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On Deferred Statistical Convergence of Fuzzy Variables

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Abstract

In this paper, within framework credibility theory, we examine several notions of convergence and statistical convergence of fuzzy variable sequences. The convergence of fuzzy variable sequences such as the notion of convergence in credibility, convergence in distribution, convergence in mean, and convergence uniformly virtually certainly via postponed Cesàro mean and a regular matrix are researched using fuzzy variables. We investigate the connections between these concepts. Significant results on deferred statistical convergence for fuzzy variable sequences are thoroughly investigated.

Keywords: Deferred statistical convergence; Fuzzy variable sequence; Credibility measure

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1. Introduction and Preliminaries

Fuzzy theory has made significant progress on the mathematical underpinnings of fuzzy set theory, which was pioneered by Zadeh in 1965. Fuzzy theory may be used to a wide range of real-world challenges. Many researchers, for example, Dubois and Prade (1998) and Nahmias (1978), have created possibility theory. A fuzzy variable is a function that maps from a credibility space to a collection of real values. The convergence of fuzzy variables is an important component of credibility theory, which may be used to real-world engineering and financial challenges. Kaufmann (1975) has investigated fuzzy variables, possibility distributions, and membership functions. Possibility measure is a key notion in possibility theory; however, it is not self-dual. It is commonly defined as a supremum preserving set function on the power set of a nonempty set. Because a self-dual measure is essential in both theory and practice, Liu and Liu (2002) developed a self-duality credibility measure. Because it has certain essential characteristics with the possibility measure, the credibility measure serves as a substitute for it in the fuzzy world. Particularly, since Liu began his examination of credibility theory, several specific contents have been investigated (see Liu (2002), Liu (2007), Li and Liu (2006), Li and Liu (2008), Liu and Liu (2003), Zhao et al. (2006), Kwakernaak (1978), Wang and Liu (2003) and Liu (2006)). Given the importance of sequence convergence in credibility theory; Liu (2003) offered four types of convergence concepts for fuzzy variables: credibility convergence, almost certainly convergence, mean convergence, and distribution convergence. Jiang (2011) and Ma (2014) have studied on numerous convergence properties of credibility distribution for fuzzy variables based on credibility theory.

Wang and Liu (2003) explored the links between mean convergence, credibility convergence, almost uniform convergence, distribution convergence, and almost surely convergence. Furthermore, many academics highlighted convergence principles in classical measure theory, credibility theory, and probability theory, as well as investigated their relationships. Readers that are interested can look into Liu and Wang (2006), You et al. (2019), Chen et al. (2015), Xia (2011), and Lin (2000).

Fast (1951) proposed statistical convergence as an extension of ordinary convergence. After the studies of Fridy (1985), statistical convergence became one of the most active fields of research in summability theory. Statistical convergence has been investigated in fuzzy number space. Other research in this area, as well as various applications of statistical convergence, may be found in Küçükaslan and Yılmaztürk (2016), Belen and Mohiuddine (2013), Mohiuddine et al. (2019a), Mohiuddine et al. (2019b), Mohiuddine et al. (2017), Mohiuddine et al. (2010), Savaş and Gürdal (2014) and Savaş and Gürdal (2016). For further information on the sequence spaces, readers should consult the monographs by Başar (2012) and Mursaleen and Başar (2020), as well as recent publication by Talo and Başar (2010) for the background on the sequence spaces.

The definitions and properties needed in this paper are shown in Liu and Liu (2002), Wang and Liu (2003), Liu (2003), Lin (2000), Li and Liu (2006), Agnew (1932), Küçükaslan and Yılmaztürk (2016) and Kolk (1993).

The aim of the present paper is to investigate the new kind of convergence for fuzzy variables sequences. The following is how the paper is structured. The literature review is covered in Section

1 of the introduction. The key findings are then demonstrated in Section 2. That is, we intend to investigate the concept of deferred statistical convergence of fuzzy variables and to develop essential features of deferred statistical convergence in credibility. Section 3 concludes with the findings of the acquired results.

2. Deferred statistical convergence in credibility

First, we'll go over some key terms. All along the paper, let $\varpi, \varpi_1, \varpi_2, \ldots$ be fuzzy variables (FVs) determined on credibility space $(\Theta, \mathcal{P}(\Theta), \mathrm{Cr})$ and $f : \mathbb{R} \to \mathbb{R}$ be a convex function.

Definition 2.1.

In credibility, the sequence $\{\varpi_i\}$ is known as deferred statistically convergent almost surely (d.s.a.s.) to the FV ϖ if and only if there exists $A \in \mathcal{P}(\Theta)$ having $\mathrm{Cr}\{A\} = 1$ such that

$$\lim_{t\to\infty} \frac{1}{q_t - p_t} \left|\left\{p_t < i \leq q_t : |\varpi_i(\phi) - \varpi(\phi)| \geq \rho\right\}\right| = 0,$$

for all $\phi \in A$ and $\rho > 0$. In this case, we use $\varpi_i \overset{DSt}{\to} \varpi$, a.s.

Definition 2.2.

In credibility, the sequence $\{\varpi_i\}$ is known as deferred statistically convergent to ϖ if and only if there is a $A \in \mathcal{P}(\Theta)$ having $\mathrm{Cr}\{A\} = 1$ so that

$$\lim_{t\to\infty} \frac{1}{q_t - p_t} \left|\left\{p_t < i \leq q_t : \mathrm{Cr}\{|\varpi_i - \varpi| \geq \rho\} \geq \sigma\right\}\right| = 0,$$

for all $\rho, \sigma > 0$. In that case, we use $DS(\mathrm{Cr}) - \lim \varpi_i = \varpi$.

Definition 2.3.

Presume that $\{\varpi_i\}$ is a sequence of FVs having finite expected values identified on $(\Theta, \mathcal{P}(\Theta), Cr)$. The sequence $\{\varpi_i\}$ is known as deferred statistically convergent in mean to ϖ provided that

$$\lim_{t\to\infty} \frac{1}{q_t - p_t} \left|\left\{p_t < i \leq q_t : E[|\varpi_i - \varpi|] \geq \rho\right\}\right| = 0,$$

for each $\rho > 0$. In that case, we use $DS(\mathrm{E}) - \lim \varpi_i = \varpi$.

Definition 2.4.

Let $\Phi, \Phi_1, \Phi_2, \ldots$ be the credibility distributions of FVs $\varpi, \varpi_1, \varpi_2, \ldots$, respectively. The sequence $\{\varpi_i\}$ is deferred statistically convergent in distribution to ϖ provided that for each $\rho > 0$,

$$\lim_{t\to\infty} \frac{1}{q_t - p_t} \left|\left\{p_t < i \leq q_t : |\Phi_i(y) - \Phi(y)| \geq \rho\right\}\right| = 0,$$

for all y. Here $\Phi(y) := \mathrm{Cr}\{\phi \in \Theta : \varpi(\phi) \leq y\}$ is continuous.

Definition 2.5.

The sequence $\{\varpi_i\}$ is known as deferred statistically convergent uniformly almost surely (d.s.u.a.s.) in credibility space (CS) to ϖ provided that for all $\rho > 0$, $\exists\sigma > 0$ and a sequence $\{A'_i\} \in \mathcal{P}(\Theta)$ having $\mathrm{Cr}\{A'_i\} = 1$ so that

$$\lim_{t\to\infty} \tfrac{1}{q_t - p_t} \left|\{p_t < i \leq q_t : |\mathrm{Cr}(A'_i)| \geq \rho\}\right| = 0$$
$$\Rightarrow \lim_{t\to\infty} \tfrac{1}{q_t - p_t} \left|\{p_t < i \leq q_t : |\varpi_i(y) - \varpi(y)| \geq \sigma\}\right| = 0,$$

for all $y \in A'_i$. In this case, we use $\varpi_i \overset{DSt}{\to} \varpi$, u.a.s.

Definition 2.6.

The sequence $\{\varpi_i\}$ is named to be deferred statistical Cauchy sequence almost surely (a.s.) provided that for each $\rho > 0$, there is a $A \in \mathcal{P}(\Theta)$ having $\mathrm{Cr}\{A\} = 1$ and $M = M(\rho)$ such that for all $\phi \in A$,

$$\lim_{t\to\infty} \frac{1}{q_t - p_t} \left|\{p_t < i \leq q_t : |\varpi_i(\phi) - \varpi_M(\phi)| \geq \rho\}\right| = 0.$$

Example 2.1.

Consider the CS $(\Theta, \mathcal{P}, \mathrm{Cr})$ to be $\{\phi_1, \phi_2, ...\}$ with $\mathrm{Cr}\{\phi_t\} = \frac{1}{2}$ for $t = 1, 2, ...$ The FV are given by

$$\varpi_i(\phi_t) = \begin{cases} \frac{1}{t}, \text{ if } i = t, \\ 0, \text{ if not.} \end{cases}$$

For any $\rho > 0$, taking $A = \Theta$ and $M = \left[\frac{1}{\rho}\right] + 1$, we get

$$\lim_{t\to\infty} \frac{1}{q_t - p_t} \left|\left\{p_t < i \leq q_t : |\varpi_i(\phi) - \varpi_M(\phi)| \geq \frac{1}{M} > \rho\right\}\right| = 0,$$

for each $\phi \in A$. Then, the sequence $\{\varpi_i\}$ is a deferred statistical Cauchy sequence a.s.

Definition 2.7.

The sequence $\{\varpi_i\}$ is named to be deferred statistically Cauchy (DStCa) sequence in credibility provided that there is a $A \in \mathcal{P}(\Theta)$ having $\mathrm{Cr}\{A\} = 1$ and $M = M(\sigma)$ so that

$$\lim_{t\to\infty} \frac{1}{q_t - p_t} \left|\{p_t < i \leq q_t : \mathrm{Cr}\{|\varpi_i - \varpi_M| \geq \rho\} \geq \sigma\}\right| = 0,$$

for all $\rho, \sigma > 0$.

Example 2.2.

Establish the CS $(\Theta, \mathcal{P}, \mathrm{Cr})$ to be $\{\phi_1, \phi_2, ...\}$ having $\mathrm{Cr}(\phi_1) = \frac{1}{2}$ and $\mathrm{Cr}(\phi_j) = \frac{1}{j}$, for $j = 2, 3, ...$ The FVs are determined by

$$\varpi_i(\phi_j) = \begin{cases} j, \text{ if } i = j, \\ 0, \text{ if not.} \end{cases}$$

For any $\sigma > 0$, considering $\rho \in (0,1)$ and taking $M = \left[\frac{2}{\sigma}\right] + 1$, we get

$$\lim_{t\to\infty} \frac{1}{q_t - p_t} \left|\left\{p_t < i \leq q_t : \mathrm{Cr}\left\{|\varpi_i - \varpi_M| \geq \rho\right\} \geq \sigma\right\}\right| = 0.$$

Hence, $\{\varpi_i\}$ is said a DStCa sequence in credibility sense.

Theorem 2.1.

Presume $\{\varpi_i\}$ be FV sequence. Then, $DS\,(\mathrm{Cr}) - \lim \varpi_i = \varpi$ if and only if $\{\varpi_i\}$ is DStCa sequence in credibility sense.

Proof:

Assume $DS\,(\mathrm{Cr}) - \lim \varpi_i = \varpi$. So, there is a $A \in \mathcal{P}\,(\Theta)$ supplying $\mathrm{Cr}\,\{A\} = 1$ so that

$$\lim_{t\to\infty} \frac{1}{q_t - p_t} \left|\left\{p_t < i \leq q_t : \mathrm{Cr}\left\{|\varpi_i - \varpi| \geq \rho\right\} \geq \sigma\right\}\right| = 0,$$

for each $\rho, \sigma > 0$. Select $M \in \mathbb{N}$ such that $\mathrm{Cr}\left\{|\varpi_M - \varpi| \geq \rho\right\} \geq \sigma$. Describe the sets K_1, K_2 and K_3 as follows:

$$\begin{aligned} K_1 &= \left\{p_t < i \leq q_t : \mathrm{Cr}\left\{|\varpi_i - \varpi_M| \geq \rho\right\} \geq \sigma\right\}, \\ K_2 &= \left\{p_t < i \leq q_t : \mathrm{Cr}\left\{|\varpi_i - \varpi| \geq \rho\right\} \geq \sigma\right\}, \\ K_3 &= \left\{p_t < i \leq q_t : \mathrm{Cr}\left\{|\varpi_M - \varpi| \geq \rho\right\} \geq \sigma\right\}. \end{aligned}$$

Clearly, $K_1 \subseteq K_2 \cup K_3$. As a result, $\delta\,(K_1) \leq \delta\,(K_2) + \delta\,(K_3) = 0$, since $DS\,(\mathrm{Cr}) - \lim \varpi_i = \varpi$. Hence, $\{\varpi_i\}$ is DStCa sequence in credibility sense.

Conversely, take $\{\varpi_i\}$ as a DStCa sequence in credibility. So, $\delta\,(K_1) = 0$. So, for the subsequent set

$$T = \left\{p_t < i \leq q_t : \mathrm{Cr}\left\{|\varpi_i - \varpi_M| \geq \rho\right\} < \sigma\right\},$$

we get $\delta\,(T) = 1$.

For all $\rho > 0$, there exists $\rho' \in \left(0, \frac{\rho}{2}\right]$ so that

$$\mathrm{Cr}\left\{|\varpi_i - \varpi_M| \geq \rho\right\} \leq 2\mathrm{Cr}\left\{|\varpi_i - \varpi| \geq \rho'\right\} < \sigma. \tag{1}$$

In addition, if $DS\,(\mathrm{Cr}) - \lim \varpi_i \neq \varpi$, then $\delta\,(K_2) = 1$. As a result, for the set

$$P = \left\{p_t < i \leq q_t : \mathrm{Cr}\left\{|\varpi_i - \varpi| \geq \rho\right\} < \sigma\right\},$$

we get $\delta\,(P) = 0$. Thus, from (1), for the set

$$G = \left\{p_t < i \leq q_t : \mathrm{Cr}\left\{|\varpi_i - \varpi_M| \geq \rho\right\} < \sigma\right\},$$

we obtain $\delta\,(G) = 0$, which gives that $\delta\,(K_1) = 1$ and so it causes a contradiction that $\{\varpi_i\}$ is DStCa sequence in credibility. Hence, $DS\,(\mathrm{Cr}) - \lim \varpi_i = \varpi$. ∎

Theorem 2.2.

The sequence $\{\varpi_i\}$ is named to be deferred statistical $T_t\,(\mathrm{Cr})$-summable to ϖ if and only if there exists $A \in \mathcal{P}\,(\Theta)$ supplying $\mathrm{Cr}\,\{A\} = 1$ so that

$$\lim_{t\to\infty} \frac{1}{q_t - p_t} \left| \left\{ p_t < i \leq q_t : \sum_{i=p_t+1}^{q_t} \mathrm{Cr}\,\{|\varpi_i - \varpi| \geq \rho\} \geq \sigma \right\} \right| = 0,$$

for any $\sigma, \rho > 0$, where

$$T_t\,(\mathrm{Cr}) = \frac{1}{q_t - p_t} \sum_{i=p_t+1}^{q_t} \mathrm{Cr}\,\{|\varpi_i| \geq \rho\}\,.$$

In that case, we use $\varpi_i \overset{DS-T_t(Cr)}{\to} \varpi$.

Example 2.3.

Take $(\Theta, \mathcal{P}\,(\Theta)\,, \mathrm{Cr})$ to be $\{\phi_1, \phi_2, ...\}$ with $\mathrm{Cr}\,(\phi_1) = \mathrm{Cr}\,(\phi_2) = \frac{1}{2}$ and $\mathrm{Cr}\,(\phi_j) = \frac{1}{2j}$, for $j = 3, 4, ...$ The FV are identified by

$$\varpi_i\,(\phi_j) = \begin{cases} 1, \text{ if } i = j, \\ 0, \text{ if not.} \end{cases}$$

Take $\varpi = 0$. So, for any $\sigma \in \left[\frac{1}{2}, 1\right)$ and $\rho \in (0, 1)$, we get

$$\sum_{i=p_t+1}^{q_t} \mathrm{Cr}\,\{|\varpi_i - \varpi| \geq \rho\} \leq \frac{t}{2t} = \frac{1}{2}.$$

Thus

$$\lim_{t\to\infty} \frac{1}{q_t - p_t} \left| \left\{ p_t < i \leq q_t : \sum_{i=p_t+1}^{q_t} \mathrm{Cr}\,\{|\varpi_i - 0| \geq \rho\} \geq \sigma \right\} \right| = 0.$$

In other words, $\varpi_i \overset{DS-T_t(Cr)}{\to} \varpi$.

Theorem 2.3.

In credibility, $\{\varpi_i\}$ deferred statistically converges to ϖ if $\{\varpi_i\}$ is deferred statistical $T_t\,(\mathrm{Cr})$-summable to ϖ.

Proof:

Assume that $\rho, \sigma > 0$. We have

$$\sum_{i=p_t+1}^{q_t} \mathrm{Cr}\,\{|\varpi_i - \varpi| \geq \rho\} \geq \mathrm{Cr}\,\{|\varpi_i - \varpi| \geq \rho\}\,.$$

Thus

$$\left| \left\{ p_t < i \leq q_t : \sum_{i=p_t+1}^{q_t} \mathrm{Cr}\,\{|\varpi_i - \varpi| \geq \rho\} \geq \sigma \right\} \right| \geq |\{p_t < i \leq q_t : \mathrm{Cr}\,\{|\varpi_i - \varpi| \geq \rho\} \geq \sigma\}|\,.$$

So, $\varpi_i \overset{DS-T_t(\mathrm{Cr})}{\to} \varpi$ clearly indicates $DS\,(\mathrm{Cr}) - \lim \varpi_i = \varpi$. ∎

Example 2.4.

The presence of deferred statistical convergence in credibility sense does not entail the presence of deferred statistical $T_t\,(\mathrm{Cr})$-summable. For example, consider $(\Theta, \mathcal{P}(\Theta), \mathrm{Cr})$ to be $\{\phi_1, \phi_2, ...\}$ with $\mathrm{Cr}(\phi_j) = \frac{j}{2j+1}$ for $j = 1, 2, ...$ The FVs are identified by

$$\varpi_i(\phi_j) = \begin{cases} 1, \text{ when } i = j, \\ 0, \text{ if not.} \end{cases}$$

Suppose that $\varpi = 0$. We have

$$\lim_{t\to\infty} \frac{1}{q_t - p_t} \left|\{p_t < i \le q_t : \mathrm{Cr}\{|\varpi_i - 0| \ge \rho\} \ge \sigma\}\right| = 0,$$

for any $\sigma \in \left[\frac{1}{2}, 1\right)$ and $\rho \in (0, 1)$. Hence, $DS\,(\mathrm{Cr}) - \lim \varpi_i = \varpi$. At the same moment,

$$\lim_{t\to\infty} \frac{1}{q_t - p_t} \left|\left\{p_t < i \le q_t : \sum_{i=p_t+1}^{q_t} \mathrm{Cr}\{|\varpi_i - 0| \ge \rho\} \ge \sigma\right\}\right| = 1.$$

This produces the intended outcome.

Deferred statistical convergence in credibility supplies some classic axioms of convergence in credibility.

$(\mathbf{H})$ When $DS\,(\mathrm{Cr}) - \lim \varpi_i = \varpi_1$ and $DS\,(\mathrm{Cr}) - \lim \varpi_i = \varpi_2$, then $\varpi_1 = \varpi_2$ in credibility.

$(\mathbf{U})$ When there is a subset $T = \{m_1 < m_2 < ...\} \subseteq \mathbb{N}$ such that $DS\,(\mathrm{Cr}) - \lim \varpi_{m_i} = \varpi$, then $DS\,(\mathrm{Cr}) - \lim \varpi_i = \varpi$.

Theorem 2.4.

The axioms $(\mathbf{H})$ and $(\mathbf{U})$ are satisfied by deferred statistical convergence in credibility sense.

Proof:

The axiom $(\mathbf{U})$ is obviously supplied by deferred statistical convergence in credibility. Suppose that there is a subset $T = \{m_1 < m_2 < ...\} \subseteq \mathbb{N}$ such that $DS\,(\mathrm{Cr}) - \lim \varpi_{m_i} = \varpi$, i.e., for all $\phi \in A$, any $\rho, \sigma > 0$, there is a $A \in \mathcal{P}(\Theta)$ having $\mathrm{Cr}\{A\} = 1$ and $i_0 = i_0(\rho)$ so that

$$\mathrm{Cr}\{|\varpi_{m_i} - \varpi| \ge \rho\} < \sigma,$$

for all $i > i_0$. Let $T = \{m_{i_0+1}, m_{i_0+2}, ...\}$. So, there exists $A \in \mathcal{P}(\Theta)$ having $\mathrm{Cr}\{A\} = 1$ so that

$$\lim_{t\to\infty} \frac{1}{q_t - p_t} \left|\{i \in T : \mathrm{Cr}\{|\varpi_i - \varpi| \ge \rho\} \ge \sigma\}\right| = 0,$$

for each $\rho, \sigma > 0$. Hence, we acquire

$$\lim_{t\to\infty} \frac{1}{q_t - p_t} \left|\{p_t < i \le q_t : \mathrm{Cr}\{|\varpi_i - \varpi| \ge \rho\} \ge \sigma\}\right| = 0,$$

for each $\rho, \sigma > 0$. Thus, the axiom $(\mathbf{U})$ is satisfied.

Suppose that $DS\,(\mathrm{Cr}) - \lim \varpi_i = \varpi_1$ and $DS\,(\mathrm{Cr}) - \lim \varpi_i = \varpi_2$. So, there is a $A \in \mathcal{P}\,(\Theta)$ supplying $\mathrm{Cr}\,\{A\} = 1$ so that

$$\lim_{t\to\infty} \frac{1}{q_t - p_t} \left|\left\{i \in T : \mathrm{Cr}\left\{|\varpi_i - \varpi_1| \geq \rho\right\} \geq \sigma\right\}\right| = 0,$$

and

$$\lim_{t\to\infty} \frac{1}{q_t - p_t} \left|\left\{i \in T : \mathrm{Cr}\left\{|\varpi_i - \varpi_2| \geq \rho\right\} \geq \sigma\right\}\right| = 0,$$

for every $\rho, \sigma > 0$. Sets B_1 and B_2 should be set up as follows:

$$B_1 = \left\{p_t < i \leq q_t : \mathrm{Cr}\left\{|\varpi_i - \varpi_1| \geq \rho\right\} \geq \sigma\right\},$$

and

$$B_2 = \left\{p_t < i \leq q_t : \mathrm{Cr}\left\{|\varpi_i - \varpi_2| \geq \rho\right\} \geq \sigma\right\}.$$

Now let $i \in B_1 \cup B_2$. Then, we acquire

$$\mathrm{Cr}\left\{|\varpi_i - \varpi_1| \geq \rho\right\} < \sigma, \mathrm{Cr}\left\{|\varpi_i - \varpi_2| \geq \rho\right\} < \sigma.$$

Therefore

$$\begin{aligned}\mathrm{Cr}\left\{|\varpi_1 - \varpi_2| \geq \rho\right\} &= \mathrm{Cr}\left\{|\varpi_1 - \varpi_i + \varpi_i - \varpi_2| \geq \rho\right\} \\ &\leq \mathrm{Cr}\left\{|\varpi_i - \varpi_1| \geq \tfrac{\rho}{2}\right\} + \mathrm{Cr}\left\{|\varpi_i - \varpi_2| \geq \tfrac{\rho}{2}\right\} \\ &< 2\sigma.\end{aligned}$$

Because $\sigma > 0$ is arbitrary, we can have $\mathrm{Cr}\left\{|\varpi_1 - \varpi_2| \geq \rho\right\} = 0$, which gives $\varpi_1 = \varpi_2$ in credibility. ■

Theorem 2.5.

If $\varpi_i \overset{DSt}{\to} \varpi$, a.s., then $f\,(\varpi_i) \overset{DSt}{\to} f\,(\varpi)$, a.s.

Proof:

From the hypothesis, we acquire

$$|f\,(\varpi_i) - f\,(\varpi)| \leq c\,|\varpi_i - \varpi|\,.$$

Since $\varpi_i \overset{DSt}{\to} \varpi$, a.s., there exists $A \in \mathcal{P}\,(\Theta)$ having $\mathrm{Cr}\,\{A\} = 1$ so that

$$\lim_{t\to\infty} \frac{1}{q_t - p_t} \left|\left\{p_t < i \leq q_t : |\varpi_i\,(\phi) - \varpi\,(\phi)| \geq \rho\right\}\right| = 0,$$

for all $\phi \in A$ and $\rho > 0$. So, for any $\rho > 0$, there is a $A \in \mathcal{P}\,(\Theta)$ having $\mathrm{Cr}\,\{A\} = 1$ so that $|\varpi_i\,(\phi) - \varpi\,(\phi)| < \frac{\rho}{c}$. Then

$$|f\,(\varpi_i\,(\phi)) - f\,(\varpi\,(\phi))| \leq c\,|\varpi_i\,(\phi) - \varpi\,(\phi)| < c\frac{\rho}{c} < \rho,$$

namely,

$$\lim_{t\to\infty} \frac{1}{q_t - p_t} \left|\left\{p_t < i \leq q_t : |f\,(\varpi_i\,(\phi)) - f\,(\varpi\,(\phi))| \geq \rho\right\}\right| = 0,$$

for all $\phi \in A$ and $\rho > 0$. So, we acquire $f\,(\varpi_i) \overset{DSt}{\to} f\,(\varpi)$, a.s. ■

Theorem 2.6.

If $DS\,(\mathrm{Cr})-\lim \varpi_i=\varpi$, then $DS\,(\mathrm{Cr})-\lim f\left(\varpi_i\right)=f\left(\varpi\right)$.

Proof:

Considering that $DS\,(\mathrm{Cr})-\lim \varpi_i=\varpi$, so there is a $A\in\mathcal{P}\left(\Theta\right)$ having $\mathrm{Cr}\left\{A\right\}=1$ so that

$$\lim_{t\to\infty}\frac{1}{q_t-p_t}\left|\left\{p_t<i\leq q_t:\mathrm{Cr}\left\{\left|\varpi_i-\varpi\right|\geq\frac{\rho}{c}\right\}\geq\sigma\right\}\right|=0,$$

for all $\rho,\sigma>0$. Therefore, we have

$$\lim_{t\to\infty}\frac{1}{q_t-p_t}\left|\left\{p_t<i\leq q_t:\mathrm{Cr}\left\{\left|\varpi_i-\varpi\right|\geq\frac{\rho}{c}\right\}<\sigma\right\}\right|=1,$$

and

$$\left|f\left(\varpi_i\right)-f\left(\varpi\right)\right|\leq c\left|\varpi_i-\varpi\right|<c\frac{\rho}{c}<\rho.$$

So,

$$\lim_{t\to\infty}\frac{1}{q_t-p_t}\left|\left\{p_t<i\leq q_t:\mathrm{Cr}\left\{\left|f\left(\varpi_i\right)-f\left(\varpi\right)\right|\geq\rho\right\}<\sigma\right\}\right|=1.$$

As a result, for all $\sigma,\rho>0$, we have

$$\lim_{t\to\infty}\frac{1}{q_t-p_t}\left|\left\{p_t<i\leq q_t:\mathrm{Cr}\left\{\left|f\left(\varpi_i\right)-f\left(\varpi\right)\right|\geq\rho\right\}\geq\sigma\right\}\right|=0,$$

implying that $DS\,(\mathrm{Cr})-\lim f\left(\varpi_i\right)=f\left(\varpi\right)$. ∎

Theorem 2.7.

If $DS\,(\mathrm{E})-\lim \varpi_i=\varpi$, then $DS\,(\mathrm{E})-\lim f\left(\varpi_i\right)=f\left(\varpi\right)$.

Proof:

From the assumption, we get

$$\lim_{t\to\infty}\frac{1}{q_t-p_t}\left|\left\{p_t<i\leq q_t:E\left[\left|\varpi_i-\varpi\right|\right]\geq\rho\right\}\right|=0,$$

and

$$\lim_{t\to\infty}\frac{1}{q_t-p_t}\left|\left\{p_t<i\leq q_t:\mathrm{Cr}\left\{\left|\varpi_i-\varpi\right|\geq\rho\right\}\geq\sigma\right\}\right|=0,$$

for all $\rho>0$ and $\sigma,\rho>0$. From Theorem 2.6, we have

$$\lim_{t\to\infty}\frac{1}{q_t-p_t}\left|\left\{p_t<i\leq q_t:\mathrm{Cr}\left\{\left|f\left(\varpi_i\right)-f\left(\varpi\right)\right|\geq\rho\right\}\geq\sigma\right\}\right|=0.$$

At the same time, we may conclude that $\left|f\left(\varpi_i\right)-f\left(\varpi\right)\right|$ is bounded. $\left|f\left(\varpi_i\right)-f\left(\varpi\right)\right|$ is thus uniformly essentially bounded. As a result,

$$\lim_{t\to\infty}\frac{1}{q_t-p_t}\left|\left\{p_t<i\leq q_t:E\left[\left|f\left(\varpi_i\right)-f\left(\varpi\right)\right|\right]\geq\rho\right\}\right|=0,$$

indicating that $DS\,(\mathrm{E})-\lim f\left(\varpi_i\right)=f\left(\varpi\right)$. ∎

Corollary 2.1.

If $DS\,(\mathrm{Cr}) - \lim \varpi_i = \varpi$, then $f(\varpi_i) \overset{DSt}{\to} f(\varpi)$, a.s.

Proof:

If $DS\,(\mathrm{Cr}) - \lim \varpi_i = \varpi$, then $\varpi_i \overset{DSt}{\to} \varpi$, a.s. Since f is a convex function, we obtain $f(\varpi_i) \overset{DSt}{\to} f(\varpi)$, a.s. ∎

Corollary 2.2.

If $DS\,(\mathrm{E}) - \lim \varpi_i = \varpi$, then $DS\,(\mathrm{Cr}) - \lim f(\varpi_i) = f(\varpi)$.

Proof:

If $DS\,(\mathrm{E}) - \lim \varpi_i = \varpi$, then $DS\,(\mathrm{Cr}) - \lim \varpi_i = \varpi$. Considering that f is a convex function, we acquire $DS\,(\mathrm{Cr}) - \lim f(\varpi_i) = f(\varpi)$. ∎

Theorem 2.8.

If $DS\,(\mathrm{Cr}) - \lim \varpi_i = \varpi$ and $f : \mathbb{R} \to \mathbb{R}$ be a continuous function, then $DS\,(\mathrm{Cr}) - \lim f(\varpi_i) = f(\varpi)$.

Proof:

Considering that f is a continuous function, so for all $\rho > 0$, there is a $\delta > 0$ so that $|\varpi_i - \varpi| < \delta$ gives $|f(\varpi_i) - f(\varpi)| < \rho$. Therefore, $|f(\varpi_i) - f(\varpi)| \geq \rho$ gives $|\varpi_i - \varpi| \geq \delta$. So one can write,

$$\{|f(\varpi_i) - f(\varpi)| \geq \rho\} \subset \{|\varpi_i - \varpi| \geq \delta\}.$$

Utilizing credibility both sides,

$$\mathrm{Cr}\{|f(\varpi_i) - f(\varpi)| \geq \rho\} \leq \mathrm{Cr}\{|\varpi_i - \varpi| \geq \delta\},$$

which gives

$$\{p_t < i \leq q_t : \mathrm{Cr}\{|f(\varpi_i) - f(\varpi)| \geq \rho\} \geq \sigma\} \subset \{p_t < i \leq q_t : \mathrm{Cr}\{|\varpi_i - \varpi| \geq \delta\} \geq \sigma\}.$$

Since $DS\,(\mathrm{Cr}) - \lim \varpi_i = \varpi$, we have

$$\lim_{t\to\infty} \frac{1}{q_t - p_t} |\{p_t < i \leq q_t : \mathrm{Cr}\{|\varpi_i - \varpi| \geq \delta\} \geq \sigma\}| = 0.$$

Thus, we acquire

$$\lim_{t\to\infty} \frac{1}{q_t - p_t} |\{p_t < i \leq q_t : \mathrm{Cr}\{|f(\varpi_i) - f(\varpi)| \geq \rho\} \geq \sigma\}| = 0,$$

which means $DS\,(\mathrm{Cr}) - \lim f(\varpi_i) = f(\varpi)$. ∎

Theorem 2.9.

Presume that $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. When $DS\,(\mathrm{E}) - \lim \varpi_i = \varpi$, then $DS\,(\mathrm{E}) - \lim f(\varpi_i) = f(\varpi)$.

Proof:

Let $DS\,(\mathrm{E}) - \lim \varpi_i = \varpi$. So, for every $\rho > 0$, we have

$$\lim_{t\to\infty} \frac{1}{q_t - p_t} \left|\{p_t < i \leq q_t : E\left[|\varpi_i - \varpi|\right] \geq \rho\}\right| = 0.$$

For any $\sigma > 0$,

$$\lim_{t\to\infty} \frac{1}{q_t - p_t} \left|\{p_t < i \leq q_t : \mathrm{Cr}\left\{|\varpi_i - \varpi| \geq \rho\right\} \geq \sigma\}\right| = 0.$$

From Theorem 2.8, we have

$$\lim_{t\to\infty} \frac{1}{q_t - p_t} \left|\{p_t < i \leq q_t : \mathrm{Cr}\left\{|f\left(\varpi_i\right) - f\left(\varpi\right)| \geq \rho\right\} \geq \sigma\}\right| = 0.$$

At the same time, we may deduce that $|f\left(\varpi_i\right) - f\left(\varpi\right)|$ is bounded. This means that $|f\left(\varpi_i\right) - f\left(\varpi\right)|$ is uniformly essentially bounded. As a result,

$$\lim_{t\to\infty} \frac{1}{q_t - p_t} \left|\{p_t < i \leq q_t : E\left[|f\left(\varpi_i\right) - f\left(\varpi\right)|\right] \geq \rho\}\right| = 0,$$

indicating that $DS\,(\mathrm{E}) - \lim f\left(\varpi_i\right) = f\left(\varpi\right)$. ■

Corollary 2.3.

Presume that $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. When $DS\,(\mathrm{Cr}) - \lim \varpi_i = \varpi$, then $f\left(\varpi_i\right) \overset{DSt}{\to} f\left(\varpi\right)$, a.s.

Proof:

Since f is a continuous function, we get $DS\,(\mathrm{Cr}) - \lim f\left(\varpi_i\right) = f\left(\varpi\right)$. When $DS\,(\mathrm{Cr}) - \lim f\left(\varpi_i\right) = f\left(\varpi\right)$, then $f\left(\varpi_i\right) \overset{DSt}{\to} f\left(\varpi\right)$, a.s. ■

Corollary 2.4.

Presume that $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. When $DS\,(\mathrm{E}) - \lim \varpi_i = \varpi$, then $DS\,(\mathrm{Cr}) - \lim f\left(\varpi_i\right) = f\left(\varpi\right)$.

Proof:

Since f is a continuous function, we get $DS\,(\mathrm{E}) - \lim f\left(\varpi_i\right) = f\left(\varpi\right)$. If $DS\,(\mathrm{E}) - \lim f\left(\varpi_i\right) = f\left(\varpi\right)$, then $DS\,(\mathrm{Cr}) - \lim f\left(\varpi_i\right) = f\left(\varpi\right)$. ■

Now, we identify deferred A-summable mean of a FV sequence $\{\varpi_i\}$ as

$$s_t = \left(A\varpi\right)_t = \sum_{i=p_t+1}^{q_t} a_{ti}\varpi_i = 0.$$

Definition 2.8.

The FV sequence $\{\varpi_t\}$ is known as deferred A-statistically convergent of order β to ϖ if for each $\rho > 0$, there is a $A \in \mathcal{P}(\Theta)$ such that

$$\lim_{t\to\infty} \frac{1}{\sigma^{\beta}(p_t, q_t)} \left|\left\{p_t < i \leq q_t : \left\|(A\varpi)_i(\phi) - \varpi(\phi)\right\| \geq \rho\right\}\right| = 0,$$

for all $\phi \in A$, where $\sigma(p_t, q_t) = \mathcal{C}(q_t - p_t)$, and $\mathcal{C}$ is a constant independent of ϖ.

It is noted that the fuzzy variable sequence $\{\varpi_i\}$ is deferred A-statistically convergent of order β to ϖ if and only if $\{(A\varpi)_i\}$ is deferred statistically convergent of order β to ϖ.

Definition 2.9.

The sequence $\{\varpi_t\}$ is known as deferred A-statistically bounded of order β, if there is a real number Q such that

$$\lim_{t\to\infty} \frac{1}{\sigma^{\beta}(p_t, q_t)} \left|\left\{p_t < i \leq q_t : \left\|(A\varpi)_i\right\| > Q\right\}\right| = 0.$$

Definition 2.10.

The sequence $\{\varpi_t\}$ is known as deferred A-statistically convergent of order β almost surely to ϖ if and only if there is a $A \in \mathcal{P}(\Theta)$ having $\mathrm{Cr}\{A\} = 1$ so that

$$\lim_{t\to\infty} \frac{1}{\sigma^{\beta}(p_t, q_t)} \left|\left\{p_t < i \leq q_t : \left\|(A\varpi)_i(\phi) - \varpi(\phi)\right\| \geq \rho\right\}\right| = 0,$$

for every $\phi \in A$ and $\rho > 0$. In this case, we type $\varpi_t \to \varpi(d.s.A.c.a.s)$.

Definition 2.11.

The sequence $\{\varpi_t\}$ is known as deferred A-statistically convergent of order β in credibility to ϖ if and only if there is a $A \in \mathcal{P}(\Theta)$ supplying $\mathrm{Cr}\{A\} = 1$ such that

$$\lim_{t\to\infty} \frac{1}{\sigma^{\beta}(p_t, q_t)} \left|\left\{p_t < i \leq q_t : \mathrm{Cr}\left(\left\|(A\varpi)_i - \varpi\right\| \geq \rho\right) \geq \sigma\right\}\right| = 0,$$

for each $\rho, \sigma > 0$.

Definition 2.12.

The sequence $\{\varpi_t\}$ is known as deferred A-statistically convergent of order β in mean to ϖ if and only if there is a $A \in \mathcal{P}(\Theta)$ having $\mathrm{Cr}\{A\} = 1$ such that

$$\lim_{t\to\infty} \frac{1}{\sigma^{\beta}(p_t, q_t)} \left|\left\{p_t < i \leq q_t : E\left[\left\|(A\varpi)_i - \varpi\right\| \geq \rho\right]\right\}\right| = 0,$$

for every $\rho > 0$.

Definition 2.13.

Let $\Phi, \Phi_1, \Phi_2, ...$ be the credibility distributions of fuzzy variables $\varpi, \varpi_1, \varpi_2, ...$, respectively. Then, the fuzzy variable sequence $\{\varpi_t\}$ is known as deferred A-statistically convergent of order β

in distribution to ϖ if for all $\rho > 0$,

$$\lim_{t\to\infty} \frac{1}{\sigma^{\beta}(p_t, q_t)} \left|\left\{p_t < i \leq q_t : \left\|(A\Phi)_i(y) - \Phi(y)\right\| \geq \rho\right\}\right| = 0,$$

for each y. Here $\Phi(y)$ is continuous.

Definition 2.14.

The sequence $\{\varpi_t\}$ is known as deferred A-statistically convergent of order β uniformly almost surely to ϖ provided that for all $\rho > 0$, there is a $\sigma > 0$ and a sequence $\{A_i'\} \in \mathcal{P}(\Theta)$ supplying $\mathrm{Cr}\{A_i'\} = 1$ so that

$$\lim_{t\to\infty} \tfrac{1}{\sigma^{\beta}(p_t,q_t)} \left|\left\{p_t < i \leq q_t : \left|\mathrm{Cr}\{A_i'\}\right| \geq \rho\right\}\right| = 0,$$

$$\Rightarrow \lim_{t\to\infty} \tfrac{1}{\sigma^{\beta}(p_t,q_t)} \left|\left\{p_t < i \leq q_t : \left\|(A\varpi)_i(y) - \varpi(y)\right\| \geq \sigma\right\}\right| = 0,$$

for all $y \in A_i'$.

(**i**) If $\beta = 1$ and $A = CI$, then the notion of deferred A-statistically convergence of FV sequence turns into deferred statistical convergence of FV sequence.

(**ii**) If $p_t = 0$, $q_t = t$ and $\beta = 1$, $A = CI$, then notion of deferred A-statistically convergence of FV sequence turns into natural statistical convergence of FV sequence.

(**iii**) If $\beta = 1$, $p_t = 0$, $q_t = t$ and $A = (a_{ti})$, determined by

$$a_{ti} = \begin{cases} \frac{C}{t+1}, & (0 \leq i \leq t), \\ 0, & (i > t), \end{cases}$$

then the notion of deferred A-statistically convergence of FV sequence turns into Cesàro statistical convergence of FV sequence.

Now, we examine the relationships among the convergence notions.

Theorem 2.10.

If a bounded fuzzy variable sequence $\{\varpi_t\}$ deferred A-statistically converges to ϖ, then it deferred A-converges to ϖ and so, statistically deferred A-converges to ϖ. In general, however, the opposite is not true.

Proof:

Presume that the bounded sequence $\{\varpi_t\}$ is deferred A-statistically convergent. Take $A \in \mathcal{P}(\Theta)$ with $\mathrm{Cr}\{A\} = 1$. Contemplate the set

$$K_\rho = \{p_t < i \leq q_t : \|\varpi_i(\phi) - \varpi(\phi)\| \geq \rho\},$$

for every $\phi \in A$, each $\rho > 0$. Then, we acquire

$$\begin{aligned}\left\|(A\varpi)_t(\phi) - \varpi(\phi)\right\| &= \left|\sum_{i=p_t+1}^{q_t} a_{ti}\varpi_i(\phi) - \varpi(\phi)\right| \\ &= \left|\sum_{i=p_t+1}^{q_t} a_{ti}\left[\varpi_i(\phi) - \varpi(\phi)\right] + \sum_{i=p_t+1}^{q_t} a_{ti}\varpi(\phi) - \varpi(\phi)\right| \\ &\leq \left|\sum_{i\in K_\rho} a_{ti}\left[\varpi_i(\phi) - \varpi(\phi)\right] + \sum_{i\notin K_\rho} a_{ti}\left[\varpi_i(\phi) - \varpi(\phi)\right]\right| \\ &\quad + \left|\varpi(\phi)\right|\left|\sum_{i=p_t+1}^{q_t} a_{ti} - 1\right| \\ &\leq \sup_i\left[\varpi_i(\phi) - \varpi(\phi)\right]\left|\sum_{i\in K_\rho} a_{ti}\right| + \rho\left|\sum_{i\notin K_\rho} a_{ti}\right| \\ &\quad + \left|\varpi(\phi)\right|\left|\sum_{i=p_t+1}^{q_t} a_{ti} - 1\right|.\end{aligned}$$

Now, getting $t \to \infty$, i.e., $q_t = \infty$ and utilizing the notion of deferred A-statistically convergence, and regularity situations of a_{ti}, we deduce that $\{\varpi_t\}$ is deferred A-statistically converges to ϖ and so, statistically deferred A-converges to ϖ. For the contrary, we can examine the subsequent example: for $\beta = 1$, $p_t = 0$, $q_t = t$, contemplate the infinite matrix $A = C(1,1) = (d_{ti})$, as

$$d_{ti} = \begin{cases} \frac{1}{t+1}, & (0 \leq i \leq t), \\ 0, & (i > t), \end{cases}$$

and the sequence $\{\varpi_t\}$ as

$$\varpi_t = \begin{cases} 1, & (t \text{ is odd}), \\ 0, & (t \text{ is even}). \end{cases}$$

According to the definition, we can obtain

$$(A\varpi)_t = \begin{cases} \frac{1}{2}, & (t \text{ is odd}), \\ \frac{t}{2(t+1)}, & (t \text{ is even}). \end{cases}$$

As a result, the sequence $\{\varpi_t\}$ deferred A-converges to $\frac{1}{2}$, but not deferred A-statistically convergent. ■

Theorem 2.11.

If the FV sequence $\{\varpi_t\}$ deferred A-statistically converges of order β in mean to ϖ, then it deferred A-statistically converges of order β in credibility to ϖ. In general, however, the opposite is not true.

Proof:

Presume the sequence $\{\varpi_t\}$ deferred A-statistically converges of order β in mean to ϖ. Then, for each $\rho > 0$, we have

$$\lim_{t\to\infty} \frac{1}{\sigma^\beta(p_t, q_t)} \left|\left\{p_t < i \leq q_t : E\left[\left\|(A\varpi)_i - \varpi\right\| \geq \rho\right]\right\}\right| = 0.$$

Using the Markov inequality, we can acquire

$$\lim_{t\to\infty}\frac{1}{\sigma^{\beta}(p_t,q_t)}\left|\left\{p_t<i\leq q_t:\mathrm{Cr}\left(\left\|(A\varpi)_i-\varpi\right\|\geq\rho\right)\geq\sigma\right\}\right|$$
$$\leq\lim_{t\to\infty}\frac{1}{\sigma^{\beta}(p_t,q_t)}\left|\left\{p_t<i\leq q_t:\left(\frac{E\left(\left\|(A\varpi)_i-\varpi\right\|\right)}{\rho}\right)\right\}\geq\sigma\right|,$$

for all given $\rho>0$ and $\sigma>0$. According to the inequality, we infer that if $\{\varpi_t\}$ deferred A-statistically convergent of order β in mean, then it deferred A-statistically convergent of order β in credibility to ϖ. For the contrary, we can examine the subsequent example.

Take $(\Theta,\mathcal{P}(\Theta),\mathrm{Cr})$ to be $\{\phi_1,\phi_2,...\}$ such that $M_1(\phi)=\sup_{\phi_t\in A}=\frac{1}{q_t-p_t+1}$ and $M_2(\phi)=\sup_{\phi_t\in A^c}=\frac{1}{q_t-p_t+1}$ with

$$\mathrm{Cr}\{A\}=\begin{cases}M_1(\phi), & \text{if } M_1(\phi)<0.5,\\ 1-M_2(\phi), & \text{if } M_2(\phi)<0.5,\\ 0.5, & \text{otherwise.}\end{cases}$$

From the above credibility space connected with the fuzzy variables, determined by

$$(A\varpi)_t(\phi)=\begin{cases}q_t-p_t+1, & \text{if } \phi=\phi_t,\\ 0, & \text{otherwise,}\end{cases}$$

$(t=1,2,3,...)$, and $\varpi\equiv 0$. For given $\rho,\sigma>0$ and $i\geq 2$, one can obtain

$$\begin{aligned}&\lim_{t\to\infty}\frac{1}{\sigma^{\beta}(p_t,q_t)}\left|\left\{p_t<i\leq q_t:\mathrm{Cr}\left(\left\|(A\varpi)_i-\varpi\right\|\geq\rho\right)\geq\sigma\right\}\right|\\&=\lim_{t\to\infty}\frac{1}{\sigma^{\beta}(p_t,q_t)}\left|\left\{p_t<i\leq q_t:\mathrm{Cr}\left(\phi:\left\|(A\varpi)_i(\phi)-\varpi(\phi)\right\|\geq\rho\right)\geq\sigma\right\}\right|\\&=\lim_{t\to\infty}\frac{1}{\sigma^{\beta}(p_t,q_t)}\left|\left\{p_t<i\leq q_t:\mathrm{Cr}\left(\phi:\left\|(A\varpi)_i(\phi)\right\|\geq\rho\right)\geq\sigma\right\}\right|\\&=0.\end{aligned}$$

As a result, the sequence $\{\varpi_t\}$ deferred A-statistically converges in credibility to 0. It is acquired that for all $i\geq 2$, the fuzzy distribution of $\left\|(A\varpi)_t-\varpi\right\|=\left\|(A\varpi_t)\right\|$ is

$$\Phi_t(y)=\begin{cases}0, & \text{if } y<0,\\ 1-\frac{1}{(q_t-p_t+1)}, & \text{if } 0\leq y<q_t-p_t+1,\\ 1, & \text{if } y\geq q_t-p_t+1.\end{cases}$$

Now, for each $t\geq 2$, we get

$$\begin{aligned}&\lim_{t\to\infty}\frac{1}{\sigma^{\beta}(p_t,q_t)}\left|\left\{p_t<i\leq q_t:E\left[\left\|(A\varpi)_i-\varpi\right\|-1\right]\right\}\right|\\&=\lim_{t\to\infty}\frac{1}{\sigma^{\beta}(p_t,q_t)}\left[\int_0^{q_t-p_t+1}\left[1-\left(1-\frac{1}{(q_t-p_t+1)}\right)\right]dy-1\right]\\&=0.\end{aligned}$$

This gives that the sequence $\{\varpi_t\}$ does not deferred A-statistically converge of order β in mean to 0. ■

Remark 2.1.

If $\{\varpi_t\}$ deferred A-statistically converge a.s., then it does not necessarily deferred A-statistically converge in credibility.

Proof:

Think the CS $(\Theta, \mathcal{P}(\Theta), \mathrm{Cr})$ to be $\{\phi_1, \phi_2, ...\}$ such that $M_1(\phi) = \sup_{\phi_t \in A} = \frac{q_t - p_t}{2q_t - 2p_t + 1}$ and $M_2(\phi) = \sup_{\phi_t \in A^c} = \frac{q_t - p_t}{2q_t - 2p_t + 1}$ with

$$\mathrm{Cr}\{A\} = \begin{cases} M_1(\phi), & \text{if } M_1(\phi) < 0.5, \\ 1 - M_2(\phi), & \text{if } M_2(\phi) < 0.5, \\ 0.5, & \text{otherwise.} \end{cases}$$

Determine a fuzzy variable as

$$(A\varpi)_t(\phi) = \begin{cases} q_t - p_t, & \text{if } \phi = \phi_t, \\ 1, & \text{if not,} \end{cases}$$

for $t \geq 1$ and $\varpi \equiv 0$. At that time, the sequence deferred A-statistically converges a.s. to ϖ. On the other hand, it does not deferred A-statistically converge in credibility to ϖ. It stems from the fact that

$$\begin{aligned} &\lim_{t\to\infty} \frac{1}{\sigma^\beta(p_t, q_t)} \left|\left\{p_t < i \leq q_t : \mathrm{Cr}\left(\|(A\varpi)_i - \varpi\| \geq \rho\right) \geq \tfrac{1}{2}\right\}\right| \\ &= \lim_{t\to\infty} \frac{1}{\sigma^\beta(p_t, q_t)} \left|\left\{p_t < i \leq q_t : \mathrm{Cr}\left(\phi : \|(A\varpi)(\phi)_i - \varpi(\phi)\| \geq \rho\right) \geq \tfrac{1}{2}\right\}\right| \\ &= \lim_{t\to\infty} \frac{1}{\sigma^\beta(p_t, q_t)} \left|\left\{p_t < i \leq q_t : \mathrm{Cr}(\phi_i) \geq \tfrac{1}{2}\right\}\right| \\ &= \tfrac{1}{2}. \end{aligned}$$

■

Remark 2.2.

If $\{\varpi_t\}$ deferred A-statistically converge a.s. to ϖ, then it does not deferred A-statistically converge in mean to ϖ.

Proof:

Think the credibility space $(\Theta, \mathcal{P}(\Theta), \mathrm{Cr})$ to be $\{\phi_1, \phi_2, ...\}$ such that with $\mathrm{Cr}\{A\} = \sum_{\phi_t \in \sigma} \frac{1}{2^t}$. Establish the fuzzy variable by

$$(A\varpi)_t(\phi) = \begin{cases} 2^t, & \text{if } \phi = \phi_t, \\ 0, & \text{otherwise,} \end{cases}$$

for $i \geq 1$ and $\varpi \equiv 0$. Here, the sequence deferred A-statistically converge a.s. to ϖ. Emphases that the fuzzy distribution of $\|(A\varpi)_i\|$ are

$$\Phi_t(y) = \begin{cases} 0, & \text{if } y < 0, \\ 1 - \frac{1}{2^t}, & \text{if } 0 \leq y < 2^t, \\ 1, & \text{if } y \geq 2^t. \end{cases}$$

for $t \geq 1$. Then, we get

$$\lim_{t\to\infty} \frac{1}{\sigma^\beta(p_t, q_t)} \left|\left\{p_t < i \leq q_t : E\left[\|(A\varpi)_i - \varpi\| \geq 1\right]\right\}\right| = 0.$$

So, this gives that the sequence $\{\varpi_t\}$ does not deferred A-statistically converge in mean to ϖ. ■

Theorem 2.12.

$\{\varpi_t\}$ deferred A-statistically converge a.s. to ϖ if and only if for all $\rho, \sigma > 0$, we get

$$\lim_{t\to\infty} \frac{1}{\sigma^\beta (p_t, q_t)} \left| \left\{ p_t < i \le q_t : \mathrm{Cr}\left(\bigcap_{i=1}^{\infty} \bigcup_{t=i}^{\infty} \|(A\varpi)_i - \varpi\| \ge \rho \right) \ge \sigma \right\} \right| = 0.$$

Proof:

According to the notion of deferred A-statistically convergence a.s., there is a $A \in \mathcal{P}(\Theta)$ having $\mathrm{Cr}\{A\} = 1$ so that

$$\lim_{t\to\infty} \frac{1}{\sigma^\beta (p_t, q_t)} \left| \{ p_t < i \le q_t : \|(A\varpi)_i - \varpi\| \ge \rho \} \right| = 0,$$

for any $\rho > 0$. Hence, for all $\rho > 0$, there is an i so that $\|(A\varpi)_t - \varpi\| < \rho$ where $p_t < i < q_t$ and for $\phi \in A$, that is the same as

$$\lim_{t\to\infty} \frac{1}{\sigma^\beta (p_t, q_t)} \left| \left\{ p_t < i \le q_t : \mathrm{Cr}\left(\bigcap_{i=1}^{\infty} \bigcup_{t=i}^{\infty} \|(A\varpi)_i - \varpi\| < \rho \right) \ge 1 \right\} \right| = 0.$$

Then,

$$\lim_{t\to\infty} \frac{1}{\sigma^\beta (p_t, q_t)} \left| \left\{ p_t < i \le q_t : \mathrm{Cr}\left(\bigcap_{i=1}^{\infty} \bigcup_{t=i}^{\infty} \|(A\varpi)_i - \varpi\| \ge \rho \right) \ge \sigma \right\} \right| = 0,$$

is obtained from the duality axiom of crebility measure. ∎

Theorem 2.13.

Take $\varpi, \varpi_1, \varpi_2, ...$ as FVs. Then, $\{\varpi_t\}$ deferred A-statistically converge a.s. to the FV ϖ if and only if for any $\rho, \delta > 0$, we get

$$\lim_{t\to\infty} \frac{1}{\sigma^\beta (p_t, q_t)} \left| \left\{ p_t < i \le q_t : \mathrm{Cr}\left(\bigcup_{t=i}^{\infty} \|(A\varpi)_i - \varpi\| \ge \rho \right) \ge \delta \right\} \right| = 0.$$

Proof:

If $\{\varpi_t\}$ deferred A-statistically converges a.s. to ϖ, then for any $\delta > 0$ there exists H such that $\mathrm{Cr}\{H\} < \delta$ and $\{\varpi_t\}$ deferred A-statistically converges uniformly a.s. to ϖ on $\mathcal{P}(\Theta) - H$. So, for any $\rho > 0$, there is a $i \le t$ such that $\|(A\varpi)_t - \varpi\| < \rho$ for $\phi \in \mathcal{P}(\Theta) - H$. That is

$$\bigcup_{t=i}^{\infty} \{ \|(A\varpi)_t - \varpi\| < \rho \} \subset H.$$

According to the subadditivity axiom that

$$\lim_{t\to\infty} \frac{1}{\sigma^\beta (p_t, q_t)} \left| \left\{ p_t < i \le q_t : \mathrm{Cr}\left(\bigcup_{t=i}^{\infty} \{ \|(A\varpi)_i - \varpi\| \ge \rho \} \right) \right\} \right| \le \delta \left(\mathrm{Cr}\{H\} \right) < \delta.$$

This gives the proof of the first part.

On the contrary, if

$$\lim_{t\to\infty}\frac{1}{\sigma^{\beta}\left(p_t,q_t\right)}\left|\left\{p_t<i\leq q_t:\mathrm{Cr}\left(\bigcup_{t=i}^{\infty}\{\|(A\varpi)_i-\varpi\|\geq\rho\}\right)\geq\delta\right\}\right|=0,$$

for any ρ, then for given $p\geq 1$ and $\delta>0$, there is p_i so that

$$\delta\left(\mathrm{Cr}\left(\bigcup_{t=p_i}^{\infty}\left\{\|(A\varpi)_i-\varpi\|\geq\frac{1}{p}\right\}\right)\right)<\frac{\delta}{2^p}.$$

Let

$$H=\bigcup_{p=1}^{\infty}\bigcup_{t=i}^{\infty}\left\{\|(A\varpi)_i-\varpi\|\geq\frac{1}{p}\right\}.$$

Then

$$\delta\left(\mathrm{Cr}\left\{H\right\}\right)\leq\sum_{p=1}^{\infty}\delta\left(\mathrm{Cr}\left(\bigcup_{t=p_i}^{\infty}\left\{\|(A\varpi)_i-\varpi\|\geq\frac{1}{p}\right\}\right)\right)\leq\sum_{p=1}^{\infty}\frac{\delta}{2^p}.$$

Additionally, we acquire

$$\sup_{\phi\in\mathcal{P}(\Theta)-H}\|(A\varpi)_t-\varpi\|<\frac{1}{p},$$

for all $p=1,2,...$ and all $t>p_i$. The theorem's proof is complete. ∎

Theorem 2.14.

If $\{\varpi_t\}$ deferred statistically A-converge uniformly a.s. to ϖ, then $\{\varpi_t\}$ deferred statistically A-converge a.s. to ϖ.

Proof:

From Theorem 2.13, if the FV sequence $\{\varpi_i\}$ statistically deferred A-converge uniformly a.s. to the FV ϖ, then we have

$$\lim_{t\to\infty}\frac{1}{\sigma^{\beta}\left(p_t,q_t\right)}\left|\left\{p_t<i\leq q_t:\mathrm{Cr}\left(\bigcup_{t=i}^{\infty}\|(A\varpi)_i-\varpi\|\geq\rho\right)\geq\delta\right\}\right|=0.$$

Since

$$\delta\left(\mathrm{Cr}\left(\bigcap_{i=1}^{\infty}\bigcup_{t=i}^{\infty}\{\|(A\varpi)_i-\varpi\|\geq\rho\}\right)\right)\leq\delta\left(\mathrm{Cr}\left(\bigcup_{t=i}^{\infty}\{\|(A\varpi)_i-\varpi\|\geq\rho\}\right)\right),$$

we obtain that

$$\delta\left(\mathrm{Cr}\left(\bigcap_{t=1}^{\infty}\bigcup_{i=t}^{\infty}\{\|(A\varpi)_i-\varpi\|\geq\rho\}\right)\right)=0.$$

As a result, $\{\varpi_t\}$ deferred statistically A-converges a.s. in credibility to ϖ. ∎

3. Conclusion

In this paper, considering deferred Cesàro mean and a regular matrix A, we investigated different types of convergence. Moreover, we obtained some interesting results. This study's findings are more generic and a natural extension of the traditional convergence of fuzzy variable sequences.

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