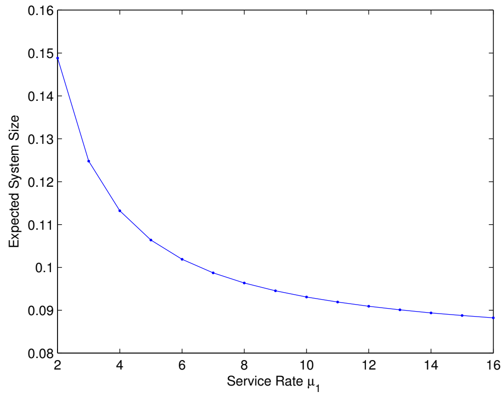

Figure 2. Service rate (μ_1) vs Expected System Size - M/M/1

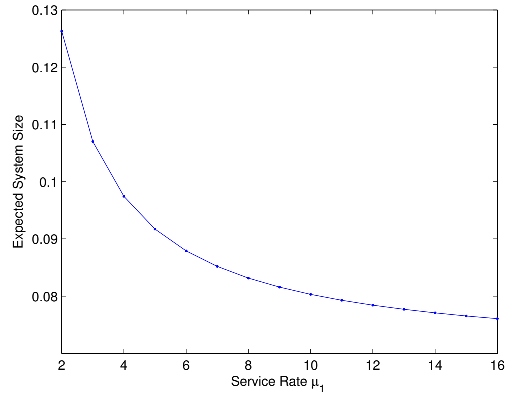

Figure 3. Service rate (μ_1) vs Expected System Size - M/Ek/1

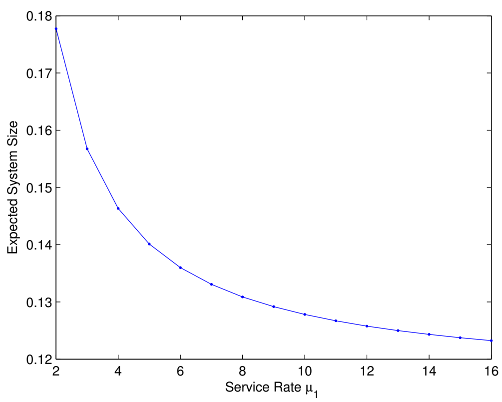

Figure 4. Service rate (μ_1) vs Expected System Size - M/Hk/1

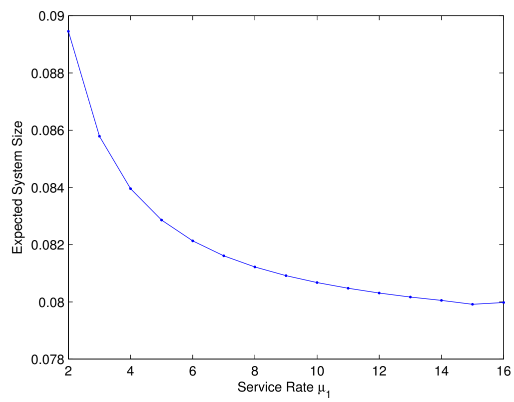

Figure 5. Service rate (μ_1) vs Expected System Size - Ek/M/1

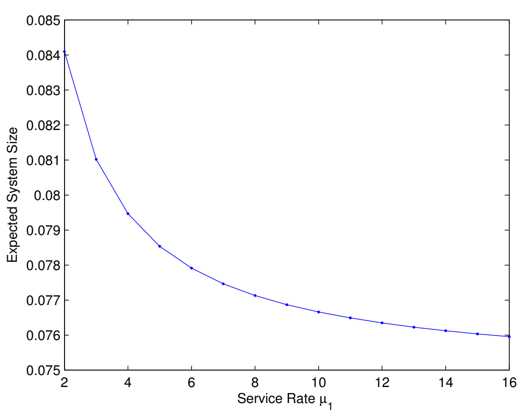

Figure 6. Service rate (μ_1) vs Expected System Size - Ek/Ek/1

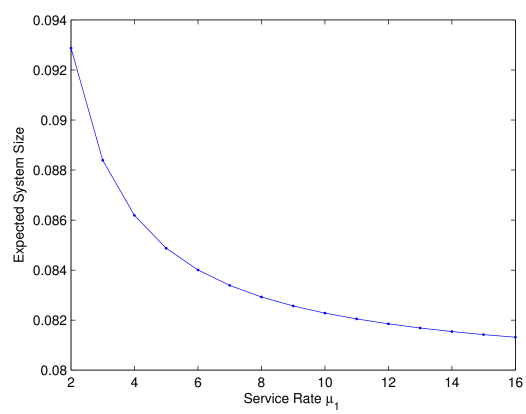

Figure 7. Service rate (μ_1) vs Expected System Size - Ek/Hk/1

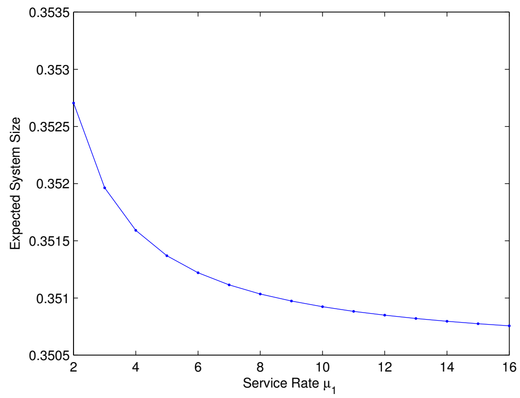

Figure 8. Service rate (μ_1) vs Expected System Size - Hk/M/1

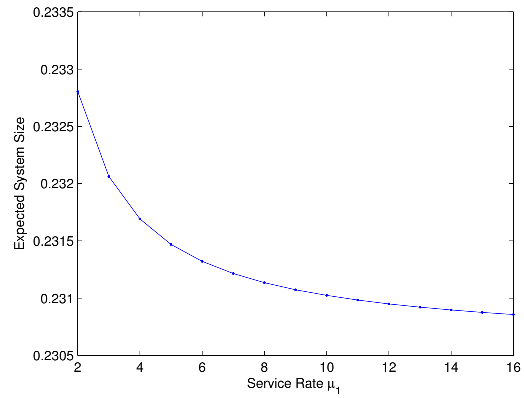

Figure 9. Service rate (μ_1) vs Expected System Size - Hk/Ek/1

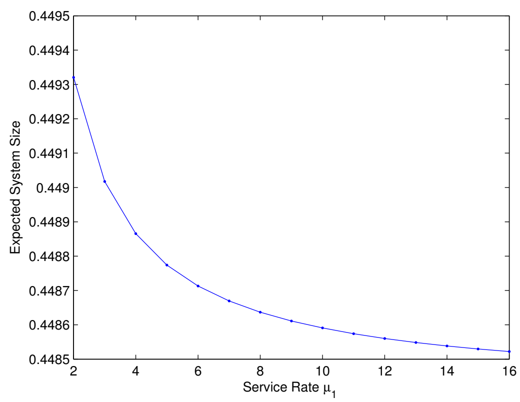

Figure 10. Service rate (μ_1) vs Expected System Size - Hk/Hk/1

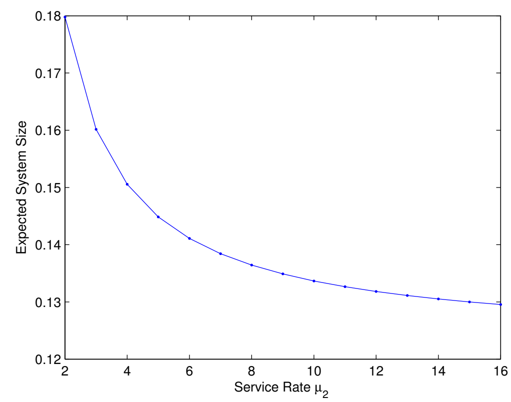

Figure 11. Service rate (μ_2) vs Expected System Size - M/M/1

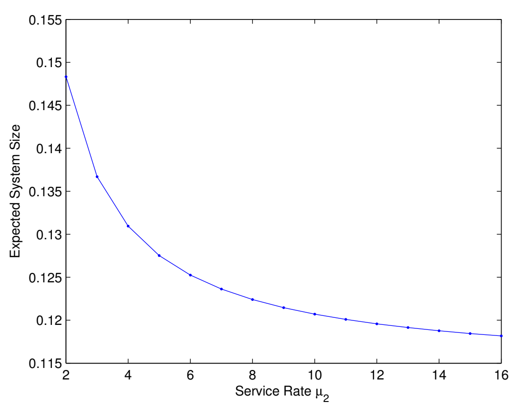

Figure 12. Service rate (μ_2) vs Expected System Size - M/Ek/1

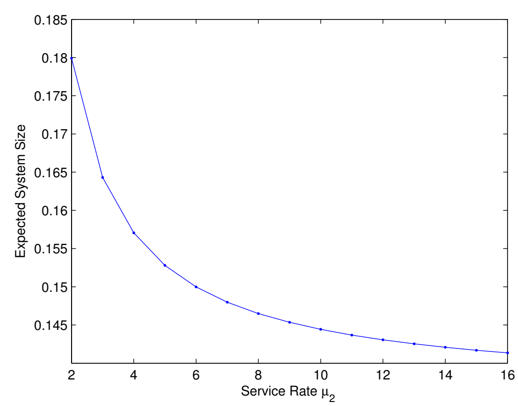

Figure 13. Service rate (μ_2) vs Expected System Size - M/Hk/1

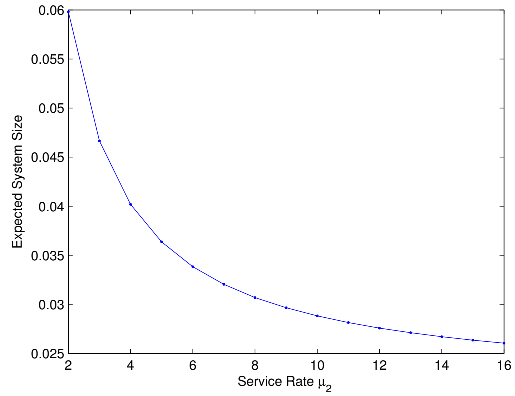

Figure 14. Service rate (μ_2) vs Expected System Size - Ek/M/1

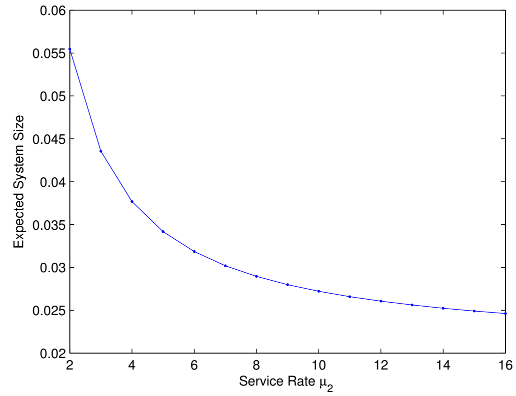

Figure 15. Service rate (μ_2) vs Expected System Size - Ek/Ek/1

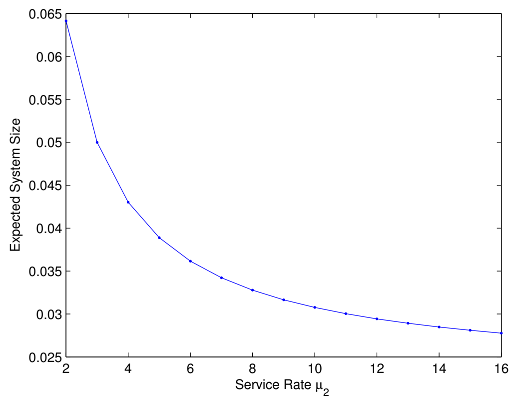

Figure 16. Service rate (μ_2) vs Expected System Size - Ek/Hk/1

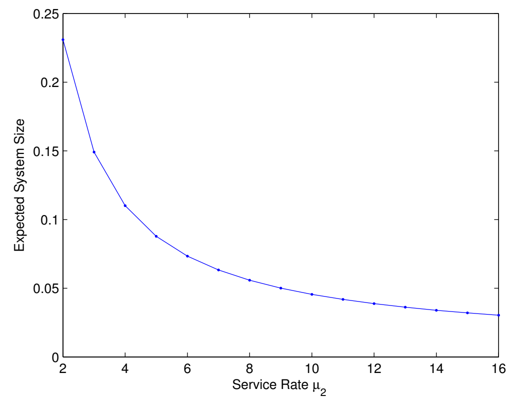

Figure 17. Service rate (μ_2) vs Expected System Size - Hk/M/1

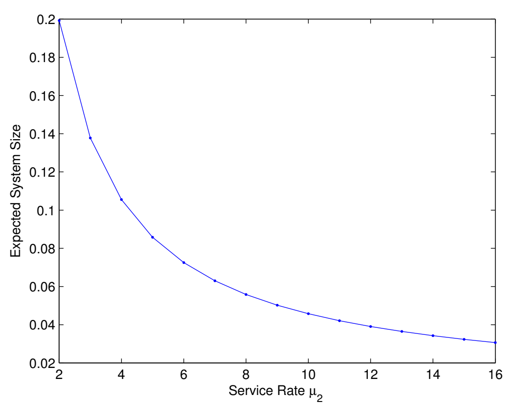

Figure 18. Service rate (μ_2) vs Expected System Size - Hk/Ek/1

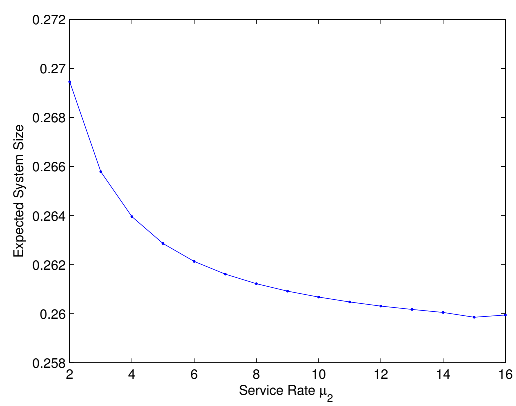

Figure 19. Service rate (μ_2) vs Expected System Size - Hk/Hk/1

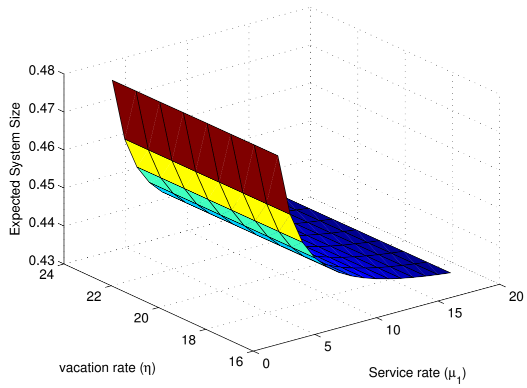

Figure 20. Vacation Rate (η) and Service rate (μ_1) vs Expected System Size - M/M/1

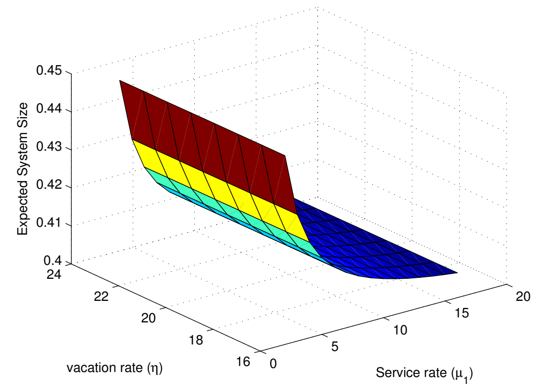

Figure 21. Vacation Rate (η) and Service rate (μ_1) vs Expected System Size - M/Ek/1

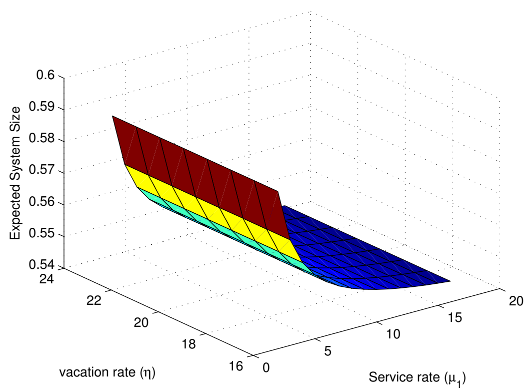

Figure 22. Vacation Rate (η) and Service rate (μ_1) vs Expected System Size - M/Hk/1

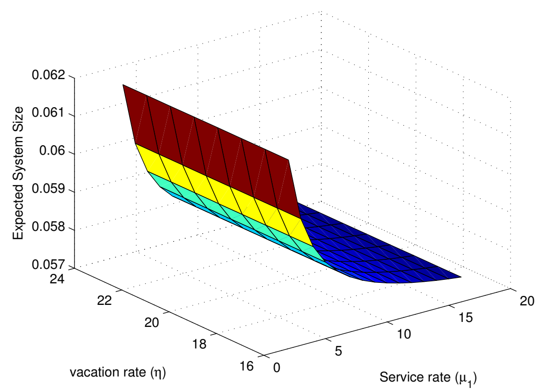

Figure 23. Vacation Rate (η) and Service rate (μ_1) vs Expected System Size - Ek/M/1

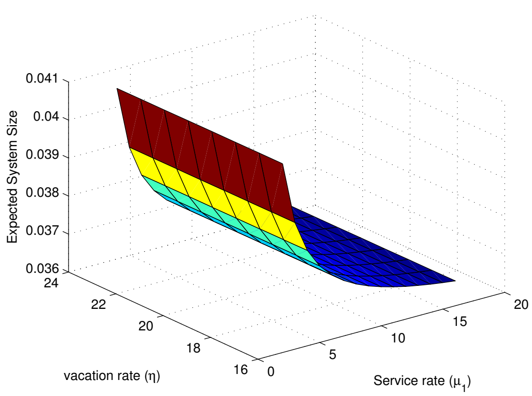

Figure 24. Vacation Rate (η) and Service rate (μ_1) vs Expected System Size - Ek/Ek/1

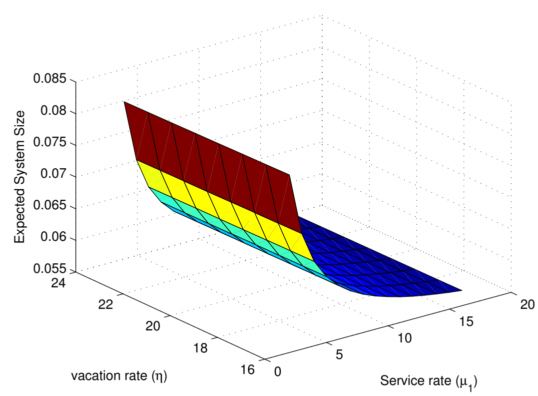

Figure 25. Vacation Rate (η) and Service rate (μ_1) vs Expected System Size - Ek/Hk/1

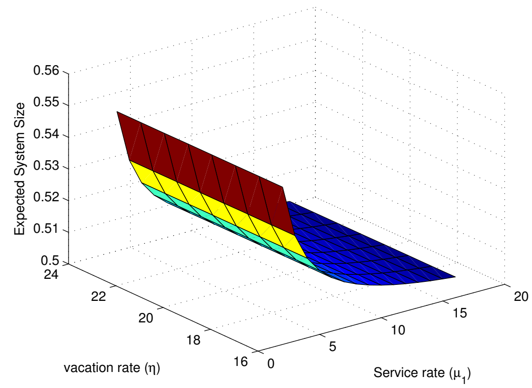

Figure 26. Vacation Rate (η) and Service rate (μ_1) vs Expected System Size - Hk/M/1

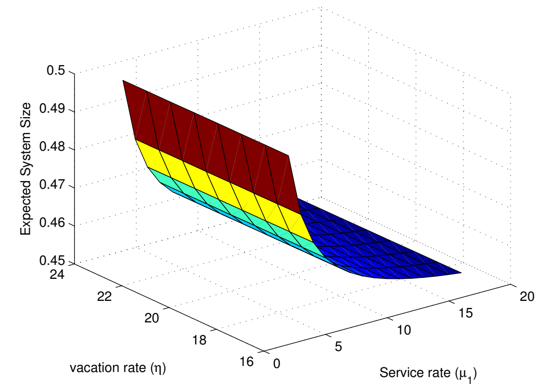

Figure 27. Vacation Rate (η) and Service rate (μ_1) vs Expected System Size - Hk/Ek/1

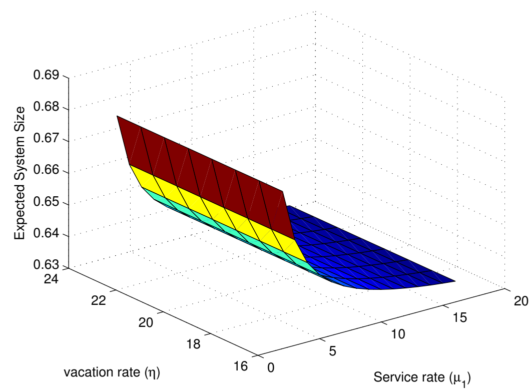

Figure 28. Vacation Rate (η) and Service rate (μ_1) vs Expected System Size - M/M/1

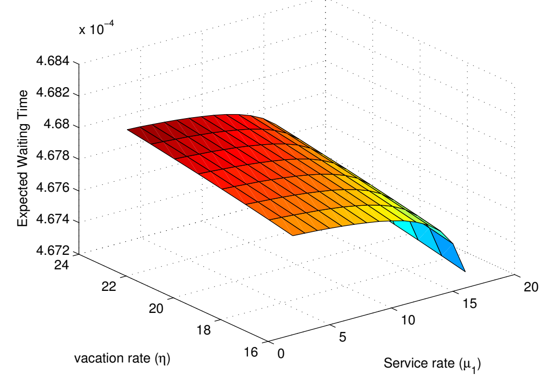

Figure 29. Vacation Rate (η) and Service rate (μ_1) vs Expected Waiting Time - M/M/1

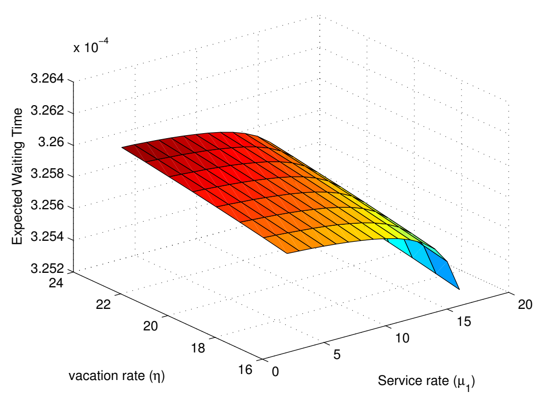

Figure 30. Vacation Rate (η) and Service rate (μ_1) vs Expected Waiting Time - M/Ek/1

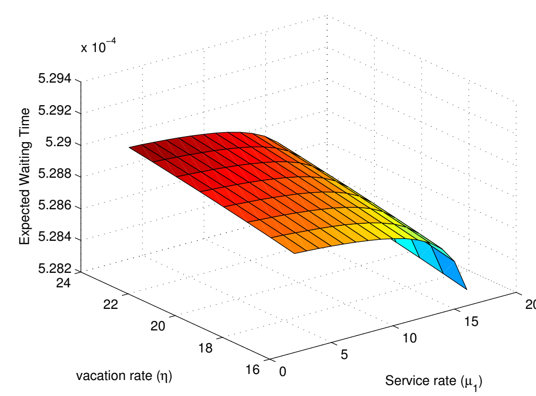

Figure 31. Vacation Rate (η) and Service rate (μ_1) vs Expected Waiting Time - M/Hk/1

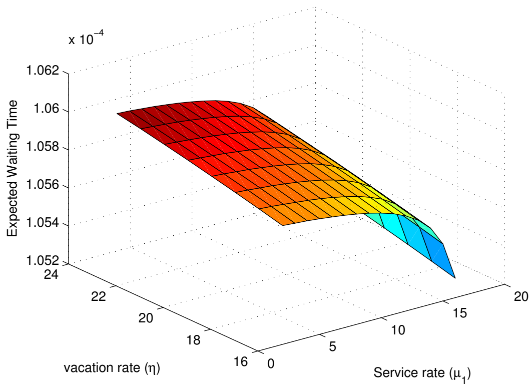

Figure 32. Vacation Rate (η) and Service rate (μ_1) vs Expected Waiting Time - Ek/M/1

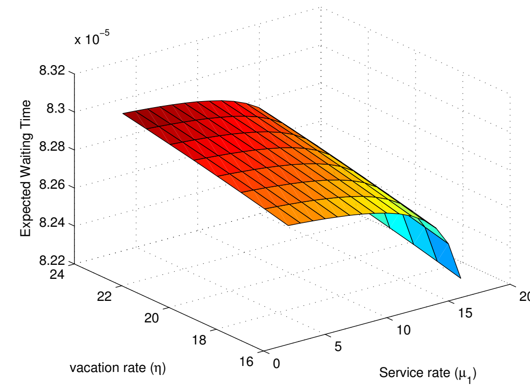

Figure 33. Vacation Rate (η) and Service rate (μ_1) vs Expected Waiting Time - Ek/Ek/1

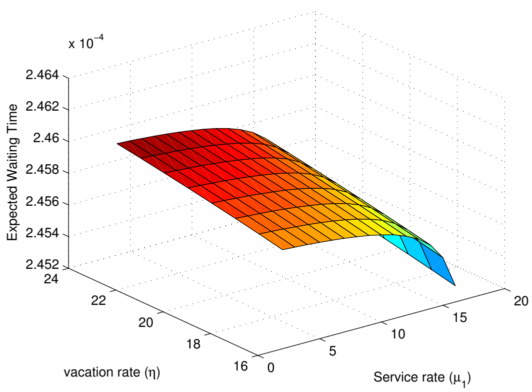

Figure 34. Vacation Rate (η) and Service rate (μ_1) vs Expected Waiting Time - Ek/Hk/1

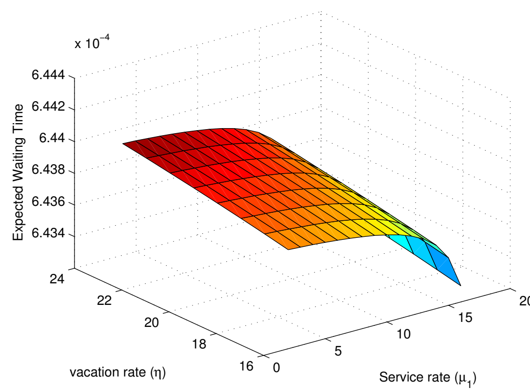

Figure 35. Vacation Rate (η) and Service rate (μ_1) vs Expected Waiting Time - Mk/M/1

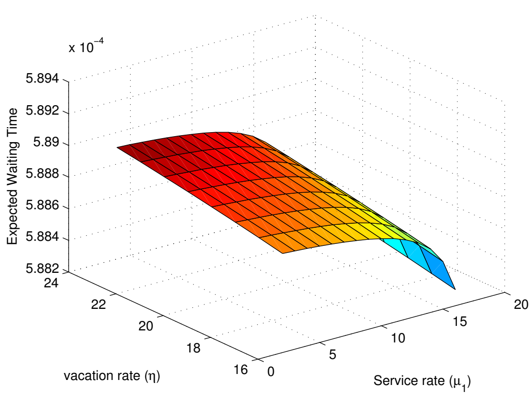

Figure 36. Vacation Rate (η) and Service rate (μ_1) vs Expected Waiting Time - Hk/Ek/1

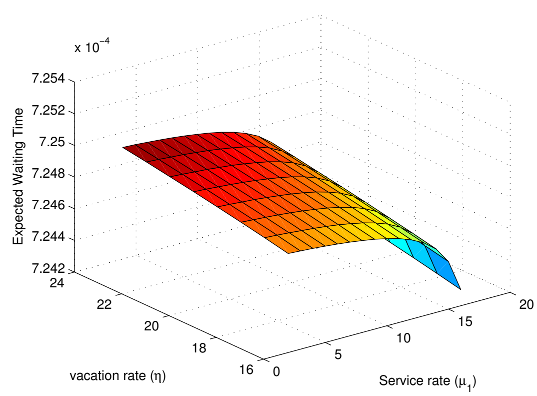

Figure 37. Vacation Rate (η) and Service rate (μ_1) vs Expected Waiting Time - Mk/Mk/1

9. Conclusion

The regular and primary servers in this study provide two types of services with multiple vacations, optional service, consultation, and interruptions, with the arrival following a Markovian arrival process and the service following a phase-type distribution. During the service, if the regular server has any doubts regarding the service, the server will seek clarification from the main server. As a customer service should not be prolong, the consultation and interruptions are restricted in number. When the system size is zero, the main server can either take a vacation or wait in the service station for regular server consultation. The invariant probability vector was utilized to assess the busy period and estimated waiting time for this system, which was represented by a QBD process. The stability criterion also takes into account some numerical data and graphical analysis. This model can be investigated further in a multi-server environment with batch arrival.

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