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Solving Multi-Objective Linear Fractional Programming Problems via Zero-Sum Game

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Abstract

This study presents a hybrid algorithm consisting of game theory and the first order Taylor series approach to find compromise solutions to multi-objective linear fractional programming (MOLFP) problems. The proposed algorithm consists of three phases including different techniques: in the first phase, the optimal solution to each LFP problem is found using the simplex method; in the second phase, a zero-sum game is solved to determine the weights of the objective functions via the ratio matrix obtained from a payoff matrix; in the last phase, fractional objective functions of the MOLFP problem are linearized using the 1st order Taylor series. A compromise solution is found by solving the single-objective LP problem constructed in the third phase by using the weights. This algorithm can provide compromise solutions to the problem by constructing different ratio matrices in the second phase. The novelty of this study is that the decision-makers can choose the most suitable solution for their strategy among the compromise solutions. Numerical examples are provided to illustrate the efficiency of the algorithm.

Keywords: Linear fractional programming; Multi-objective problem; Game theory; Taylor series

MSC 2010 No.: 90C32, 90C29, 41A58, 90C05

1. Introduction

Fractional programming is used for modeling real-life problems such as industrial planning, production planning, financial and corporate planning, healthcare, and hospital planning.

In recent years, several solution techniques and methods are proposed for solving the MOLFP problems in the literature. Chakraborty and Gupta (2002) explored a solution procedure for finding an efficient solution to the MOLFP problems based on a fuzzy set theoric approach and reduced the complexity of solving the considered problems. Costa (2005) developed an interactive method for computing the preferred non-dominated solution in MOLFP problems using some branch and bound techniques. The aim of the computation phase of the algorithm is to optimize one of the fractional objective functions while constraining the others. Guzel and Sivri (2005) presented a method via goal programming for finding an efficient solution to the MOLFP problems. Wu (2009) focused on a solution procedure for implementing the weighted max-ordering approach to obtain a weakly efficient solution to a MOLFP problem. The proposed approach needs a solution to a min-max auxiliary problem and thus he used the Taylor series method to linearize the auxiliary problem for computing efficiently.

Lotfi et al. (2010) proposed an LP approach to test the strongly and weakly efficient solutions in the MOLFP problems by applying a simple geometrical interpretation. Dangwal et al. (2012) used Taylor polynomial series approach to find a solution for the MOLFP problems via the vague set. Dheyab (2012) proposed a complementary method where the LFP problem is transformed into an LP problem by maximizing and minimizing the numerator and denominator, respectively, of the fractional objective function being maximized. Stanojević and Stanojević (2013) presented two procedures using the efficiency test introduced in the study of Lotfi et al. (2010) for generating strongly and weakly efficient solutions in MOLFP problems starting from any feasible solution. Sulaiman and Abdulrahim (2013) presented a number of transformation techniques from the MOLFP problem to the single-objective LFP problem by using average mean and average median values of objective functions to find the optimal solution and solved the problem by the modified simplex method. Jain (2014) presented a method using the Gauss elimination technique to derive a numerical solution of the MOLFP problem by extending his previous study proposed for finding a solution to the MOLP problem. Porchelvi et al. (2014) presented an algorithm for solving MOLFP problems for both crisp and fuzzy cases using the complementary method proposed in the study of Dheyab (2012). In the algorithm, any objective function of the MOLP problem is optimized subject to the original constraints and the additional constraints, which are the remaining objective functions. Tantawy (2014) proposed a feasible direction method only applicable only for a special class of MOLFP problems to find all efficient solutions.

De and Deb (2015) used the Taylor series approach to solve MOLFP problems in the fuzzy environment. Taylor series approach is used to transform the MOLFP problems into the MOLP problems by introducing imprecise aspiration levels to each objective, and the additive weighted method is used to find the solution. Hossein-Abadi and Payan (2016) proposed a linearization procedure to present an interactive method for solving an MOLFPP which includes a simple calculation process. The final solution is intended to meet the judgments of the decision-maker by interacting with one.

Pramy and Islam (2017) proposed a method, modifying the studies of Dheyab (2012) and Porchelvi et al. (2014), presenting multiple efficient solutions by solving the MOLFP problems. The method provides the decision-makers flexibility to choose a better option among alternatives. Perić et al. (2017) presented a solution method to the MOLFP problems via the goal programming method by analyzing the applicability of linearization techniques, which are Taylor's polynomial linearization approximation, the method of variable change, and a modification of the method of variable change. Nahar and Alim (2017) suggested a statistical average approach where a single-objective function is developed from multi-objective functions to optimize the objective function, compared the proposed technique with some other techniques, such as arithmetic averaging and geometric averaging, and showed the effectiveness of the approach. Bhati et al. (2017) presented a review of the MOFP problems excluding various technical parts of fractional programming. In the review, the MOFP problems are classified into two classes: general MOFP problems and MOLFP problems. Then, these classes were subclassified based on the basis of the proposed algorithm and optimality criteria.

In this paper, a hybrid algorithm is proposed to find compromise solutions to the MOLFP problems by using the game theory and the 1st order Taylor series approach. The algorithm contains three different techniques in the phases. In the first phase, the optimal solution to each LFP problem is found by applying the simplex method. In the second phase, the weights of objective functions are determined by solving a zero-sum game. In the last phase, linearized objective functions of the MOLFP problem are determined by using the Taylor series, a single-objective LP problem is constructed via the weights obtained, and a compromise solution is found by solving this LP problem. The proposed algorithm provides compromise solutions by constructing different ratio matrices in the second phase. As a result, the novelty of the study is that the decision-makers can choose a better option among solutions according to their satisfaction level.

The rest of the paper is organized as follows. In Section 2, the proposed algorithm is given in steps, and numerical examples are illustrated in Section 3. Finally, Section 4 presents the conclusion and contribution of the paper.

2. Proposed algorithm

Consider A MOLFP problem introduced by Kornbluth and Steuer (1981):

$$Optimize\ (z_1(x), ..., z_k(x)) = \frac{z_{kN}(x)}{z_{kD}(x)} = \frac{C_{kN}x}{C_{kD}x}, \tag{1a}$$

subject to

$$g_i(x) \leq b_i, \quad i = 1, ..., m, \tag{1b}$$

where $k = (1, 2, ..., l)$ is the number of objective functions; $x = (x_1, ..., x_n)$ is a vector of decision variables; $g_i(x)$ $(i = 1, ..., m)$ are constraints; b_i are right-hand side of the constraints. $z_{kN}(x)$ and $z_{kD}(x)$ are nominators and denominators, respectively, of each fractional objective function (1a). Also, C_{kN} is a vector of costs in the nominator, and C_{kD} is a vector of costs in the denominator of the objective functions.

In this section, the proposed algorithm is introduced in three phases.

First phase: This phase is itemized for solving the MOLFP problem (1). The following steps will be iterated for each objective function. In other words, here the steps are presented for a single objective function ($k = 1$) and will be carried out for each objective function.

Step 1: Rewrite the constraints (1b) in the standard form.
Step 2: Separate the objective function into two parts relative to the nature of the optimization:

- If $z_1(x)$ is maximized, then $max\ z_{1N}(x)$ and $min\ z_{1D}(x)$.
- If $z_1(x)$ is minimized, then $min\ z_{1N}(x)$ and $max\ z_{1D}(x)$.

Step 3: Construct the initial simplex table and start the iteration for $t = 0$.
Step 4: Determine reduced costs for the objective functions constructed in Step 2. Here, reduced costs of z_{1N}^t and z_{1D}^t are presented as $(c_{1Nj} - z_{1Nj})$ and $(c_{1Dj} - z_{1Dj})$ $(j = 1, ..., n)$, respectively.
Step 5 Find the original objective function value z_1^t by substituting the basic variables into $z_1(x)$, i.e.,

$$z_1^t = \frac{z_{1N}^t}{z_{1D}^t}.$$

Step 6: Satisfy the following inequality:

$$\frac{z_{1N}^t + \lambda(c_{1Nj} - z_{1Nj})}{z_{1D}^t + \lambda(c_{1Dj} - z_{1Dj})} \geq z_1^t, \tag{2}$$

where the ratio on the left-hand side of (2) should be greater than or equal to the value of z_1^t if the considered LFP problem is maximized. If the LFP is a minimization problem, the ratio should be less than or equal to the value of z_1^t.
Step 7: Rewrite the inequality (2) as

$$z_{1D}^t(c_{1Nj} - z_{1Nj}) - z_{1N}^t(c_{1Dj} - z_{1Dj}) \{\geq\leq\} 0. \tag{3}$$

Then, find an entering variable via the simplex table satisfying the inequality (3).

- If there is no entering variable in the simplex table satisfying the inequality (3), STOP. The solution last found is the optimal solution of the considered LFP problem: $x^{1*} = (x_1^{1*}, ..., x_n^{1*})$.
- Else, find a leaving variable applying the minimum ratio test, make row operations in the simplex table for the next iteration ($t = t + 1$), and go to Step 4.

At the end of the first phase, the individual optimal solution to each LFP problem is found.

Second phase: The following steps are itemized for finding the weight of each objective function via a zero-sum game to convert the MOLFP problem into a single-objective LP problem.

Step 8: Take the individual optimal solution of each LFP problem and the closest feasible point to each individual optimal solution obtained in order to increase the number of strategies.

Step 9: Construct a payoff matrix as given in Table 1. Each optimal solution is substituted into each objective function, respectively. Note that if the payoff matrix has at least one negative entry, it must be converted to a positive matrix. This matrix is obtained by adding successive integer of the absolute value of the smallest negative entry of the matrix.

Table 1. The payoff matrix

	$z_1(x)$	...	$z_k(x)$
x^{1*}	$z_1(x^{1*})$	$\cdots$	$z_k(x^{1*})$
x^{2*}	$z_1(x^{2*})$	$\cdots$	$z_k(x^{2*})$
$\vdots$	$\vdots$	$\ddots$	$\vdots$
x^{k*}	$z_1(x^{k*})$	$\cdots$	$z_k(x^{k*})$

Step 10: Form all possible row ratios among all rows of the payoff matrix. The ratio matrix is given in Table 2.

Table 2. The ratio payoff matrix

	$z_1(x)$	...	$z_k(x)$
$R(x^{1*}/x^{2*})$	$z_1^{1/2}$	$\cdots$	$z_k^{1/2}$
$R(x^{1*}/x^{3*})$	$z_1^{1/3}$	$\cdots$	$z_k^{1/3}$
$\vdots$	$\vdots$	$\ddots$	$\vdots$
$R(x^{1*}/x^{k*})$	$z_1^{1/k}$	$\cdots$	$z_k^{1/k}$

Here, the size of the ratio matrix is determined by $(2k-1)$ rows and k columns.

Step 11: Solve k player zero-sum game having $(2k-1)$ strategies and find the weights w_k for each objective function. Here, the objective functions represent the players, and the ratios represent the strategies.

Third phase: In the last phase, each objective function of the MOLFP problem (1) will be linearized using the Taylor series. Then, each linearized objective function will be multiplied with the corresponding weight to obtain a single-objective LP problem.

Step 12: Expand each objective function $z_k(x)$ to the Taylor series at its own optimal solution x^{k*} as follows:

$$z_{kL}(x) = z_k(x^{k*}) + \sum_{j=1}^{n} \frac{\delta z_k(x)}{\delta x_j}(x_j - x_j^{k*}), \tag{4}$$

where the subscript L denotes the linearization of the $z_k(x)$.

Step 13: Multiply each linearized objective function with the corresponding weight to determine a single-objective LP problem satisfying the original constraints. That is,

$$Optimize \quad z(x) = w_1 z_{1L}(x) + ... + w_k z_{kL}(x) \tag{5a}$$

subject to

$$g_j(x) \leq b_j, \quad j = 1, ..., m. \tag{5b}$$

Step 14: Find the optimal solution of (5), correspondingly of (1), and STOP.

It is important to note that it is possible to have more than one ratio matrix by different permutations from rows of the ratio matrix. Therefore, a compromise solution set can be generated to present to the decision-maker.

3. Numerical examples

Example 3.1.

Consider the MOLFP problem:

$$Max \quad z_1 = \frac{-3x_1 + 2x_2}{x_1 + x_2 + 3}, \tag{6a}$$

$$Max \quad z_2 = \frac{7x_1 + x_2}{5x_1 + 2x_2 + 1}, \tag{6b}$$

subject to

$$x_1 - x_2 \geq 1, \tag{6c}$$

$$2x_1 + 3x_2 \leq 15, \tag{6d}$$

$$x_1 \geq 3. \tag{6e}$$

First phase:

Step 1: The constraints are rewritten in the standard form as follows:

$$x_1 - x_2 - x_3 + x_6 = 1, \tag{7a}$$

$$2x_1 + 3x_2 + x_4 = 15, \tag{7b}$$

$$x_1 - x_5 + x_7 = 3. \tag{7c}$$

Step 2: Since two of the constraints have "$\geq$", the first objective function will be rewritten as

$$Max \quad z_{1N} = -3x_1 + 2x_2 - M(x_6 + x_7), \tag{8a}$$

$$Min \quad z_{1D} = x_1 + x_2 + M(x_6 + x_7), \tag{8b}$$

where the indices N and D refer to nominator and denominator, respectively, of the objective function. The optimal solution of the constructed LFP problem having z_1 is found by applying the following iterations.

Table 3. Initial simplex table for the LFP problem having z_1

$Max \quad z_{1N}$			-3	2	0	0	0	-M	-M	
$Min \quad z_{1D}$			1	2	0	0	0	M	M	
C_{1N}	C_{1D}	x_B	x_1	x_2	x_3	x_4	x_5	x_6	x_7	b
-M	M	x_6	1	-1	-1	0	0	1	0	1
0	0	x_4	2	3	0	1	0	0	0	15
-M	M	x_7	1	0	0	0	-1	0	1	3
$c_{1Nj} - z_{1Nj}$			2M-3	M+2	-M	0	-M	0	0	
$d_{1Dj} - z_{1Dj}$			1-2M	2+M	M	0	M	0	0	

Step 3: The initial simplex table is constructed as given in Table 3.

Steps 4-7: It is seen from Table 3 that the reduced costs of $z_{1N}(x)$ and $z_{1D}(x)$ are determined in the last two rows, respectively. The objective function value z_1^0 is found as 0 by considering the right-hand side values in Table 3. Since the constructed LFP problem having z_1 is maximized,

$$\frac{z_{1N}^0 + \lambda(c_{1Nj} - z_{1Nj})}{z_{1D}^0 + \lambda(c_{1Dj} - z_{1Dj})} \geq 0 \tag{9}$$

is considered, and the following inequality is formed:

$$(c_{1Nj} - z_{1Nj}) \geq 0. \tag{10}$$

The entering variable is determined from Table 3 such as to satisfy (10). It is seen that x_1 is the entering variable which satisfies the inequality (10). x_6 is specified as the leaving variable by applying the minimum ratio test.

First iteration:

Steps 4-7: The iterated simplex table is constructed as given in Table 4.

Table 4. First iterated simplex table for the LFP problem having z_1

$Max \quad z_{1N}$			-3	2	0	0	0	-M	-M	
$Min \quad z_{1D}$			1	2	0	0	0	M	M	
C_{1N}	C_{1D}	x_B	x_1	x_2	x_3	x_4	x_5	x_6	x_7	b
-3	1	x_1	1	-1	-1	0	0	1	0	1
0	0	x_4	0	5	2	1	0	-2	0	13
-M	M	x_7	0	1	1	0	-1	-1	1	2
$c_{1Nj} - z_{1Nj}$			0	M-1	M-3	0	-M	3-2M	0	
$d_{1Dj} - z_{1Dj}$			0	3-M	1-M	0	M	2M-1	0	

It is seen that the reduced costs for each objective function, i.e., $z_{1N}(x)$ and $z_{1D}(x)$, are determined from the last two rows, respectively. Also, by considering the right-hand side values, the objective function value z_1^1 is found as -0.75. Therefore,

$$\frac{z_{1N}^1 + \lambda(c_{1Nj} - z_{1Nj})}{z_{1D}^1 + \lambda(c_{1Dj} - z_{1Dj})} \geq -0.75, \tag{11}$$

is written, and the following inequality is obtained as

$$4(c_{1Nj} - z_{1Nj}) + 3(c_{1Dj} - z_{1Dj}) \geq 0. \tag{12}$$

To determine the entering variable, the last two rows of Table 4 are used such that to satisfy (12). Thus, x_2 and x_7 are determined the entering and leaving variables, respectively.

Second iteration:

Steps 4-7: The simplex table is revised as in Table 5.

Table 5. Second iterated simplex table for the LFP problem having z_1

$Max \quad z_{1N}$			-3	2	0	0	0	
$Min \quad z_{1D}$			1	2	0	0	0	
c_{1N}	c_{1D}	x_B	x_1	x_2	x_3	x_4	x_5	b
-3	1	x_1	1	0	0	0	-1	3
0	0	x_4	0	0	-3	1	5	3
2	2	x_2	0	1	1	0	-1	2
$c_{1Nj} - z_{1Nj}$			0	0	-2	0	-1	
$d_{1Dj} - z_{1Dj}$			0	0	-2	0	3	

The objective function value z_1^2 is found as -0.625; thus

$$8(c_{1Nj} - z_{1Nj}) + 5(c_{1Dj} - z_{1Dj}) \geq 0, \tag{13}$$

can be determined. By considering (13), the entering and leaving variables are determined as x_5 and x_4, respectively.

Third iteration: Third iterated simplex table for the LFP problem having z_1 is given in Table 6.

Table 6. Third iterated simplex table for the LFP problem having z_1.

$Max \quad z_{1N}$			-3	2	0	0	0	
$Min \quad z_{1D}$			1	2	0	0	0	
c_{1N}	c_{1D}	x_B	x_1	x_2	x_3	x_4	x_5	b
-3	1	x_1	1	0	$-3/5$	$1/5$	0	$18/5$
0	0	x_5	0	0	$-3/5$	$1/5$	1	$3/5$
2	2	x_2	0	1	$2/5$	$1/5$	0	$13/5$
$c_{1Nj} - z_{1Nj}$			0	0	$-13/5$	$1/5$	0	
$d_{1Dj} - z_{1Dj}$			0	0	$-1/5$	$-3/5$	0	

Steps 4-7: $z_{1N}(x)$ and $z_{1D}(x)$ are determined, and after finding the objective function value z_1^3 as -0.609,

$$23(c_{1Nj} - z_{1Nj}) + 14(c_{1Dj} - z_{1Dj}) \geq 0, \tag{14}$$

is obtained in similar way.

Since there is no entering variable, the algorithm is over at the end of third iteration. The optimal solution of the LFP problem having z_1 is found as $x = (18/5, 13/5)$, and the optimal value is $z^* = -14/23$.

The same steps in the first phase are iterated for the LFP problem having z_2. At the end of the fourth iteration, the optimal solution for the LFP problem having z_2 is $(15/2, 0)$, and the optimal value is $105/77$.

Second phase:

Steps 8-9: The payoff matrix is constructed. Here, the MOLFP problem (6) has two players and two strategies. To increase the number of strategies, the point obtained before the individual optimal solution of each single-objective LFP problem will be taken. Thus, there are four strategies as $(3, 2)$, $(18/5, 13/5)$, $(15/2, 0)$, and $(3, 0)$. These points are substituted into each objective function, and the payoff matrix is constructed as presented in Table 7.

Table 7. The payoff matrix

	z_1	z_2
$(3, 2)$	-0.625	1.15
$(18/5, 13/5)$	-0.608	1.148
$(15/2, 0)$	-2.14	1.329
$(3, 0)$	-1.5	-1.312

Since the payoff matrix has negative entries, the value 3.14 is added to all entries to make all of them positive, and Table 8 is obtained.

Table 8. The positive payoff matrix

	z_1	z_2
$(3, 2)$	2.515	4.29
$(18/5, 13/5)$	2.532	4.288
$(15/2, 0)$	1	4.469
$(3, 0)$	1.64	1.828

Steps 10-11: The first ratio matrix can be obtained as in Table 9.

The weights are found as 0.59 and 0.41 for the objective functions z_1 and z_2, respectively, by solving the zero-sum game given in Table 9. These weights will be used in the third phase.

Table 9. The ratio payoff matrix

	z_1	z_2
$R(x^{1*}/x^{2*})$	0.99	1.0
$R(x^{1*}/x^{3*})$	2.515	0.96
$R(x^{1*}/x^{4*})$	1.53	2.35

Third phase:

Step 12: Each fractional objective function is expanded to the Taylor series to linearize. Therefore, the first and second objective functions are presented as follows:

$$\begin{aligned} z_{1L}(x) &= \frac{-14}{23} + \frac{\partial z_1}{\partial x_1}|_{(2.6,3.6)}(x_1 - 3.6) + \frac{\partial z_1}{\partial x_2}|_{(2.6,3.6)}(x_2 - 2.6) \\ &= -0.259x_1 + 0.283x_2, \end{aligned} \tag{15}$$

$$\begin{aligned} z_{2L}(x) &= \frac{15}{11} + \frac{\partial z_2}{\partial x_1}(x_1 - 7.5) + \frac{\partial z_2}{\partial x_2}(x_2 - 0) \\ &= 0.005x_1 - 0.045x_2. \end{aligned} \tag{16}$$

Steps 13-14: The linearized objective functions are multiplied with the corresponding weights, and a single-objective LP problem is obtained satisfying the original constraints of (6). Thus, the following LP problem is solved:

$$Max \quad z = 0.59(-0.259x_1 + 0.283x_2) + 0.41(0.005x_1 - 0.045x_2), \tag{17a}$$

subject to

$$x_1 - x_2 \geq 1, \tag{17b}$$

$$2x_1 + 3x_2 \leq 15, \tag{17c}$$

$$x_1 \geq 3. \tag{17d}$$

The optimal solution is found as $(3, 2)$, and the optimal value is 0.155. This example was solved in the study of Guzel and Sivri (2005), and the point $(3, 2)$ was also found as a compromise solution of the problem.

The optimal solutions and objective function values can be found from different ratio matrices depending on the order of the rows to present the decision-maker. Therefore, the weights and a set of compromise solutions are presented in Table 10.

Table 10. Different optimal values related to different ratios

Weights		Optimal Solution		Objective Function Value
z_1	z_2	x_1	x_2	z
0.44	0.56	3	2	0.1356
0.08	0.92	3	0	0.48
0.58	0.42	3	2	0.1536
1	0	3.6	2.6	0.1966

Example 3.2.

Consider the following MOLFP problem:

$$Max \quad z_1 = \frac{3x_1 + x_2 - 1}{2x_1 - x_2 + 1}, \tag{18a}$$

$$Min \quad z_2 = \frac{2x_1 + 5x_2 + 3}{x_1 - 2x_2 + 2}, \tag{18b}$$

subject to

$$5x_1 + 3x_2 \leq 60, \tag{18c}$$

$$x_1 \leq 5. \tag{18d}$$

First phase:

Steps 1-2: The constraints are rewritten in the standard form, and then the objective functions z_1 and z_2 are separated as follows:

$$Max \quad z_{1N} = 3x_1 + x_2 - 1, \tag{19a}$$

$$Min \quad z_{1D} = 2x_1 - x_2 + 1, \tag{19b}$$

$$Min \quad z_{2N} = 2x_1 + 5x_2 + 3, \tag{19c}$$

$$Max \quad z_{2D} = x_1 - 2x_2 + 2, \tag{19d}$$

where the indices N and D refer to nominators and denominators of the objective functions.

Step 3: The initial simplex table for the constructed LP problem having z_1 is constructed in Table 11.

Table 11. Initial simplex table for the LFP problem having z_1

$Max \quad z_{1N}$			3	1	0	0	
$Min \quad z_{1D}$			2	-1	0	0	
C_{1N}	C_{1D}	x_B	x_1	x_2	x_3	x_4	b
0	0	x_3	5	3	1	0	60
0	0	x_4	1	0	0	1	5
$c_{1Nj} - z_{1Nj}$			3	1	0	0	
$d_{1Dj} - z_{1Dj}$			2	-1	0	0	

Steps 4-7: Considering the right-hand side values in Table 11, the objective function value z_1^0 is found as -1. Since the constructed LFP problem having z_1 is maximized,

$$\frac{z_{1N}^0 + \lambda(c_{1Nj} - z_{1Nj})}{z_{1D}^0 + \lambda(c_{1Dj} - z_{1Dj})} \geq -1, \tag{20}$$

is considered, and

$$(c_{1Nj} - z_{1Nj}) + (c_{1Dj} - z_{1Dj}) \geq 0, \tag{21}$$

is formed. It is seen that x_1 is the entering variable which satisfies the inequality (21), and x_4 is the leaving variable.

First iteration:

Steps 4-7: The first iterated simplex table is constructed such that the basis vectors are x_3 and x_1, and then the objective function value z_1^1 is found as $14/11$. Therefore,

$$\frac{z_{1N}^1 + \lambda(c_{1Nj} - z_{1Nj})}{z_{1D}^1 + \lambda(c_{1Dj} - z_{1Dj})} \geq \frac{14}{11}, \tag{22}$$

is written, and

$$11(c_{1Nj} - z_{1Nj}) - 14(c_{1Dj} - z_{1Dj}) \geq 0, \tag{23}$$

is obtained. Thus, x_2 and x_3 are determined as the entering and leaving variables, respectively.

Second iteration:

Steps 4-7: In the second iterated simplex table, the basis vectors are x_2 and x_1. The objective function value is found as $z_1^2 = -77/2$. Therefore,

$$2(c_{1Nj} - z_{1Nj}) - 77(c_{1Dj} - z_{1Dj}) \geq 0, \tag{24}$$

is determined. By considering (24), the entering and leaving variables are found as x_4 and x_1, respectively.

Third iteration:

Steps 4-7: x_2 and x_4 are the basis vectors in the third iterated simplex table, and the objective function value z_1^3 is found as -1. Thus,

$$(c_{1Nj} - z_{1Nj}) + (c_{1Dj} - z_{1Dj}) \geq 0, \tag{25}$$

is obtained. It is seen that x_1 is the entering variable which satisfies the inequality (25), and x_4 is the leaving variable.

After the third iteration, since the previous vectors started to be obtained continually, the operations are finalized. When the objective function values obtained via these basis vectors are compared, it is seen that x_2 and x_4 make the objective function greater. Thus, the basis vectors remain as x_2 and x_4. Therefore, the optimal solution is $(0, 20)$, and the optimal value is $z_1^* = -1$.

The same steps in the first phase are iterated for the LFP problem having z_2. At the initial iteration, the optimal solution is found as $(0, 0)$, and the optimal value is $z_2^* = 1.5$.

Second phase:

Steps 8-9: By using the corner point $(5, 35/3)$ and the optimal points $(0, 20)$ and $(0, 0)$, the payoff matrix is constructed and presented in Table 12.

Table 12. The payoff matrix

	z_1	z_2
$(0, 20)$	-1	-2.71
$(5, 35/3)$	-38.5	-4.37
$(0, 0)$	-1	1.5

Since the payoff matrix has negative entries, the value 39.5 is added to all entries to make them positive, and Table 13 is obtained.

Table 13. The positive payoff matrix

	z_1	z_2
$(0, 20)$	38.5	36.79
$(5, 35/3)$	1	35.13
$(0, 0)$	38.5	41

Step 10: A ratio matrix can be obtained in Table 14.

Table 14. The ratio payoff matrix

	z_1	z_2
$R(x^{1*}/x^{2*})$	38.5	1.05
$R(x^{1*}/x^{3*})$	1	0.897

By solving the zero-sum game given in Table 14, the weights are found to be 0 and 1 for the objective functions z_1 and z_2, respectively. These weights will be used in the third phase.

Third phase:

Step 12: Since the full weight is assigned to the objective function z_2, the fractional objective function z_2 is expanded to the Taylor series to linearize at the point $(0,0)$:

$$\begin{aligned} z_{2L}(x) &= \frac{3}{2} + \frac{\partial z_2}{\partial x_1}|_{(0,0)}(x_1 - 0) + \frac{\partial z_2}{\partial x_2}|_{(0,0)}(x_2 - 0) \\ &= 0.25x_1 + 4x_2 + 1.5. \end{aligned} \tag{26}$$

Steps 13-14: The following LP problem is constructed and solved:

$$Min \quad z = 0.25x_1 + 4x_2 + 1.5, \tag{27a}$$

subject to

$$5x_1 + 3x_2 \leq 60, \tag{27b}$$

$$x_1 \leq 5. \tag{27c}$$

The optimal solution is found as $(0,0)$, and the optimal value is -1. Thus, one of the compromise solutions for the MOLFP problem is $(0,0)$.

By constructing all other possible ratio matrices, the weights for the objective functions z_1 and z_2 are determined to be 1 and 0, respectively. Accordingly, the full weight is assigned to z_1; thus the fractional objective function is expanded to the Taylor series at the point $(0,20)$. Finally, another compromise solution for the MOLFP problem is found as $(5,0)$.

4. Conclusion

In this paper, a hybrid algorithm including three phases, which are the simplex method, game theory, and the Taylor series approach, is presented to find compromise solutions to a MOLFP problem. Compromise solutions are found considering weights determined via the zero-sum game. These solutions provide various options to decision-makers considering different degrees of importance of the objectives. As a result, decision-makers can choose the most suitable one among solutions according to the satisfaction levels, i.e., weights, determined for the objectives which is the novelty of the study.

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