[Applications and Applied Mathematics: An International Journal (AAM)](https://digitalcommons.pvamu.edu/aam)

Volume 19 | Issue 1 Article 1

12-2022

(R1974) A Multi Server Markovian Working Vacation Queue With Server State Dependent Rates and with Partial Breakdown

A. Sundaramoorthy
CK College of Engineering and Technology

R. Kalyanaraman
Annamalai University

Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466

Applications and Applied Mathematics:
An International Journal
(AAM)

Vol. 19, Issue 1 (June 2024), 21 pages

A Multi Server Markovian Working Vacation Queue with Server State Dependent Rates and with Partial Breakdown

[1]A. Sundaramoorthy and [2]R. Kalyanaraman

[1]Department of Mathematics
CK College of Engineering and Technology
Cuddalore-607003
Tamilnadu, India
[2]Department of Mathematics
Annamalai University
Annamalainagar-608002
Tamilnadu, India
[1]a.sundarmaths92@gmail.com; [2]r.kalyan24@rediff.com

Received: March 12, 2022; Accepted: March 6, 2024

Abstract

In this article, we consider an $M/M/C$ queue in which the arrival rate and service rate depends on the state of the system. In addition, the servers takes working vacation and the system may breakdown. Whenever breakdown takes place, the repair process immediately commences. During the repair period the customers are given service in a reduced service rate. Based on the vacation termination point, two models have been defined. The steady state probability vector of the number of customers in the queue and the stability condition are obtained using Matrix-Geometric method. The stationary waiting time distributions have been obtained. Some illustrative examples are also provided.

Keywords: Markovian queue; Multi server; Partial breakdown; Steady-state probability vector; Working vacation; State dependent arrival rate; Matrix-Geometric method

MSC 2010 No.: 90B22, 60K25 and 60K30

1. Introduction

For many practical queueing situations, it can bee seen that (i) the system has more than one server, (ii) the server may do some other work during their free hours, (iii) the arrival rate of customer varies, (iv) the service rate to customer also varies, and (v) the system may breakdown. In this paper we designed a queueing system to consider all the above points. In brief the system discussed in this article is an $M/M/C$ queueing system with working vacation to server and state dependent arrival service rates.

Recent years have seen an increasing interest in queueing systems with server working vacation due to their applications in telecommunication systems, manufacturing systems, and computer systems. In many real life queueing situations, it can be seen that the server works during its rest period, if the necessity occurs, called working vacation period. But, in such a queue, the server works with variable service rate, in particular reduced service rate, rather than completely stops service during vacation period. Also in real life, we have encountered that the systems may failed. With all these points in mind, we have proposed and analyzed the model in this paper.

Servi and Finn (2002) have first analyzed an $M/M/1$ queue with multiple working vacation in which the vacation times are exponentially distributed. During the vacation period, the server serves in reduced rate; Wu and Takagi (2006) extend this work for $M/G/1$ queue. Kim et al. (2003) analyzed the queue length distribution of the $M/G/1$ queue with working vacations. Liu et al. (2007) examined stochastic decomposition structure of the queue length and waiting time in an $M/M/1$ working vacation queue. Xu et al. (2009) extended the $M/M/1$ working vacation queue to an $M^{[x]}/M/1$ working vacation queue. Li et al. (2009) used the matrix analytic method to analyze an $M/G/1$ queue with exponential working vacation under a specific assumption. Lin and Ke (2009) consider a multi server queue with single working vacation. Jain and Jain (2010) investigated a single working vacation model with server break down. Ke et al. (2010) have given a short survey on vacation models in recent years. Haghighi and Mishev (2016) analyzed busy periods of a single-server Poisson queueing system with splitting and batch delayed-feedback.

The $C-$ server Markovian queue with exponentially distributed vacation was first studied by Levy and Yechiali (1976). The same model has been studied by Vinod (1986) using matrix geometric method. Chao and Zhao (1998) investigated a multi server model and provided an algorithm for finding the stationery distribution and performance measures. Tian et al. (1999) and Zhang and Tian (2003) established stochastic decomposition results for a multi-server Markovian queue with vacation.

Yechiali and Naor (1971) have considered a unreliable single-server exponential queueing model with arrival state depending on operational state or breakdown state of the server. Fond and Ross (1977) analyzed the same model with the assumption that any arrival finding the server busy is lost, and they obtained the steady-state proportion of customer's lost. Shogan (1979) has deals with a single server queueing model with arrival rate depending on server state. Shanthikumar (1982) has analyzed a single server Poisson queue with arrival rate depending on the state of the server. Haghighi et al. (1986) discussed the Multi-Server Markovian Queueing System with Balking and

Reneging. Jayaraman et al. (1994) analyzed a general bulk service queue with arrival rate depending on server breakdowns. Haghighi (1998) analyzed an Analysis of a Parallel Multi-Processor System with Task Split and Feedback. Tian and Yue (2002) discussed the queueing system with variable arrival rate. The authors studied the model by using the principle of quasi-birth and death process (QBD) and matrix-geometric method. Furthermore, they calculated some performance measures. The Matrix-geometric method approach is a useful tool for solving the more complex queueing problems. The Matrix-geometric method has been applied by many researchers to solve various queueing problems in different frameworks. Neuts (1981) explained various matrix geometric solutions of stochastic models. The Matrix-geometric approach is utilized to develop the computable explicit formula for the probability distributions of the queue length and other system characteristic.

Haghighi and Mishev (2008) have considered a busy period of a single-server Poisson queueing system with splitting and batch delayed-feedback. Kalyanaraman and Sundaramoorthy (2019) studied a Markovian working vacation queue with state dependent rates and partial breakdown. To the best of our knowledge, in the study of working vacation queue, the existing literatures focus mainly on queueing systems with server state independent arrival rates; in this work we deviate from these works by assuming server state dependent arrival rates with C servers.

In this paper, we consider two $M/M/C$ and with partial breakdown queues with multiple working vacation. For both the models the arrival rate and service rate depend on the server states. The models has been analyzed using matrix geometric method. The rest of this paper is organized as follows. In Section 2, we give the model description, as quasi-birth-death process. In Section 3, we present the steady state solution using matrix geometric method for Model-I. In Section 4, we present the steady state solution using matrix geometric method for Model-II. In Section 5, we present particular model. Section 6 gives some performance measures. In Section 7, we have derived the stationary waiting time distribution in queue. In Section 8, we presents some numerical examples. The last section ends with a conclusion.

2. The Mathematical Model

We consider a multi-server queueing system with the following characteristics:

(1) The system alternate between three states: the regular state, the working vacation state, and breakdown state.

(2) The arrival process follows Poisson with parameter λ during regular state.

(3) In regular state, the server serves customers based on an exponential distribution with rate μ.

(4) All the servers take vacation if there are no customer in the system at a service completion point. If there are less than C customers in the system, that is, less than C servers are busy, the remaining server waits.

(5) During vacation, the arrival follows Poisson with rate λ_1 $(\lambda_1 < \lambda)$.

(6) Vacation period follows negative exponential with rate θ and the vacation policy is multiple vacation policy; that is, the servers continue vacation until the servers finds at least one customer at a vacation completion point (Model-I) / the server finds at least $C-$ customers at a vacation completion point (Model-II).

(7) When the servers are in vacation, if a customer arrives, one of the servers serve the customers using exponential distribution with rate μ_1 $(\mu_1 < \mu)$. As this vacation period ends, the server instantaneously switches over to the normal service rate μ, if there is at least one customer waiting for service (Model-I) / if there is at least $C-$ customers waiting for service (Model-II). Upon completion of a service at a vacation period, the server will (i) continue the current vacation if it is not finished and no customer is waiting for service (Model-I)/$(C-1)$ customers is waiting for service (Model-II); (ii) continue the service with rate μ_1 if the vacation has not expired and if there is at least one customer waiting for service.

(8) During service of customers in the normal busy period, the system may breakdown. The number of breakdowns follows Poisson process with parameter α. Once the system break downs, the repair to the system server starts immediately and the duration of repaired period follows negative exponential with rate β.

(9) During repair period customers arrive according to Poisson process with rate λ_2 $(\lambda_2 < \lambda_1)$.

(10) During repair period the server serves the customers, the service period follows negative exponential with rate μ_2 $(\mu > \mu_1 > \mu_2)$

(11) The first come first served (FCFS) service rule is followed to select the customers for service.

The models defined in this article can be studied as a Quasi Birth and Death (QBD) process. The following notations are necessary for the analysis:

Let $L(t)$ be the number of customers in the queue at time t and let

$$J(t) = \begin{cases} 0, \text{if the servers are on working vacation,} \\ 1, \text{if the servers are busy,} \\ 2, \text{if the servers are on partial breakdown,} \end{cases}$$

be the server state at time t.

Let $X(t) = (L(t), J(t))$. Then $\{(X(t)) : \ t \geq 0\}$ is a Continuous Time Markov Chain (CTMC) with state space $S = \{(i, j) : \ i \geq 0; j = 0, 1\}$, where i denotes the number of customer in the queue and j denotes the server state.

Using the lexicographical sequence for the states, the rate matrix Q (Model-I)/Q_1 (Model-II) has been formed, is the infinitesimal generator of the Markov chain.

3. Model-I

In this section, we completely analyze Model-I,

$$
Q = \begin{array}{cc}
 & \begin{array}{cccccccccc} 0 & 1 & 2 & 3 & \dots & C-2 & C-1 & C & C+1 & \dots \end{array} \\
\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ \vdots \\ C-2 \\ C-1 \\ C \\ C+1 \\ \vdots \end{array} &
\begin{pmatrix}
B_0 & A_0 & & & & & & & & \\
B_{10} & B_{11} & A_0 & & & & & & & \\
 & B_{21} & B_{22} & A_0 & & & & & & \\
 & & B_{31} & B_{32} & & & & & & \\
 & & \ddots & \ddots & \ddots & & & & & \\
 & & & & & B_{(C-2)2} & A_0 & & & \\
 & & & & & B_{(C-1)1} & B_{(C-1)2} & A_0 & & \\
 & & & & & & A_2 & A_1 & A_0 & \\
 & & & & & & & A_2 & A_1 & \\
 & & & & & & & \ddots & \ddots & \ddots
\end{pmatrix}
\end{array},
$$

where the sub-matrices $A_0,\ A_1$, and A_2 are of order 3×3 and are appearing as

$$A_0 = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} -(\lambda_1 + C\mu_1 + \theta) & \theta & 0 \\ 0 & -(\lambda + C\mu + \alpha) & \alpha \\ 0 & \beta & -(\lambda_2 + \beta + C\mu_2) \end{bmatrix},$$

$$A_2 = \begin{bmatrix} C\mu_1 & 0 & 0 \\ 0 & C\mu & 0 \\ 0 & 0 & C\mu_2 \end{bmatrix},$$

and the boundary matrix is defined by

$$B_0 = \begin{bmatrix} -(\lambda_1 + \theta) & \theta & 0 \\ \mu & -(\lambda + \mu + \alpha) & \alpha \\ 0 & \beta & -(\lambda_2 + \beta) \end{bmatrix},$$

$$B_{10} = \begin{bmatrix} \mu_1 & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu_2 \end{bmatrix},$$

$$B_{11} = \begin{bmatrix} -(\lambda_1 + \mu_1 + \theta) & \theta & 0 \\ 0 & -(\lambda + \mu + \alpha) & \alpha \\ 0 & \beta & -(\lambda_2 + \mu_2 + \beta) \end{bmatrix},$$

$$B_{i1} = \begin{bmatrix} i\mu_1 & 0 & 0 \\ 0 & i\mu & 0 \\ 0 & 0 & i\mu_2 \end{bmatrix}, \text{ for } i = 2, 3, 4, ...C-1,$$

$$B_{i2} = \begin{bmatrix} -(\lambda_1 + i\mu_1 + \theta) & \theta & 0 \\ 0 & -(\lambda + i\mu + \alpha) & \alpha \\ 0 & \beta & -(\lambda_2 + i\mu_2\beta) \end{bmatrix}, \text{ for } i = 2, 3, 4, ...C-1.$$

We define the matrix $A = A_0 + A_1 + A_2$. This matrix A is a 3×3 matrix and it can be written as

$$A = \begin{bmatrix} -\theta & \theta & 0 \\ 0 & -\alpha & \alpha \\ 0 & \beta & -\beta \end{bmatrix}.$$

3.1. The Steady State Solution

Let $P = (p_0, p_1, p_2, \ldots)$ be the stationary probability vector associated with Q, such that $PQ = 0$ and $Pe = 1$, where e is a column vector of $1's$ of appropriate dimension.

Let $p_i = (p_{i0}, p_{i1}, p_{i2})$ for $i \geq 0$.

If the steady state condition is satisfied, then the sub vectors p_i are given by the following equations:

$$p_0 B_0 + p_1 B_{10} = 0, \tag{1}$$

$$p_0 A_0 + p_1 B_{11} + p_2 B_{21} = 0, \tag{2}$$

$$p_i A_0 + p_{i+1} B_{(i+1)2} + p_{i+2} B_{(i+2)1} = 0, \text{ for } i = 1, 2, 3, ...C-3, \tag{3}$$

$$p_{C-2} A_0 + p_{C-1} B_{(C-1)2} + p_C A_2 = 0, \tag{4}$$

$$p_i A_0 + p_{i+1} A_1 + p_{i+2} A_2 = 0, \; i \geq C-1, \tag{5}$$

$$p_i = p_{C-1} R^{i-(C-1)}; i \geq C, \tag{6}$$

where R is the rate matrix, is the minimal non-negative solution of the matrix quadratic equation (see Neuts (1981)).

$$R^2 A_2 + R A_1 + A_0 = 0, \tag{7}$$

the matrices $A_0, \; A_1$, and A_2 are upper triangular matrices of order 3.

Substituting the equation (6) in (4), we have

$$p_{C-2} A_0 + p_{C-1}(B_{(C-1)2} + R A_2) = 0, \tag{8}$$

and the normalizing condition is

$$\sum_{i=0}^{C-2} p_i e + p_{C-1}(I-R)^{-1}e = 1. \tag{9}$$

Theorem 3.1.

The queueing system described in Section 2 is stable if and only if $\rho < 1$, where $\rho = \dfrac{(\lambda_2\alpha + \lambda\beta)}{C\mu\beta + C\mu_2\alpha}$.

Proof:

Consider the infinitesimal generator $A = \begin{bmatrix} -\theta & \theta & 0 \\ 0 & -\alpha & \alpha \\ 0 & \beta & -\beta \end{bmatrix}$, which is a square matrix of order 3, the row vector $\pi = (\pi_1, \pi_2, \pi_3)$ satisfying the condition $\pi A = 0$ and $\pi e = 1$.

Following Neuts (1981), the system is stable if and only if $\pi A_0 e < \pi A_2 e$. That is, the system is stable if and only if $\dfrac{(\lambda_2\alpha + \lambda\beta)}{C\mu\beta + C\mu_2\alpha} < 1$. ■

Theorem 3.2.

If $\rho < 1$, the matrix equation (7) has the minimal non-negative solution $R = -A_0A_1^{-1} - R^2A_2A_1^{-1}$.

Proof:

Since A is reducible, the analysis present in Neuts (1978) is not applicable. In Lucantoni (1979), a similar reducible matrix is treated for the case when the elements are probabilities.

Equation (7) can be written as,

$$A_0A_1^{-1} + RA_1A_1^{-1} + R^2A_2A_1^{-1} = 0A_1^{-1}.$$

Since A_1 is non-singular, A_1^{-1} exists,

$$R = -A_0A_1^{-1} - R^2A_2A_1^{-1}, \tag{10}$$

where

$$A_1^{-1} = \begin{bmatrix} \frac{-1}{(\lambda_1 + C\mu_1 + \theta)} & (\lambda_2 + \beta + C\mu_2)\theta S_0 & \alpha\theta S_0 \\ 0 & S_0(\lambda_2 + \beta + C\mu_2)(\lambda_1 + C\mu_1 + \theta) & S_0\alpha(\lambda_1 + C\mu_1 + \theta) \\ 0 & S_0\beta(\lambda_1 + C\mu_1 + \theta) & S_0(\lambda + C\mu + \alpha)(\lambda_1 + C\mu_1 + \theta) \end{bmatrix},$$

$$S_0 = \frac{1}{-(\lambda_1 + C\mu_1 + \theta)[(\lambda_2 + \beta + C\mu_2)(\lambda + C\mu + \alpha) - \alpha\beta]}.$$

Using Neuts and Lucantoni (1979), the matrix R is numerically computed by using the recurrence relation with $R(0) = 0$ in Equation (10).

Let $Q^* = \begin{array}{c c} & \begin{array}{cccccccc} 0 & 1 & 2 & 3 & \ldots & C-3 & C-2 & C-1 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ \vdots \\ C-2 \\ C-1 \end{array} & \begin{pmatrix} B_0 & A_0 & & & & & & \\ B_{10} & B_{11} & A_0 & & & & & \\ & B_{21} & B_{22} & A_0 & & & & \\ & & B_{31} & B_{32} & & & & \\ & & \ddots & \ddots & \ddots & & & \\ & & & & & B_{(C-2)1} & B_{(C-2)2} & A_0 \\ & & & & & & B_{(C-1)1} & B_{(C-1)2} + RA_2 \end{pmatrix} \end{array},$

also be irreducible and let $P^* = (p_0, p_1, p_2, \ldots, p_{C-1})$ be a solution of $P^*Q^* = 0$.

Solving Equations (1) and (2), we get

$$p_1 = -(p_0 B_0 B_{10}^{-1}), \tag{11}$$

$$p_2 = p_0(B_0 B_{10}^{-1} B_{11} - A_0) B_{21}^{-1}. \tag{12}$$

In this way we can calculate all the $p_i' s,\ 0 \leq i \leq C-2$.

Finally, we get

$$p_{C-1}[DA_0 + B_{(C-1)2} + RA_2] = 0, \tag{13}$$

where the matrix D is readily computed.

p_{C-1} is the left eigenvector of the matrix $DA_0 + B_{(C-1)2} + RA_2$ of order 3 corresponding to the eigenvalue zero.

It is normalized so that

$$\sum_{i=0}^{C-2} p_i e + p_{C-1}(I-R)^{-1} e = 1. \text{ and } p_i = p_{C-1} R^{i-(C-1)};\ i \geq C. \qquad \blacksquare$$

Remark 3.1.

The computation of R can be carried out using a number of well-known methods. We use Theorem 1 of Latouche and Neuts (1980). The matrix R is computed by successive substitutions in the recurrence relation:

$$R(0) = 0, \tag{14}$$

$$R(n+1) = -A_0 A_1^{-1} - [R(n)]^2 A_2 A_1^{-1} \text{ for } n \geq 0, \tag{15}$$

and is the limit of the monotonically increasing sequence of matrices $\{R(n), n \geq 0\}$.

4. Model-II

In this section, we completely analyze Model-II.

$$Q_1 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ \vdots \\ C-2 \\ C-1 \\ C \\ C+1 \\ \vdots \end{array}
\begin{array}{c}
\begin{array}{cccccccccc} 0 & 1 & 2 & 3 & \dots & C-2 & C-1 & C & C+1 & \dots \end{array} \\
\left(\begin{array}{cccccccccc}
B'_0 & A_0 & & & & & & & & \\
B_{10} & B'_{11} & A_0 & & & & & & & \\
 & B_{21} & B'_{22} & A_0 & & & & & & \\
 & & B_{31} & B'_{32} & & & & & & \\
 & & \ddots & \ddots & \ddots & & & & & \\
 & & & & & B'_{(C-2)2} & A_0 & & & \\
 & & & & & B_{(C-1)1} & B'_{(C-1)2} & A_0 & & \\
 & & & & & & A_2 & A_1 & A_0 & \\
 & & & & & & & A_2 & A_1 & \\
 & & & & & & & \ddots & \ddots & \ddots
\end{array}\right)
\end{array},$$

the boundary matrix is defined by

$$B'_0 = \begin{bmatrix} -(\lambda_1) & 0 & 0 \\ \mu & -(\lambda+\mu+\alpha) & \alpha \\ 0 & \beta & -(\lambda_2+\beta) \end{bmatrix},$$

$$B'_{11} = \begin{bmatrix} -(\lambda_1+\mu_1) & 0 & 0 \\ 0 & -(\lambda+\mu+\alpha) & \alpha \\ 0 & \beta & -(\lambda_2+\mu_2+\beta) \end{bmatrix},$$

$$B'_{i2} = \begin{bmatrix} -(\lambda_1+i\mu_1) & 0 & 0 \\ 0 & -(\lambda+i\mu+\alpha) & \alpha \\ 0 & \beta & -(\lambda_2+i\mu_2+\beta) \end{bmatrix}, \text{ for } i = 2, 3, 4, \ldots C-1.$$

4.1. The Steady State Solution

Let $P_1 = (p_0, p_1, p_2, \ldots)$ be the stationary probability vector associated with Q_1, such that $P_1 Q_1 = 0$ and $P_1 e = 1$, where e is a column vector of $1's$ of appropriate dimension.

Let $p_i = (p_{i0}, p_{i1}, p_{i2})$ for $i \geq 0$.

If the steady state condition is satisfied, then the subvectors p_i are given by the following equations:

$$p_0 B'_0 + p_1 B_{10} = 0, \tag{16}$$

$$p_0 A_0 + p_1 B'_{11} + p_2 B_{21} = 0, \tag{17}$$

$$p_i A_0 + p_{i+1} B'_{(i+1)2} + p_{i+2} B_{(i+2)1} = 0, \text{ for } i = 1, 2, 3, \ldots C-3, \tag{18}$$

$$p_{C-2}A_0 + p_{C-1}B'_{(C-1)2} + p_C A_2 = 0, \tag{19}$$

$$p_i A_0 + p_{i+1}A_1 + p_{i+2}A_2 = 0,\ i \geq C-1, \tag{20}$$

$$p_i = p_{C-1}R^{i-(C-1)}; i \geq C, \tag{21}$$

where R_1 is the rate matrix, is the minimal non-negative solution of the matrix quadratic equation (see Neuts (1981)).

$$R_1^2 A_2 + R_1 A_1 + A_0 = 0, \tag{22}$$

the matrices $A_0,\ A_1$, and A_2 are upper triangular matrices of order 3.

Substituting the equation (21) in (19), we have

$$p_{C-2}A_0 + p_{C-1}(B'_{(C-1)2} + R_1 A_2) = 0, \tag{23}$$

and the normalizing condition is

$$\sum_{i=0}^{C-2} p_i e + p_{C-1}(I - R_1)^{-1}e = 1. \tag{24}$$

Let $Q_1^* = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ \vdots \\ C-2 \\ C-1 \end{array} \begin{array}{c} \begin{array}{cccccccc} 0 & 1 & 2 & 3 & \dots & C-3 & C-2 & C-1 \end{array} \\ \begin{pmatrix} B'_0 & A_0 & & & & & & \\ B_{10} & B'_{11} & A_0 & & & & & \\ & B_{21} & B'_{22} & A_0 & & & & \\ & & B_{31} & B'_{32} & & & & \\ & & \ddots & \ddots & \ddots & & & \\ & & & & & B_{(C-2)1} & B'_{(C-2)2} & A_0 \\ & & & & & & B_{(C-1)1} & B'_{(C-1)2} + RA_2 \end{pmatrix} \end{array},$

is also irreducible and let $P_1^* = (p_0, p_1, p_2, \dots, p_{C-1})$ be a solution of $P_1^* Q_1^* = 0$.

Solving Equations (16) and (17), we get

$$p_1 = -(p_0 B'_0 B_{10}^{-1}), \tag{25}$$

$$p_2 = p_0(B'_0 B_{10}^{-1} B'_{11} - A_0)B_{21}^{-1}. \tag{26}$$

In this way we can calculate all the $p_i's,\ 0 \leq i \leq C-2$.

Finally we get

$$p_{C-1}[DA_0 + B'_{(C-1)2} + R_1 A_2] = 0, \tag{27}$$

where the matrix D is readily computed.

p_{C-1} is the left eigenvector of the matrix $DA_0 + B'_{(C-1)2} + R_1A_2$ of order 3 corresponding to the eigenvalue zero.

It is normalized so that

$$\sum_{i=0}^{C-2} p_i e + p_{C-1}(I - R_1)^{-1}e = 1 \text{ and } p_i = p_{C-1}R_1^{i-(C-1)};\ i \geq C.$$

5. Particular Model

In the above model, we assume that $C = 1$, $\lambda_1 = \lambda_2 = \lambda$, and $\mu_1 = \mu_2 = \mu$, then we get

$$p_{00} = \frac{1}{S_1 + S_2[(\lambda+\theta) - \mu r_0] - S_3[(\mu r_{01} + \theta) + \frac{1}{\mu}[(\lambda+\theta) - \mu r_0][\mu r_1 - (\lambda+\mu+\alpha)]]},$$

$$p_{01} = \frac{1}{\mu}[(\lambda+\theta) - \mu r_0]p_{00},$$

$$p_{02} = \frac{-1}{(\beta + \mu r_{21})}[(\mu r_{01} + \theta) + \frac{1}{\mu}[(\lambda+\theta) - \mu r_0][\mu r_1 - (\lambda+\mu+\alpha)]]p_{00}, \quad \text{and}$$

$$p_i = p_0 R^i;\ i \geq 1.$$

6. Performance Measures

Using straightforward calculations the following performance measures have been obtained for models discussed in this article:

(i) Mean queue length $E(L) = \sum_{i=0}^{C-1} ip_i e + p_{C-1}[R^2(I-R)^{-2} + CR(I-R)^{-1}]e$ (Model-I),

$$= \sum_{i=0}^{C-1} ip_i e + p_{C-1}[R_1^2(I-R_1)^{-2} + CR_1(I-R_1)^{-1}]e \text{ (Model-II)},$$

(ii) $E(L^2) = \sum_{i=0}^{C-1} i^2 p_i e + p_{C-1}[2CR^2(I-R)^{-2} + C^2R(I-R)^{-1} + R^2(I+R)(I-R)^{-3}]e$ (Model-I),

$$= \sum_{i=0}^{C-1} i^2 p_i e + p_{C-1}[2CR_1^2(I-R_1)^{-2} + C^2R_1(I-R_1)^{-1} + R_1^2(I+R_1)(I-R_1)^{-3}]e$$
(Model-II),

(iii) Variance of $L = var(L) = \sum_{i=0}^{C-1} i^2 p_i e + p_{C-1}[2CR^2(I-R)^{-2} + C^2R(I-R)^{-1}$

$$+ R^2(I+R)(I-R)^{-3}]e - \left[\sum_{i=0}^{C-1} ip_ie + p_{C-1}[R^2(I-R)^{-2} + CR(I-R)^{-1}]e\right]^2 \text{ (Model-I)},$$

$$= \sum_{i=0}^{C-1} i^2p_ie + p_{C-1}[2CR_1^2(I-R_1)^{-2} + C^2R_1(I-R_1)^{-1}$$

$$+ R_1^2(I+R_1)(I-R_1)^{-3}]e - \left[\sum_{i=0}^{C-1} ip_ie + p_{C-1}[R_1^2(I-R_1)^{-2} + CR(I-R_1)^{-1}]e\right]^2 \text{ (Model-II)},$$

(iv) Probability that no customer in the queue $=p_0e$.

(v) Mean queue length when the servers are in vacation period $=\sum_{i=0}^{\infty} ip_{i0}$.

(vi) Mean queue length when the servers are in regular busy period $=\sum_{i=0}^{\infty} ip_{i1}$.

(vii) Probability that the servers are in working vacation period$= pr\{J=0\} = \sum_{i=1}^{\infty} p_{i0}$.

(viii) Probability that the servers are in regular busy period$=pr\{J=1\} = \sum_{i=1}^{\infty} p_{i1}$.

7. Stationary Waiting Time Distribution in the Queue (For Model-I and Model-II)

In this section, we derive the stationary waiting time distributions for Model-I and Model-II.

Let $W(t)$ be the distribution function for the waiting time in the queue of an arriving (tagged) customer. Note that if there is no customer in the system, the arrival receives service immediately. If at least one server is not busy then also there would be no delay in getting service. Thus, the probability that the customer gets his service without waiting is $\sum_{i=0}^{C-1} p_ie$ (where $e = \begin{bmatrix}1\\1\\1\end{bmatrix}$). Hence, with probability $1-(\sum_{i=0}^{C-1} p_ie)$, the customer has to wait before getting the service. The waiting time may be viewed as the time until absorption in a Markov chain with state space

$$\Omega_1 = \{*\} \bigcup \{C, C+1, C+2\ldots\}.$$

Here, $*$ is the absorbing state, which corresponds to taking the tagged customer into service and is obtained by lumping together the level states $0 = \{(0,0),(0,1)\}$ and $i = \{(i,0),(i,1)\};\ 1 \le i \le C-1$. For $i \ge C$, the level i is given by $i = \{(i,j), j = 0 \text{ or } 1\}$. The states other than the absorbing state correspond to the number of customers present in the system as the tagged customer arrives. Once the tagged customer joins the queue, the subsequent arrivals will not affect his waiting time in the queue. Hence, the parameter λ does not show up in the generator matrix $\widetilde{Q}$ of this Markov process, given by

$$\widetilde{Q} = \begin{array}{c} \\ * \\ C \\ C+1 \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccc} * & C & C+1 & \dots \end{array} \\ \begin{pmatrix} & & & \\ A_2 e & D & & \\ & A_2 & D & \\ & & \ddots & \end{pmatrix} \end{array},$$

where $D = \begin{bmatrix} -C\mu_1 - \theta & \theta & 0 \\ 0 & -C\mu - \alpha & \alpha \\ 0 & \beta & -C\mu_2 - \beta \end{bmatrix}$.

Now, define the vector

$$Y(t) = (Y_*(t), Y_C(t), Y_{C+1}(t), \dots),$$

where

$$Y_i(t) = (y_{i0}(t), y_{i1}(t), y_{i2}), \text{ for } i \geq C.$$

The components of the $Y_i(t)$ are the corresponding probabilities in regular state, working vacation state and breakdown state at time t, the CTMC with generator $\widetilde{Q}$ is in the respective state of level i. Note that the scalar $Y_*(t)$ is the probability that the process is in the absorbing state at time t. By the PASTA property, we get

$$Y(0) = (p_{00} + p_{01} + p_{02} \dots + p_{(C-1)0} + p_{(C-1)1} + p_{(C-1)2}, p_C, p_{C+1}, \dots).$$

Clearly,

$$W(t) = Y_*(t), \text{ for } t \geq 0.$$

The LST of $W(t)$ is given by (see Neuts (1981))

$$\widetilde{W}(s) = \sum_{i=C}^{\infty} Y_i(0) \left[(sI - D)^{-1} A_2\right]^{i-C} (sI - D)^{-1} A_2 e. \tag{28}$$

The mean waiting time can be obtained from $\widetilde{W}(s)$ as

$$E(W) = -\widetilde{W}'(0) = \sum_{i=1}^{\infty} p_{C+i} \sum_{j=0}^{i-1} U^j (-D)^{-1} U^{i-j} U e + \sum_{i=0}^{\infty} p_{C+i} U^i (-D)^{-2} A_2 e,$$

where $U = (-D)^{-1} A_2$ is a stochastic matrix. Hence, (28) can be simplified as

$$E(W) = -\widetilde{W}'(0) = \sum_{i=1}^{\infty} p_{C+i} \sum_{j=0}^{i-1} U^j (-D)^{-1} e + \sum_{i=0}^{\infty} p_{C+i} U^i (-D)^{-1} e. \tag{29}$$

Let

$$H = \sum_{i=0}^{\infty} p_{C+i} U^i.$$

Since U is stochastic, we get

$$He = p_C(I-R)^{-1}e = 1 - p_{00} - p_{01} - p_{02} - p_{10} - p_{11} - p_{12} - \cdots - p_{(C-1)0} - p_{(C-1)1} - p_{(C-1)2}.$$

This result can be used to find an approximate value of H, and hence, that of the second term in (29) to any desired degree of accuracy. Thus, only the first term in (29) demands serious computation. For this we make use of the ideas in Neuts (1981), Krishna Kumar (2005), and Neuts (1979).

Now, consider the matrix

$$U_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

which has the property that

$$UU_2 = U_2 U = U_2.$$

Then, we get

$$\sum_{j=0}^{i-1} U^j (I - U + U_2) = I - U^i + iU_2, \text{ for } i \geq 1.$$

By the classical theorem on finite Markov chains, the matrix $(I - U + U_2)$ is nonsingular (see Kemeny (1960)). In view of the last equation, the first term in (29) becomes

$$\left[\sum_{i=1}^{\infty} p_{C+i}(I - U^i + iU_2) \right] (I - U + U_2)^{-1}(-D)^{-1}e.$$

With this simplification, we get

$$E(W) = \left[p_C \left(R(I-R)^{-1} + I + R(I-R)^{-2}U_2 \right) - H \right] (I - U + U_2)^{-1}(-D)^{-1}e$$
$$+ H(-D)^{-1}e. \tag{30}$$

For Model-I, Equation (30) becomes

$$E(W) = \left[p_C \left(R(I-R)^{-1} + I + R(I-R)^{-2}U_2 \right) - H \right] (I - U + U_2)^{-1}(-D)^{-1}e$$
$$+ H(-D)^{-1}e.$$

For Model-II, a similar arguments leads to:

Generator matrix $\widetilde{Q}_1$ is given by

$$\widetilde{Q_1} = \begin{array}{c} \\ * \\ C \\ C+1 \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccc} * & C & C+1 & \dots \end{array} \\ \begin{pmatrix} & & & \\ A_2e & D & & \\ & A_2 & D & \\ & & \ddots & \end{pmatrix} \end{array},$$

where $D = \begin{bmatrix} -C\mu_1-\theta & \theta & 0 \\ 0 & -C\mu-\alpha & \alpha \\ 0 & \beta & -C\mu_2-\beta \end{bmatrix}$,

and $H_1 = \sum_{i=0}^{\infty} p_{C+i}U^i$.

Equation (30) becomes

$$E(W_1) = \left[p_C\left(R_1(I-R_1)^{-1} + I + R_1(I-R_1)^{-2}U_2\right) - H_1\right](I-U+U_2)^{-1}(-D)^{-1}e$$
$$+H(-D)^{-1}e.$$

8. Numerical Study

In this section, some examples are given to show the effect of the parameters λ, λ_1, λ_2, μ, μ_1, μ_2, θ, α, β and C on the performance measures mean queue length, $E(L^2)$, variance of L, probability that no customer in the queue, mean queue length when the servers are in vacation period, mean queue length when the servers are in regular busy period, probability that the servers are in working vacation period and probability that the servers are in regular busy period for Model-I and Model-II analyzed in this paper. The corresponding results are presented as Case (1), Case (2) and Case (3).

Case (1): If $\lambda = 0.7,\ \lambda_1 = 0.5,\ \lambda_2 = 0.3,\ \mu = 5,\ \mu_1 = 2,\ \mu_2 = 0.9,\ \theta = 2.4,\ \alpha = 0.5,\ \beta = 0.8$ and $C = 5$, the matrix $R = R_1$ is obtained using the equations (14) and (15),

$$R = \begin{bmatrix} 0.04 & 0.003930 & 0.000389 \\ 0 & 0.027525 & 0.002635 \\ 0 & 0.001868 & 0.056288 \end{bmatrix},$$

the probability vectors and performance measures are presented in Table 1 and Table 2, respectively.

Case (2): If $\lambda = 0.5,\ \lambda_1 = 0.3,\ \lambda_2 = 0.1,\ \mu = 3,\ \mu_1 = 2,\ \mu_2 = 1,\ \theta = 1.9,\ \alpha = 0.6,\ \beta = 0.9$ and $C = 5$, the matrix $R = R_1$ is obtained using the equations (14) and (15),

$$R = \begin{bmatrix} 0.025107 & 0.003150 & 0.000336 \\ 0 & 0.032214 & 0.003359 \\ 0 & 0.000997 & 0.017010 \end{bmatrix},$$

the probability vectors and performance measures are presented in Table 3 and Table 4, respectively.

Case (3): If $\lambda_1 = \lambda_2 = \lambda = 0.4,\ \mu_1 = \mu_2 = \mu = 2,\ \theta = 2.1,\ \alpha = 0.5,\ \beta = 0.9$ and $C = 5$, the matrix $R = R_1$ is obtained using the equations (14) and (15),

$$R = \begin{bmatrix} 0.032864 & 0.006803 & 0.000332 \\ 0 & 0.038190 & 0.001810 \\ 0 & 0.003258 & 0.036742 \end{bmatrix},$$

the probability vectors and performance measures are presented in Table 5 and Table 6, respectively.

9. Conclusion

The highlights of the models analyzed in this paper are: both during vacation period and breakdown period the servers serve the customers; also, the arrival rate depends on the server states. In these points these models deviate from the existing models in the literature. The models' steady state probability for obtained using Matrix-Geometric method and also waiting time distribution is carried out using generator matrix of CTMC. The model can be generalized by taking arrival time/service time follows a general distribution.

Acknowledgment:

The work of the second author was supported by University Grants Commission, New Delhi through the BSR (Fellowship) Grant No. F.25-1/2014-15(BSR)/7-254/2009.

Table 1: Probabilities

prob.	Model-I	Model-II
p_0	(0.419897, 0.225262, 0.138475)	(0.7788047, 0.0000025037, 0.000000669299425)
p_1	(0.045695, 0.055619, 0.044101)	(0.1946949, 0.0000027834, 0.000000616941634)
p_2	(0.003489, 0.004220, 0.010473)	(0.0243337, 0.0000014024, 0.000000294340907)
p_3	(0.000205, 0.000561, 0.005567)	(0.0020257, 0.0000008701, 0.000000162324511)
p_4	(0.000010, 0.000079, 0.005081)	(0.0001250, 0.0000006362, 0.000000098110377)
p_5	(0.00000039999997625, 0.00001170508221549, 0.000286211376078)	(0.00000500000032844, 0.00000050893919479, 0.0000005582327844)
p_6	(0.00000001599999955, 0.00000085839724306, 0.000016141264495)	(0.00000020000000234, 0.00000003376283431, 0.0000000642823572)
p_7	(0.00000000063999994, 0.00000005384214674, 0.000000910827623)	(0.00000000799999977, 0.00000000172733006, 0.0000000052859761)
p_8	(0.0000000000256, 0.0000000031859464, 0.000000051410786)	(0.00000000031999997, 0.00000000007997218, 0.0000000003741722)
p_9	(0.000000000001024, 0.00000000018382913, 0.000000002902215)	(0.0000000000128, 0.00000000000352873, 0.0000000000244135)
p_{10}	(0.00000000000004096, 0.00000000001048526, 0.000000000163844)	(0.000000000000512, 0.00000000000015199, 0.000000000001517)

Table 2: Performance Measures

performance measures	Model-I	Model-II
Mean queue length $E(L)$	0.105923	0.080345
$E(L^2)$	0.150697	0.111601
Variance of L=$var(L)$	0.139477	0.105146
Prob. that no customer in the queue	0.179788	0.5266399
Mean queue length when the servers are in vacation period	0.027968	0.668656
Mean queue length when the servers are in regular busy period	3.873586	0.014349
Prob. that the servers are in working vacation period	0.010195	0.492117
Probability that the servers are in regular busy period	0.9802203	0.006643

Table 3: Probabilities

prob.	**Model-I**	**Model-II**
p_0	(0.377856, 0.258197, 0.187934)	(0.8607078, 0.0000002969, 0.000000144224003)
p_1	(0.028345, 0.057181, 0.033016)	(0.1291057, 0.0000003144, 0.000000110335705)
p_2	(0.001423, 0.003629, 0.006465)	(0.0096827, 0.0000001551, 0.000000043021057)
p_3	(0.000054, 0.001261, 0.004638)	(0.0004840, 0.0000000958, 0.000000022656732)
p_4	(0.000002, 0.000028, 0.004666)	(0.0000180, 0.0000000706, 0.000000012869840)
p_5	(0.00000005021399829, 0.00000556029408472, 0.000079463381553)	(0.00000045192601306, 0.00000005898714406, 0.000000006504060)
p_6	(0.00000000126072286, 0.00000025850249858, 0.000001370366021)	(0.00000001134650685, 0.00000000333026340, 0.000000000460619)
p_7	(0.00000000003165297, 0.00000000969762581, 0.000000024178659)	(0.00000000028487676, 0.00000000014348185, 0.000000000022833)
p_8	(0.00000000000079471, 0.00000000033660516, 0.000000000443863)	(0.00000000000715240, 0.00000000000554225, 0.000000000000966)
p_9	(0.00000000000001995, 0.00000000001128843, 0.000000000008681)	(0.00000000000017958, 0.00000000000020203, 0.000000000000037)
p_{10}	(0.00000000000000050, 0.00000000000037236, 0.000000000000185)	(0.00000000000000451, 0.00000000000000711, 0.000000000000001)

Table 4: Performance Measures

performance measures	**Model-I**	**Model-II**
Mean queue length $E(L)$	0.104749	0.066346
$E(L^2)$	0.147961	0.091464
Variance of L=$var(L)$	0.136989	0.087062
Prob. that no customer in the queue	0.211808	0.4896652
Mean queue length when the servers are in vacation period	0.000196	0.725482
Mean queue length when the servers are in regular busy period	2.764004	0.012548
Prob. that the servers are in working vacation period	0.010095	0.534697
Probability that the servers are in regular busy period	0.9836982	0.008659

Table 5: Probabilities

prob.	**Model-I**	**Model-II**
p_0	(0.265965, 0.307378, 0.189096)	(0.8187282, 0.0000019301, 0.000000825972620)
p_1	(0.025078, 0.081342, 0.046068)	(0.1637437, 0.0000019313, 0.000000549951778)
p_2	(0.001616, 0.004704, 0.008929)	(0.0163734, 0.0000009185, 0.000000179194856)
p_3	(0.000079, 0.000348, 0.004424)	(0.0010909, 0.0000005586, 0.000000080925176)
p_4	(0.000003, 0.000043, 0.003808)	(0.0000541, 0.0000004146, 0.000000042110081)
p_5	(0.00000009859200389, 0.00001406904175383, 0.000139992363983)	(0.00000177794242973, 0.00000038401304892, 0.000000020258831)
p_6	(0.00000000324012772, 0.00000099406247500, 0.000005169097221)	(0.00000005843030237, 0.00000002682680567, 0.000000002029690)
p_7	(0.00000000010648356, 0.00000005482620580, 0.000000191723302)	(0.00000000192025351, 0.00000000142862977, 0.000000000142530)
p_8	(0.00000000000349948, 0.00000000271917178, 0.000000007143568)	(0.00000000006310721, 0.00000000006808722, 0.000000000008460)
p_9	(0.00000000000011501, 0.00000000012714273, 0.000000000267391)	(0.00000000000207396, 0.00000000000305713, 0.000000000000455)
p_{10}	(0.00000000000000378, 0.00000000000572753, 0.000000000010054)	(0.00000000000006816, 0.00000000000013234, 0.000000000000022)

Table 6: Performance Measures

performance measures	**Model-I**	**Model-II**
Mean queue length $E(L)$	0.105959	0.123149
$E(L^2)$	0.149804	0.174107
Variance of L=$var(L)$	0.138577	0.158941
Prob. that no customer in the queue	0.183050	0.4431838
Mean queue length when the servers are in vacation period	0.000491	0.754625
Mean queue length when the servers are in regular busy period	3.643045	0.005482
Prob. that the servers are in working vacation period	0.010388	0.577842
Probability that the servers are in regular busy period	0.985204	0.023377

REFERENCES

Chao, X. and Zhao, Y.Q. (1998). Analysis of multi-server queues with station and server vacations, European Journal of Operational Research, Vol. 110, No. 2, pp. 392-406.

Fond, S. and Ross, S. (1977). A heterogeneous arrival and service queueing loss model, Tech-Report ORC 77-12, Operations Research Center, University of California, Berkeley, CA.

Haghighi, A.M. (1998). An analysis of a parallel multi-processor system with task split and feed-back, Computers and Ops. Res., Vol. 25, No. 11, pp. 948-956.

Haghighi, A.M., Medhi., J. and Mohanty., S. G. (1986). On a multi-server Markovian queueing system with balking and reneging, Computers and Ops. Res., Vol. 13, pp. 421-425.

Haghighi, A.M. and Mishev, D.P. (2008). *Queuing Models in Industry and Business*, Nova Science Publishers, Inc., a New York.

Haghighi, A.M. and Mishev, D.P. (2016). Busy period of a single-server Poisson queueing system with splitting and batch delayed-feedback, Int. J. Mathematics in Operational Research, Vol. 8, No. 2, pp. 239-257.

Jain, M. and Jain, A. (2010). Working vacation queueing model multiple type of server breakdown, Appl. Math. Modelling, Vol. 34, No. 1, pp. 1-13.

Jayaraman, D., Nadarajan, R. and Sitrarasu, M.R. (1994). A general bulk service queue with arrival rate dependent on server breakdowns, Appl. Math. Modelling, Vol. 18, pp. 156-160.

Kalyanaraman, R. and Sundaramoorthy, A. (2019). A Markovian working vacation queue with server state dependent arrival rate and with partial breakdown, International Journal of Innovative Technology and Exploring Engineering, Vol. 8, No. 6S2, pp. 664-668.

Ke, J.C., Wu, C.H. and Zhang, Z.G. (2010). Recent development in vacation queueing models: A short surrey, Int. Jr. of Oper. Res., Vol. 7, No. 4, pp. 3-8.

Kemeny, J.C. and Snell, J.L. (1960). *Finite Markov Chains*, The University Series in Undergraduate Mathematics, D. Van Nostrand, Princeton, NJ, USA.

Kim, J.D., Choi, D.W. and Chae, K.C. (2003). Analysis of queue-length distribution of the $M/G/1$ with working vacation $(M/G/1/WV)$, In Proceeding of the International Conference on Statistics and Related, Honolulu, Hawaii, USA.

Krishna Kumar, B. and Pavai Madheswari, S. (2005). An $M/M/2$ queueing system with heterogeneous servers and multiple vacations, Mathematical and Computer Modelling, Vol. 41, No. 13, pp. 1415-1429.

Latouche, G. and Neuts, M.F. (1980). Efficient algorithmic solutions to exponential tandem queues with blocking, SIAM Jr. Algebraic Discrete Math., Vol. 1, pp. 93-106.

Levy, Y. and Yechiali, U. (1976). An $M/M/s$ queue with servers vacations, Information Systems and Operational Research, Vol. 14, No. 2, pp. 153-163.

Li, J.H., Tian, N., Zhang, Z.G. and Luh, H.P. (2009). Analysis of the $M/G/1$ queue with exponentially working vacation- a matrix analytic approach, Queueing Systems, Vol. 61, pp. 139-166.

Lin, Ch.H. and Ke, J.Ch. (2009). Multi-server system with single working vacation, Applied Mathematical Modelling, Vol. 33, pp. 2967-2977.

Liu, W., Xu, X. and Tian, N. (2007). Stochastic decompositions in the $M/M/1$ queue with working vacation, Oper. Res. Lett., Vol. 35, pp. 595-600.

Lucantoni, D.M. (1979). *A $GI/M/C$ Queue with a Different Service Rate for Customers Who Need Not Wait an Algorithmic Solution*, Technical Rep. Univ. of Delware, USA.

Neuts, M.F. (1978). Markov chains with applications in queueing theory which have a matrix-geometric invariant probability vector, Adv. Appl. Probab., Vol. 10, pp. 185-212.

Neuts, M.F. and Lucantoni, D.M. (1979). A Markovian queue with N servers subject to breakdowns and repairs, Management Science, Vol. 25, No. 9, pp. 849-861.

Neuts, M.F. (1981). *A Matrix-Geometric Solution in Stochastic Models*, Vol. 2 of John Hopkins series in the Mathematical Sciences, Johns Hopkins University Press, Baltimore, MD, USA.

Neuts, M.F. and Lucantoni, D.M. (1979). A Markovian queue with N servers subject to breakdowns and repairs, Management Science, Vol. 25, No. 9, pp. 849-861.

Servi, L.D. and Finn, S.G. (2002). $M/M/1$ queues with working vacations $(M/M/1/WV)$, Perform. Eval., Vol. 50, pp. 41-52.

Shanthikumar, J.G. (1982). Analysis of a single server queue with time and operational dependent server failures, Adv. in Mgnt. Studies, Vol. 1, pp. 339-359.

Shogan, A.W. (1979). A single server queue with arrival rate dependent on server breakdowns, Naval Res. Log. Quart., Vol. 26, pp. 487-497.

Tian, N., Li, Q. and Gao, J. (1999). Conditional stochastic decomposition in $M/M/C$ queue with server vacations, Communication in Statistics, Stochastic Models, Vol. 15, pp. 367-377.

Tian, N. and Yue, D. (2002). *Quasi-birth and Death Peocess and the Matrix Geometric Solution*, Beijing: Science Press.

Vinod, B. (1986). Exponential queues with servers vacations, Journal of the Operational Research Society, Vol. 37, No. 10, pp. 1007-1014.

Wu, D.A. and Takagi, H. (2006). $M/G/1$ queue with multiple working vacations, Perform. Eval., Vol. 63, pp. 654-681.

Xu, X., Zhang, Z. and Tian, N. (2009). Analysis for the $M^X/M/1$ working vacation queue, Int. Jr. of Infor. and Manag. Sci., Vol. 3, pp. 379-394.

Yechiali, V. and Naor, P. (1971). Queueing problems with heterogeneous arrivals and service, Oprs. Res., Vol. 19, pp. 722-734.

Zhang, Z.G. and Tian, N. (2003). Analysis of queueing systems with synchronous single vacation for some servers, Queueing Systems, Vol. 45, pp. 161-175.