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M-Regression Estimation with the k Nearest Neighbor's Smoothing Under Quasi-associated Data in Functional Statistics

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Abstract

The main goal of this paper is to study the non parametric M-estimation under quasi-associated sequence with the k Nearest Neighbor's method shortly (kNN). We construct an estimator of this nonparametric function and we study its asymptotic properties. Furthermore, a comparison study based on simulated data is also provided to illustrate the highly sensitive of the kNN approach to the presence of even a small proportion of outliers in the data.

Keywords: kNN method; Functional data; Quasi-associated data; M-Regression

MSC 2020 No.: 62G08, 62G10, 62G35

1. Introduction

It is very well recognized that robust regression in statistics is an attractive research method. It is used to overcome some of the weaknesses of classical regression, namely when outliers contain heteroscedastic data.

In the statistical literature, several papers have been devoted to the study of the nonparametric M-estimator properties. The first results concerning this topic for the asymptotic normality in both dependent and independent data are Györfi et al. (1989), Collomb and Hardle (1986), Huber (1964), Härdle and Tsybakov (1998), Robinson (1983), Boente and Fraiman (1990), Boente and Fraiman (1989) for prior results and Boente et al. (2009), Laïb and Ould Saïd (2000), Attouch et al. (2010), Attouch et al. (2012). The nonparametric robust regression estimation was firstly introduced by Azzedine et al. (2008). They obtained the almost complete convergence with rates in the independent and identically distributed (i.i.d.) case, Crambes et al. (2008) was examined the similar problem for a functional covariate, then Cai and Roussas (1992) studied its asymptotic properties under the α-mixing assumption in the L^p norm. In the case of functional and stationary ergodic data, Gheriballah et al. (2013) found almost complete convergence with rate, and for recent research, we can refer to Derrar et al. (2020) and the references therein.

For quasi-associated random variate, the M-estimation literature for nonparametric functional data analysis is not yet well documented, note that, Bulinski and Suquet (2001) introduced this type of dependency structure as a generalization of positively associated variables given by Esary et al. (1967) and negatively linked random variables considered by Jong-Dev and Proschan (1983) for real-valued random fields. Both types of association have great importance in various applied fields (see the book by Barlow and Proschan (1981) for a deeper discussion on this topic). Nonparametric estimation involving (positively and negatively) associated random variables has been extensively studied. We quote, for instance, Matula (1992), Roussas (2000), Masry (2002), Roussas (1991), Mebsout et al. (2020) and the reference therein. We refer the reader to Dedecker et al. (2007) or Doukhan et al. (2010) for some other weak dependence structures and their applications.

The study's main objective is to construct an estimator of the regression function by relinking the M-estimation approach with the quasi-association setting using the kNN method. This is motivated by the fact that the robust regression estimator has several advantages over the classical kernel regression estimator. The main profit in using a robust regression is that it allows reducing the effect of outlier data.

This famous kNN method of estimation have attracted a lot of interest in the statistical literature for evaluating multivariate data because of their flexibility and efficiency. Pushed by its attractive features, the functional kNN smoothing approach has received a growing consideration in the last years. Gyorfi et al. (2002)'s book is a thorough analysis of kNN estimators in the finite dimensional context. Work in this area was started by Cover (1968), and a large number of articles are now available in various estimating contexts, which including regression, discrimination, density and mode estimation, and clustering analysis, we make reference to Collomb (1981), Devroye and Wagner (1982), Li (1985), Moore and Yackel (1977), Devroye and Wagner (1977), Devroye et al. (1994), Beirlant et al. (2008), Laloë (2008), Burba et al. (2009), Tran et al. (2006), Lian (2011), Attouch and Bouabsa (2013), Attouch et al. (2018), Kudraszow and Vieu (2013) and Kara et al. (2017), Almanjahie et al. (2020), Bouabsa (2021) for the most recent advances and references. Note that, such a study has a great impact on practice. On the one hand, the robust regression is an essential alternative regression model that allows overcoming many drawbacks of the classical regression, such as the sensitivity to the outliers or the heteroscedasticity phenomena. Furthermore, it is well known that the kNN method is better than the classical kernel method. However, the difficulty in the kNN smoothing is the fact that the bandwidth parameter is a random variable, unlike the classical regression in which the smoothing parameter is a deterministic scalar. So, the study of the asymptotic properties of our proposed estimator is complicated, and it requires some additional tools and techniques.

In NFDA, kNN M-estimation with quasi-associated data is new. This researches's primary goal is to provide generalizations, to the k Nearest Neighbor case, the results obtained by Attaoui et al. (2015) in the quasi-associated dependency case. More precisely, we establish the almost complete convergence uniformly on the number of neighbors with rates of an estimator constructed by combining the ideas of robustness with those of smoothed regression. We point out that the main feature of our approach is to develop an alternative prediction model to the classical regression that is not sensitive to outliers or heteroscedastic data, taking into account the local data structure.

The paper is organized as follows. Section 2 is devoted to the presentation of our estimate and then the fixed notations and hypotheses are given in Section 3. We state the result and their proofs in Section 4, where, uniform almost complete convergence with rates is given in Subsection 4.1 and Subsection 4.2 is consecrated to the study of the asymptotic normality. Section 5 is devoted to simulation study to prove the efficiency of our study.

2. The functional M-estimator model

Lest's $(W_i)_{i=1,\dots,n} := (A_i, B_i)_{i=1,\dots,n}$ a series in the separable Hilbert space of stationary quasi-associated and identically distributed random variables $\varepsilon := \mathcal{G} \times \mathbb{R}$, where $\mathcal{G}$ is a separable real Hilbert space with the $\|.\|$ norm created by the internal product <.,.>. This nonparametric model, denoted by ϑ_a, is implicitly defined as a zero with respect to ϕ in the equation

$$\Lambda(a, \phi) := \mathbf{E}\left[\rho\left(B_1, \phi\right) \mid A_1 = a\right] = 0, \tag{1}$$

in which ρ is a real-valued Borel function that satisfies some requirements of regularity to be described above. We assume that ϑ_a exist and is unique for all $a \in S$ where S is a fixed compact subset of $\mathbb{R}^d$ (see, for example, Boente and Fraiman (1989)). The ϑ_a natural estimator indicated by $\widehat{\vartheta}_a$ is a zero with respect to ϕ of the equation

$$\hat{\rho}(a, \phi) = 0, \tag{2}$$

then, we have that

$$\hat{\Lambda}(a, \phi) := \frac{\sum_{i=1}^{n} L\left(h_L^{-1}\left(a - A_i\right)\right)\rho\left(B_i, \phi\right)}{\sum_{i=1}^{n} L\left(h_L^{-1}\left(a - A_i\right)\right)},$$

with L is a kernel function and $h_{L,n} = h_n$ is a series of positive real numbers which goes to zero as n goes to infinity.

2.1. The kNN M-asymptotic estimator's properties

In fact, this study focused on the asymptotic properties of the kNN M-estimator, for which the h_L scalar bandwidth parameter is replaced with a random one defined by

$$E_k = \min\left\{ h_L \in \mathbb{R}^+ \text{such that } \sum_{i=1}^{n} 1\!\!\mathrm{I}_{B(a,h_L)}\left(A_i\right) = k \right\}.$$

Principally, M-estimator of kNN is represented by

$$\tilde{\Lambda}(a,\phi) := \frac{\sum_{i=1}^{n} L\left(E_k^{-1}\left(a - A_i\right)\right)\rho\left(B_i,\phi\right)}{\sum_{i=1}^{n} L\left(E_k^{-1}\left(a - A_i\right)\right)}.$$

Certainly, the use of E_k instead of h_L increases the accuracy of the previous case because the smoothing parameter E_k strongly depends on the data unlike the first case where h_L is arbitrarily chosen independently of the sample. In addition, the kNN method facilitates the selection of the smoothing parameter because this choice would be reduced to a problem of choosing an integer k between 1 and the sample size n. However, establishing the asymptotic properties of the proposed estimator becomes more complicated than in the classical case because the smoothing parameter E_k is a random variable. Thus, the treatment of this situation requires additional mathematical tools and specific analytical arguments.

In reality, the $\tilde{\Lambda}(a,\phi)$ estimator is more suitable than $\hat{\Lambda}(a,\phi)$, since its bandwidth parameter is sophistically selected, whereas $\hat{\Lambda}(a,\phi)$ when the bandwidth parameter is arbitrarily selected independently of the results. The setting of the asymptotic properties of $\tilde{\Lambda}(a,\phi)$, is indeed more complex than the $\hat{\Lambda}(a,\phi)$ estimator since its bandwidth parameter is a random variable.

3. Principal hypotheses and notations

Until specifying our key result, in the number of neighbors $k \in (k_n^1, k_n^2)$, the almost complete consistency of $\hat{\Lambda}(a,\phi)$ is defined uniformly and we suppose that the first two conditional inverse moments of B given A, finite and strictly positive ones.

In all the paper we suppose that the sequence $(A_i, B_i)_{i=1,\ldots,n}$ is stationary quasi associated process, S is a fixed compact subset of $\mathbb{R}^d$ and Υ (respectively $\Upsilon_{i,j}$) the density of A (respectively the joint density of (A_i, A_j)). Furthermore, we set by C or C' some positive generic constants and by

$$\Delta_\ell := \sup_{s>\ell} \sum_{|i-j|\geq s} \Delta_{i,j},$$

where

$$\Delta_{i,j} = \sum_{\ell=1}^{d}\sum_{l=1}^{d}\left|\text{Cov}\left(A_i^\ell, A_j^l\right)\right| + \sum_{\ell=1}^{d}\left|\text{Cov}\left(A_i^\ell, B_j\right)\right| + \sum_{l=1}^{d}\left|\text{Cov}\left(B_i, A_j^l\right)\right| + \left|\text{Cov}\left(B_i, B_j\right)\right|.$$

In the following, we will denote by $L_1 = L\left(d\left(a, A_1\right)/h_L\right)$.

Now, we will state the following assumptions that are necessary to show our main result.

(H1) For all $p \in (0,1)$ and for all $t > 0, \mathbb{P}(A \in \mathcal{B}(a,t)) = \varphi_a(t) > 0$, we have that

$$\lim_{t\to 0}\frac{\varphi_a(pt)}{\varphi_a(t)} = \chi_a(p) < \infty.$$

(H2) The kernel L is supported within $(0, 1/2)$ and has a continuous first derivative on $(0, 1/2)$ which is such that

$$0 < C\mathbb{I}_{(0,1/2)}(\cdot) \le L(\cdot) \le C'\mathbb{I}_{(0,1/2)}(\cdot) \text{ and } L(1/2) - \int_0^{1/2} L'(p)\chi_a(p)dp > 0,$$

where $\mathbb{I}_A$ is the indicator function of the set A.

(H3) The class of functions $\kappa = \left\{\mapsto L\left(\psi^{-1}d(a,\cdot)\right), \psi > 0\right\}$ is a class which can be evaluated pointwise. in such a way that

$$\sup_G \int_0^1 \sqrt{1 + \log \mathcal{N}\left(\varepsilon||\Theta||_{G,2}, L, d_G\right)}d\varepsilon < \infty.$$

where the supremum is taken over all probability measures G on the space $\mathcal{G}$ with $G\left(\Theta^2\right) < \infty$ and where Θ is the envelope function of the set κ. Here, d_G is the $L_2(G)$-metric and $\mathcal{N}\left(\varepsilon, \kappa, d_G\right)$ is the minimal number of open balls (with respect to the $L_2(G)$-metric) with radius ε which are needed to cover the function class κ. We will denote by $\|.\|_{G,2}$ the $L_2(G)$-norm.

(H4) k_n^1 and k_n^2 sequences verified

$$\varphi_a^{-1}\left(\frac{k_n^2}{n}\right) \to 0, \quad \text{and} \quad \frac{\log n}{\min\left(n\varphi_a^{-1}\left(\frac{k_n^1}{n}\right), k_n^1\right)} \to 0.$$

(H5) The density Υ is of class $\mathscr{C}^1\left(\mathbb{R}^d\right)$, such that $\inf_{a\in S}\Upsilon(a) > C > 0$ and the joint density $\Upsilon_{i,j}$ satisfies $\sup_{|i-j|\ge 1}\left\|\Upsilon_{(A_i,A_j)}\right\|_\infty < \infty$, where $\|\cdot\|_\infty$ is the supremum norm.

(H6) There exists $\beta_0 > 0$ such that

$$\sup_{a\in S}|\vartheta_a| \le \beta_0.$$

(H7) The process $\left\{\left(A_i, B_i\right), i \in \mathbb{N}\right\}$ is quasi-associated with covariance coefficient $\Delta_\ell, \ell \in \mathbb{N}$ satisfying

$$\exists a > 0 \text{ such that } \Delta_\ell \le Ce^{-a\ell}.$$

(H8) $\begin{cases} \text{(H8.a) The function } \Lambda(\ldots) \text{ is of class } \mathscr{C}^1 \text{ on } S \times [-\beta_0, +\beta_0]\,, \text{ such that} \\ \inf\limits_{a\in S, \phi\in[-\beta_0,\beta_0]} \dfrac{\partial \Lambda}{\partial \phi}(a,\phi) > C > 0. \\ (H8.b) \text{ For each fixed } \phi \in [-\beta_0, +\beta_0]\,, \text{ the function } \Lambda(.,\phi) \text{ is continous over S.} \end{cases}$

(H9) The function ρ is strictly monotonic with respect to the second component, Lipschitz and such that, $\forall \phi \in [-\beta_0, +\beta_0]$, $\mathbb{E}(\exp(|\rho(B,\phi)|)) \leq C$, and $\forall i \neq j, \mathbb{E}\left(\left|\rho\left(B_i,\phi\right)\rho\left(B_j,\phi\right)\right| \mid A_i, A_j\right) \leq C'$.

(H10) The inverse moments of the response variable verify

$$\text{for all } m \geq 2, \mathbb{E}\left[B^{-m} \mid A = a\right] < C < \infty.$$

Remark 3.1.

Our work is the link between the work of Kara et al. (2017), Attaoui et al. (2015) and Bouabsa (2021). So these several assumptions are the same considered in all this research.

4. Results

4.1. Consistency

Now we study in this Section the almost complete consistency of $\hat{\vartheta}_a$, for a fixed $a \in S$.

Theorem 4.1.

Under Hypotheses (H1)-(H4) and (H5) the estimator $\hat{\vartheta}_a$ exists and is unique. Moreover, we have, as $n \longrightarrow \infty$ with $\varpi = \min(\varpi_1, \varpi_2)$, that

$$\sup_{a\in S}\sup_{k_n^1\leq k\leq k_n^2}\left|\hat{\vartheta}_a - \vartheta_a\right| = O\left(\varphi_a^{-1}\left(\frac{k_n^2}{n}\right)^{\varpi}\right) + O\left(\left(\frac{\log n}{n^{-\gamma}\left(k_n^1\right)^d}\right)^{1/2}\right), \quad \text{a.co.}$$

Proof:

Under some modification of Kara et al. (2017), there exits $\xi \in]0,1[$, then we have that

$$\sum_n \sum_{k=k_n^1}^{k_n^2} \mathbb{P}\left(E_k \leq \varphi_a^{-1}\left(\frac{\xi k_n^1}{n}\right)\right) < \infty, \text{ and } \sum_n \sum_{k=k_n^1}^{k_n^2} \mathbb{P}\left(E_k \geq \varphi_a^{-1}\left(\frac{k_n^2}{n\xi}\right)\right) < \infty.$$

Thus, we write, for all $\varepsilon > 0$,

$$\mathbb{P}\left\{\left[\left(\varphi_a^{-1}\left(\frac{k_n^2}{n}\right)\right)^{\varpi} + \sqrt{\frac{\log n}{n^{-\gamma}\left(k_n^1\right)^d}}\right]\sup_{a\in S}\sup_{k_n^1\leq k\leq k_n^2}\left|\tilde{\vartheta}_a - \vartheta_a\right| \geq \varepsilon\right\}$$

$$\leq \mathbb{P}\left\{\left[\left(\varphi_a^{-1}\left(\frac{k_n^2}{n}\right)\right)^{\varpi} + \sqrt{\frac{\log n}{n^{-\gamma}\left(k_n^1\right)^d}}\sup_{a\in S}\sup_{k_n^1\leq k\leq k_n^2}\left|\tilde{\vartheta}_a - \vartheta_a\right|\right.\right.$$

$$\times \mathbb{1}_{\left\{\left(\varphi_a^{-1}\left(\frac{\xi k_n^1}{n}\right) \leq h_L \leq \varphi_a^{-1}\left(\frac{k_n^2}{\xi a}\right)\right\}} \geq \frac{\epsilon}{2}\Bigg\}$$
$$+\mathbb{P}\left\{h_L \notin \left(\varphi_a^{-1}\left(\frac{\xi k_n^1}{n}\right), \quad \varphi_a^{-1}\left(\frac{k_n^2}{n\xi}\right)\right)\right\}.$$

Thus, all that remains to prove is the following asymptotic results. ∎

Proposition 4.1.

Similar to the conditions of Theorem 4.1, we have

$$\sup_{a\in S}\sup_{x_n\leq h_L\leq y_n}\left|\hat{\vartheta}_a-\vartheta_a\right| = O\left(y_n^{\varpi}\right)+O_{a.c.o.}\left(\sqrt{\frac{\log n}{n^{1-\gamma}\varphi_a\left(x_n\right)^d}}\right),$$

where, $x_n = \varphi_a^{-1}\left(\frac{\xi k_n^1}{n}\right)$ and $y_n = \varphi_a^{-1}\left(\frac{k_n^2}{n\xi}\right)$.

Proof:

The proof relies on the fact that the second variable of ρ is purely monotonic. After all, we will just give proof for the increasing case for the sake of simplicity. Under this assumption, we write

$$\sup_{x_n\leq h_L\leq y_n}\sup_{a\in S}\left|\widehat{\vartheta}_a-\vartheta_a\right| = \sup_{x_n\leq h_L\leq y_n}\sup_{a\in S}\left|\widehat{\vartheta}_a-\vartheta_a\right|\mathbb{1}_{\left\{\sup_{a\in S}\left|\widehat{\vartheta}_a\right|\leq\beta_0\right\}}$$
$$+\sup_{x_n\leq h_L\leq y_n}\sup_{a\in S}\left|\widehat{\vartheta}_a-\vartheta_a\right|\mathbb{1}_{\left\{\sup_{a\in S}\left|\widehat{\vartheta}_a\right|>\beta_0\right\}}.$$

So, to demonstrate the result, it is necessary to show that

$$\sum_n \mathbb{P}\left(\inf_{x_n\leq h_L\leq y_n}\inf_{a\in S}\widehat{\vartheta}_a < -\beta_0\right)<\infty, \quad \sum_n \mathbb{P}\left(\sup_{x_n\leq h_L\leq y_n}\sup_{a\in S}\widehat{\vartheta}_a > \beta_0\right)<\infty, \tag{3}$$

and

$$\sup_{x_n\leq h_L\leq y_n}\sup_{a\in S}\left|\widehat{\vartheta}_a-\vartheta_a\right|\mathbb{1}_{\left\{\sup_{a\in S}\left|\widehat{\vartheta}_a\right|\leq\beta_0\right\}} = O_{a.co.}\left(\left(y_n^{\varpi}\right)+\left(\frac{\log n}{n^{1-\gamma}\varphi_a\left(x_n\right)^d}\right)^{\frac{1}{2}}\right). \tag{4}$$

Since $\hat{\Lambda}(a,\cdot)$ is increasing for each $a\in S$, we must demonstrate that

$$\sum_n \mathbb{P}\left(\sup_{x_n\leq h_L\leq y_n}\sup_{a\in S}\hat{\Lambda}\left(a,-\beta_0\right)>0\right)<\infty, \quad \text{and} \sum_n \mathbb{P}\left(\inf_{x_n\leq h_L\leq y_n}\inf_{a\in S}\hat{\Lambda}\left(a,\beta_0\right)<0\right)<\infty.$$

Assumption (H6) implies that

$$\sup_{x_n\leq h_L\leq y_n}\sup_{a\in S}\Lambda\left(a,-\beta_0\right)<0, \quad \text{and} \quad \inf_{x_n\leq h_L\leq y_n}\inf_{a\in S}\Lambda\left(a,\beta_0\right)>0.$$

Assuming that we can verify

$$\sup_{x_n \le h_L \le y_n} \sup_{a \in S} \hat{\Lambda}(a, -\beta_0) \longrightarrow \sup_{x_n \le h_L \le y_n} \sup_{a \in S} \Lambda(a, -\beta_0),$$

we obtain

$$\sum_n \mathbb{P}\left(\sup_{x_n \le h_L \le y_n} \sup_{a \in S} \hat{\Lambda}(a, -\beta_0) > 0 \right) \le \sum_n \mathbb{P}\left(\left| \sup_{x_n \le h_L \le y_n} \sup_{a \in S} \hat{\Lambda}(a, -\beta_0) - \sup_{x_n \le h_L \le y_n} \sup_{a \in S} \Lambda(a, -\beta_0) \right| \ge \epsilon_1 \right) < \infty,$$

and

$$\sum_n \mathbb{P}\left(\inf_{x_n \le h_L \le y_n} \inf_{a \in S} \hat{\Lambda}(a, \beta_0) < 0 \right) \le \sum_n \mathbb{P}\left(\left| \inf_{x_n \le h_L \le y_n} \inf_{a \in S} \hat{\Lambda}(a, \beta_0) - \inf_{x_n \le h_L \le y_n} \inf_{a \in S} \Lambda(a, \beta_0) \right| \ge \epsilon_2 \right) < \infty,$$

with, $\epsilon_1 = -\sup_{x_n \le h_L \le y_n} \sup_{a \in S} \Lambda(a, -\beta_0)$ and $\epsilon_2 = \inf_{x_n \le h_L \le y_n} \inf_{a \in S} \Lambda(a, \beta_0)$.

Furthermore, we write under (H8.a) that

$$\left(\widehat{\vartheta_a} - \vartheta_a \right) \mathbb{1}_{\left\{ |\widehat{\vartheta_a} - \vartheta_a| \le \beta \right\}} = \frac{\Lambda\left(a, \widehat{\vartheta_a}\right) - \hat{\Lambda}\left(a, \widehat{\vartheta_a}\right)}{\frac{\partial \Lambda}{\partial t}(a, \alpha_n)} \mathbb{1}_{\left\{ |\widehat{\vartheta_a} - \vartheta_a| \le \beta \right\}},$$

where α_n is between $\widehat{\vartheta}_a$ and ϑ_a. As a result, the only thing left to reveal is the convergence rate of

$$\sup_{x_n \le h_L \le y_n} \sup_{a \in S} \sup_{\phi \in [-\beta_0, \beta_0]} |\hat{\Lambda}(a, \phi) - \Lambda(a, \phi)|. \tag{5}$$

The verification of (5) is focused on the decomposition shown below.

$$\begin{aligned} &\sup_{x_n \le h_L \le y_n} \sup_{a \in S} \sup_{\phi \in [-\beta_0, \beta_0]} |\hat{\Lambda}(a, \phi) - \Lambda(a, \phi)| \le \\ &\quad \frac{1}{\inf_{x_n \le h_L \le y_n} \inf_{a \in S} \left| \hat{\Lambda}_D(a) \right|} \left\{ \sup_{x_n \le h_L \le y_n} \sup_{a \in S} \sup_{\phi \in [-\beta_0, \beta_0]} \left| \hat{\Lambda}_N(a, \phi) - \mathbb{E}\left[\hat{\Lambda}_N(a, \phi) \right] \right| \right. \\ &\quad + \sup_{x_n \le h_L \le y_n} \sup_{a \in S} \sup_{\phi \in [-\beta_0, \beta_0]} \left| \mathbb{E}\left[\hat{\Lambda}_N(a, \phi) \right] - H(a, \phi) \right| \\ &\quad + \sup_{x_n \le h_L \le y_n} \sup_{a \in S} \sup_{\phi \in [-\beta_0, \beta_0]} \left| \Lambda(a, \phi) \left(\Upsilon(a) - \mathbb{E}\left[\hat{\Lambda}_D(a) \right] \right) \right| \\ &\quad \left. + \sup_{x_n \le h_L \le y_n} \sup_{a \in S} \sup_{\phi \in [-\beta_0, \beta_0]} \left| \Lambda(a, \phi) \left(\mathbb{E}\left[\hat{\Lambda}_D(a) \right] - \hat{\Lambda}_D(a) \right) \right| \right\}, \end{aligned} \tag{6}$$

where

$$\hat{\Lambda}_N(a,\phi) := \frac{1}{n\mathbb{E}\left[L_1\right]^d} \sum_{i=1}^{n} L\left(h_L^{-1}\left(a-A_i\right)\right) \rho_a\left(B_i,\phi\right),$$

$$\hat{\Lambda}_D(a) := \frac{1}{n\mathbb{E}\left[L_1\right]^d} \sum_{i=1}^{n} L\left(h_L^{-1}\left(a-A_i\right)\right),$$

and

$$H(a,\phi) := \Lambda(a,\phi)\Upsilon(a).$$

After this, the accompanying Lemmas and Corollary are used to prove Proposition 4.1. ∎

Lemma 4.1.

Under Hypotheses (H2), (H4)-(H5) and (H8), as $n \longrightarrow \infty$, we have that

$$\sup_{x_n \le h_L \le y_n} \sup_{a \in S} \sup_{\phi \in [-\beta_0, \beta_0]} \left| \mathbb{E}\left[\hat{\Lambda}_N(a,\phi)\right] - H(a,\phi) \right| = O(y_n^{\varpi}).$$

Proof:

$$\begin{aligned} \mathbb{E}\left[\hat{\Lambda}_N(a,\phi)\right] &= \frac{1}{\mathbb{E}\left[L_1\right]^d} \int_{\mathbb{R}^d} \mathbb{E}\left[\rho(B,\phi) \mid (A=u)\right] L\left(\frac{a-u}{h_L}\right) \Upsilon(u)du \\ &= \frac{1}{\mathbb{E}\left[L_1\right]^d} \int_{\mathbb{R}^d} \Lambda(u,\phi) L\left(\frac{a-u}{h_L}\right) \Upsilon(u)du \\ &= \int_{\mathbb{R}^d} H(a-h_L z,\phi)L(z)dz. \end{aligned}$$

We use the fact that $\mathbb{E}\left[L_1\right] \le C\varphi_a(h_L)$, then since both Υ and Λ are of class $\mathscr{C}^1$, a Taylor expansion of $H(a-h_L z,\phi)$ with (H2), permit to write

$$\sup_{x_n \le h_L \le y_n} \sup_{a \in S} \sup_{\phi \in [-\beta_0, \beta_0]} \left| \mathbb{E}\left[\hat{\Lambda}_N(a,\phi)\right] - H(a,\phi) \right| = O(y_n^{\varpi}).$$ ∎

Lemma 4.2.

Under Hypotheses (H1) (H4), (H5)-(H7), we have

$$\sup_{x_n \le h_L \le y_n} \sup_{a \in S} \sup_{\phi \in [-\beta_0, \beta_0]} \left| \hat{\Lambda}_N(a,\phi) - \mathbb{E}\left[\hat{\Lambda}_N(a,\phi)\right] \right| = O\left(\sqrt{\frac{\log n}{n^{1-\gamma}\varphi_a\left(x_n\right)^d}}\right) \quad \text{a.co.}$$

Proof:

We use a truncation approach by introducing the following random variable, since ρ might not be bounded,

$$\hat{\Lambda}_N^*(a,\phi) = \frac{1}{n\mathbb{E}\left[L_1\right]^d} \sum_{i=1}^{n} L\left(h_L^{-1}\left(a-A_i\right)\right) \rho_a\left(B_i,\phi\right) \mathbb{I}_{|\rho_a(B_i,\phi)|<\theta_n} \text{ with } , \theta_n = n^{\gamma/6}.$$

The defined result is then a consequence of the intermediate results that follow,

$$\sup_{x_n \leq h_L \leq y_n} \sup_{a \in S} \sup_{\phi \in [-\beta_0, \beta_0]} \left| \mathbb{E}\left[\hat{\Lambda}_N^*(a,\phi)\right] - \mathbb{E}\left[\hat{\Lambda}_N(a,\phi)\right] \right| = O_{\text{a.co}}\left(\sqrt{\frac{\log n}{n^{1-\gamma}\varphi_a\left(x_n\right)^d}}\right), \tag{7}$$

$$\sup_{x_n \leq h_L \leq y_n} \sup_{a \in S} \sup_{\phi \in [-\beta_0, \beta_0]} \left| \hat{\Lambda}_N^*(a,\phi) - \hat{\Lambda}_N(a,\phi) \right| = O_{\text{a.co}}\left(\sqrt{\frac{\log n}{n^{1-\gamma}\varphi_a\left(x_n\right)^d}}\right), \tag{8}$$

and

$$\sup_{x_n \leq h_L \leq y_n} \sup_{a \in S} \sup_{\phi \in [-\beta_0, \beta_0]} \left| \hat{\Lambda}_N^*(a,\phi) - \mathbb{E}\left[\hat{\Lambda}_N^*(a,\phi)\right] \right| = O_{\text{a.co.}}\left(\sqrt{\frac{\log n}{n^{1-\gamma}\varphi_a\left(x_n\right)^d}}\right). \tag{9}$$

We start by proving (9). Since S is compact we write

$$S \subset \bigcup_{j=1}^{\wp_n} B\left(a_\ell, \tau_n\right),$$

with $\wp_n = O\left(n^\beta\right)$ and $\tau_n = O\left(\wp_n^{-1}\right)$, where $t = \dfrac{\delta(d+2)}{2} + \dfrac{1}{2} + \dfrac{\gamma}{6}$ and $\delta \leq (1-\gamma-\xi_2)/d$.

Let us now consider all $a \in S$,

$$\ell(a) = \arg\min_{\ell \in \{1, \ldots \wp_n\}} \|a - a_\ell\|,$$

and we take into consideration the following decomposition,

$$\begin{aligned}
&\sup_{x_n \leq h_L \leq y_n} \sup_{a \in S} \sup_{\phi \in [-\beta_0, \beta_0]} \left| \hat{\Lambda}_N^*(a,\phi) - \mathbb{E}\left[\hat{\Lambda}_N^*(a,\phi)\right] \right| \\
&\quad \leq \underbrace{\sup_{x_n \leq h_L \leq y_n} \sup_{a \in S} \sup_{\phi \in [-\beta_0, \beta_0]} \left| \hat{\Lambda}_N^*(a,\phi) - \hat{\Lambda}_N^*\left(a_{\ell(a)},\phi\right) \right|}_{Z_1} \\
&\quad + \underbrace{\sup_{x_n \leq h_L \leq y_n} \sup_{a \in S} \sup_{\phi \in [-\beta_0, \beta_0]} \left| \hat{\Lambda}_N^*\left(a_{\ell(a)},\phi\right) - \mathbb{E}\left[\hat{\Lambda}_N^*\left(a_{\ell(a)},\phi\right)\right] \right|}_{Z_2} \\
&\quad + \underbrace{\sup_{x_n \leq h_L \leq y_n} \sup_{a \in S} \sup_{\phi \in [-\beta_0, \beta_0]} \left| \mathbb{E}\left[\hat{\Lambda}_N^*\left(a_{\ell(a)},\phi\right)\right] - \mathbb{E}\left[\hat{\Lambda}_N^*(a,\phi)\right] \right|}_{Z_3}.
\end{aligned} \tag{10}$$

- First, we utilize the compactness of $[-\beta_0, \beta_0]$, for Z_2 and we're writing

$$[-\beta_0, \beta_0] \subset \bigcup_{j=1}^{z_n} (\phi_j - g_n, \phi_j + g_n), \tag{11}$$

with $g_n = n^{-1/2}$ and $z_n = O\left(n^{1/2}\right)$. Set

$$\mathscr{N}_n = \{\phi_j - g_n, \phi_j + g_n, 1 \leq j \leq z_n\}. \tag{12}$$

For $1 \leq j \leq z_n$, by the $\mathbb{E}\left[\hat{\Lambda}_N^*(a, \cdot)\right]$ and $\hat{\Lambda}_N^*(a, \cdot)$ monotony, we get

$$\mathbb{E}\left[\hat{\Lambda}_N^*\left(a_{\ell(a)}, \phi_j - g_n\right)\right] \leq \sup_{\phi \in (\phi_j - g_n, \phi_j + g_n)} \mathbb{E}\left[\hat{\Lambda}_N^*\left(a_{\ell(a)}, \phi\right)\right] \leq \mathbb{E}\left[\hat{\Lambda}_N^*\left(a_{\ell(a)}, \phi_j + g_n\right)\right], \tag{13}$$

$$\hat{\Lambda}_N^*\left(a_{\ell(a)}, \phi_j - g_n\right) \leq \sup_{\phi \in (\phi_j - g_n, \phi_j + g_n)} \hat{\Lambda}_N^*\left(a_{\ell(a)}, \phi\right) \leq \hat{\Lambda}_N^*\left(a_{\ell(a)}, \phi_j + g_n\right). \tag{14}$$

Furthermore, by assumption (H9), for any $\phi_1, \phi_2 \in [-\beta_0, \beta_0]$, we have that

$$\left|\mathbb{E}\left[\hat{\Lambda}_N^*\left(a_{\ell(a)}, \phi_1\right)\right] - \mathbb{E}\left[\hat{\Lambda}_N^*\left(a_{\ell(a)}, \phi_2\right)\right]\right| \leq C\left|\phi_1 - \phi_2\right|. \tag{15}$$

As a result, we can deduce from (11) and (15) that

$$\begin{aligned} &\sup_{a \in S} \sup_{\phi \in [-\beta_0, \beta_0]} \left|\hat{\Lambda}_N^*\left(a_{\ell(a)}, \phi\right) - \mathbb{E}\left[\hat{\Lambda}_N^*\left(a_{\ell(a)}, \phi\right)\right]\right| \\ &\leq \max_{1 \leq \ell \leq \rho_n} \max_{1 \leq j \leq z_n} \max_{\phi \in \{\phi_j - g_n, \phi_j + g_n\}} \left|\hat{\Lambda}_N^*\left(x_\ell, \phi\right) - \mathbb{E}\left[\hat{\Lambda}_N^*\left(a_\ell, \phi\right)\right]\right| + 2Cg_n. \end{aligned} \tag{16}$$

We will find out using a simple algebraic equation

$$g_n = o\left(\sqrt{\frac{\log n}{n\varphi_a(x_n)^d}}\right). \tag{17}$$

Then, for some positive real η sufficiently large, it is sufficient to demonstrate that

$$\begin{aligned} &\sup_{x_n \leq h_L \leq y_n} \max_{1 \leq \ell \leq \wp_n} \max_{1 \leq j \leq z_n} \max_{\phi \in \{\phi_j - g_n, \phi_j + g_n\}} \left|\hat{\Lambda}_N^*\left(a_\ell, \phi\right) - \mathbb{E}\left[\hat{\Lambda}_N^*\left(a_\ell, \phi\right)\right]\right| \\ &= O_{\text{a.co}}\left(\sqrt{\frac{\log n}{n^{1-\gamma}\varphi_a\left(x_n\right)^d}}\right). \end{aligned} \tag{18}$$

To do just that, we employ a Bernstein-type inequality for dependent random variables, after all, we've written

$$\hat{\Lambda}_N^*\left(a_\ell, \phi\right) - \mathbb{E}\left[\hat{\Lambda}_N^*\left(a_\ell, \phi\right)\right] = \sum_{i=1}^n \Omega_i,$$

where

$$\Omega_i = \frac{1}{n\mathbb{E}\left[L_1\right]} \Psi\left(A_i, B_i\right),$$

with

$$\Psi(u,v) = \rho(v,\phi)L\left(h_L^{-1}\left(a_\ell - u\right)\right)\mathbb{I}_{(|\rho(v,\phi)|<\theta_n)} \\ - \mathbb{E}\left[\rho\left(B_1,\phi\right)L\left(h_L^{-1}\left(a_\ell - A_1\right)\right)\mathbb{I}_{(|\rho(B_1,\phi)|<\theta_n)}\right], u\in\mathbb{R}^d, v\in\mathbb{R}.$$

So,

$$\|\Psi\|_\infty \leq C\theta_n\|L\|_\infty \text{ and} \\ \mathrm{Lip}(\Psi) \leq \left(\|L\|_\infty \mathrm{Lip}(\rho) + \theta_n h_L^{-1}\mathrm{Lip}(L)\right) \leq C\theta_n h_L^{-1}\mathrm{Lip}(L).$$

The inequality of Newmann and Kallabis is focused on the asymptotic analysis of $\mathrm{Var}\left(\sum_{i=1}^n \Omega_i\right)$ and $\mathrm{Cov}\left(\Omega_{m_1}\ldots\Omega_{m_u},\Omega_{\phi_1}\ldots\Omega_{\phi_v}\right)$, for all $(m_1,\ldots,m_u)\in\mathbb{N}^u$ and $(\phi_1,\ldots,\phi_v)\in\mathbb{N}^v$. We start by studying the variance term,

$$\mathrm{Var}\left(\sum_{i=1}^n \Omega_i\right) = n\,\mathrm{Var}\left(\Omega_1\right) + \sum_{i=1}^n\sum_{\substack{j=1\\j\neq i}}^n \mathrm{Cov}\left(\Omega_i,\Omega_j\right). \tag{19}$$

Under (H9), we have

$$\begin{aligned}\mathrm{Var}\left(\Omega_1\right) &\leq \frac{1}{n^2\mathbb{E}\left[L_1\right]^{2d}}\mathbb{E}\left[\left|\rho\left(B_1,\phi\right)L_1\left(a_\ell\right)\right|^2\right] \\ &\leq C'\frac{1}{n^2\mathbb{E}\left[L_1\right]^{2d}}\mathbb{E}\left[\left|L_1\left(a_\ell\right)\right|^2\right] \\ &\leq C'n^{-2}\mathbb{E}\left[L_1\right]^{-d}\int_{\mathbb{R}^d} L^2(u)\Upsilon\left(a_\ell - h_L u\right)du.\end{aligned} \tag{20}$$

For all $\sup\limits_{x_n\leq h_L\leq y_n}$, we get that

$$\mathrm{Var}\left(\Omega_1\right) = O\left(n^{-2}\varphi_a\left(x_n\right)^{-d}\right). \tag{21}$$

Now, let us evaluate the asymptotic behavior of the sum in the right-hand side of (19). For this we use the technique developed by Masry (2002). Indeed, we need the following decomposition

$$\sum_{i=1}^n\sum_{\substack{j=1\\j\neq i}}^n \mathrm{Cov}\left(\Omega_i,\Omega_j\right) = \sum_{i=1}^n\sum_{\substack{j=1\\0<|i-j|\leq\varrho_n}}^n \mathrm{Cov}\left(\Omega_i,\Omega_j\right) + \sum_{i=1}^n\sum_{\substack{j=1\\|i-j|>\varrho_n}}^n \mathrm{Cov}\left(\Omega_i,\Omega_j\right),$$

with (ϱ_n) is a sequence of positive integer as n goes to infinity tending to infinity.

For $|i-j| \leq \varrho_n$, we utilize (H2)-(H5), (H9), to write that

$$\begin{aligned}
\mathbb{E}\left[\left|\Omega_i \Omega_j\right|\right] \leq & C \frac{1}{n^2 \mathbb{E}\left[L_1\right]^{2d}}\left(\mathbb{E}\left[\left|\rho\left(B_i, \phi\right) L_i\left(a_\ell\right) \rho\left(B_j, z\right) L_j\left(a_\ell\right)\right|\right]\right. \\
& \left.+\left(\mathbb{E}\left[\left|\rho\left(B_1, \phi\right) L_1\left(a_\ell\right)\right|\right]\right)^2\right), \\
\leq & C \frac{1}{n^2 \mathbb{E}\left[L_1\right]^{2d}}\left(\mathbb{E}\left[L_i\left(a_\ell\right) L_j\left(a_\ell\right)\right]+\left(\mathbb{E}\left[L_1\left(a_\ell\right)\right]\right)^2\right), \\
\leq & \frac{1}{n^2}\left(\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} L(u) L(v) \Upsilon_{\left(A_i, A_j\right)}\left(a_\ell-h_L u, a_\ell-h_L v\right) d u d v\right) \\
& \left.+\left(\int_{\mathbb{R}^d} L(u) \Upsilon\left(a_\ell-h_L u\right) d u\right)^2\right). \\
= & O\left(n^{-2}\right).
\end{aligned}$$

Then, we obtain

$$\begin{aligned}
\sum_{i=1}^n \sum_{\substack{j=1 \\ 0<1-j| \leq \varrho_n}}^n \operatorname{Cov}\left(\Omega_i, \Omega_j\right) & \leq n \varrho_n\left(\mathbb{E}\left[\Omega_i \Omega_j\right]\right), \\
& \leq C n^{-1} \varrho_n.
\end{aligned}$$

On the other hand, for $|i-j|>\varrho_n$, we utilize (H7) and the quasi-association of the sequence $\left(A_i, B_i\right)$ to write

$$\begin{aligned}
\sum_{i=1}^n \sum_{j=1}^n \operatorname{Cov}\left(\Omega_i, \Omega_j\right) & \leq \theta_n^2 n^{-2} \mathbb{E}\left[L_1\right]^{-2(d+1)} \sum_{i=1}^n \sum_{\substack{j=1 \\
|i-j|>\varrho_n}}^n \Delta_{i, j} \\
& \leq \theta_n^2 n^{-1} \mathbb{E}\left[L_1\right]^{-2(d+1)} \Delta_{\varrho_n} \\
& \leq \theta_n^2 n^{-1} \mathbb{E}\left[L_1\right]^{-2(d+1)} e^{-a \varrho_n}.
\end{aligned}$$

So

$$\sum_{i=1}^n \sum_{\substack{j=1 \\ i \neq j}}^n \operatorname{Cov}\left(\Omega_i, \Omega_j\right) \leq C\left(n^{-1} \varrho_n+\theta_n^2 n^{-1} \mathbb{E}\left[L_1\right]^{-2(d+1)} e^{-a \varrho_n}\right).$$

Take $\varrho_n=\frac{1}{a} \log \left(a \theta_n^2 \mathbb{E}\left[L_1\right]^{-2(d+1)}\right)$. Then, by (H4), we obtain

$$n \mathbb{E}\left[L_1\right]^d \sum_{i=1}^n \sum_{j=1}^n \operatorname{Cov}\left(\Omega_i, \Omega_j\right) \rightarrow 0. \tag{22}$$

Finally, by mixing (21) and (22), we get for all $\sup\limits_{x_n \leq h_L \leq y_n}$ that

$$\operatorname{Var}\left(\sum_{i=1}^n \Omega_i\right)=O\left(\frac{1}{n \varphi_a\left(x_n\right)^d}\right).$$

Now, for all $\left(m_1, \ldots, m_u\right) \in \mathbb{N}^u$ and $\left(\phi_1, \ldots, \phi_v\right) \in \mathbb{N}^v$, we treat with the covariance term in (19). To do that, we consider the following cases,

– If $\phi_1 = m_u$, we get

$$\begin{aligned}|\mathrm{Cov}\left(\Omega_{m_1}\dots\Omega_{m_u},\Omega_{\phi_1}\dots\Omega_{\phi_v}\right)| &\leq \left(\frac{C\theta_n\|L\|_\infty}{n\mathbb{E}\left[L_1\right]^d}\right)^{u+v}\mathbb{E}\left|L_1^2\left(a_\ell\right)\right| \\ &\leq \mathbb{E}\left[L_1\right]^d\left(\frac{C\theta_n}{n\mathbb{E}\left[L_1\right]^d}\right)^{u+v}.\end{aligned}$$

– If $\phi_1 > m_u$, we utilize the quasi-association condition, and we obtain

$$\begin{aligned}&|\mathrm{Cov}\left(\Omega_{m_1}\dots\Omega_{m_u},\Omega_{\phi_1}\dots\Omega_{\phi_v}\right)| \\ &\leq \left(\frac{\theta_n\mathbb{E}\left[L_1\right]^{-1}\mathrm{Lip}(L))}{n\mathbb{E}\left[L_1\right]^d}\right)^2\left(\frac{2\theta_n\|L\|_\infty}{n\mathbb{E}\left[L_1\right]^d}\right)^{u+v-2}\sum_{i=1}^{u}\sum_{j=1}^{v}\Delta_{m_i,\phi_j} \\ &\leq \mathbb{E}\left[L_1\right]^{-2}\left(\frac{C\theta_n}{n\mathbb{E}\left[L_1\right]^d}\right)^{u+v}v\Delta_{\phi_1-m_u} \\ &\leq \mathbb{E}\left[L_1\right]^{-2}\left(\frac{C\theta_n}{n\mathbb{E}\left[L_1\right]^d}\right)^{u+v}ve^{-a(\phi_1-m_u)}.\end{aligned} \tag{23}$$

On the other side, we get

$$\begin{aligned}|\mathrm{Cov}\left(\Omega_{m_1}\dots\Omega_{m_u},\Omega_{\phi_1}\dots\Omega_{\phi_v}\right)| &\leq \left(\frac{2C\theta_n\|L\|_\infty}{n\mathbb{E}\left[L_1\right]^d}\right)^{u+v-2}\left(|\mathbb{E}\Omega_{m_u}\Omega_{\phi_1}|\right) \\ &\leq \left(\frac{C\theta_n}{n\mathbb{E}\left[L_1\right]^d}\right)^{u+v}\mathbb{E}\left[L_1\right]^{2d}.\end{aligned} \tag{24}$$

Then, we take the $\frac{d}{2d+2}$ -power of (23) and the $\frac{d+2}{2d+2}$ power of (24)

$$|\mathrm{Cov}\left(\Omega_{m_1},\dots\Omega_{m_u},\Omega_{\phi_1},\dots\Omega_{\phi_v}\right)| \leq \mathbb{E}\left[L_1\right]^d\left(\frac{C\theta_n}{n\mathbb{E}\left[L_1\right]^d}\right)^{u+v}ve^{-\frac{ad}{2d+2}(\phi_1-m_u)}.$$

We apply Kallabis and Neumann (2006) (Theorem 2.1) for the variables $\Omega_i, i = 1,\dots,n$, with $\sup\limits_{x_n\leq h_L\leq y_n}$,

$$L_n = \frac{C\theta_n}{n\sqrt{h^d}},\quad M_n = \frac{C\theta_n}{n\mathbb{E}\left[L_1\right]^d},\quad \text{and } A_n = \mathrm{Var}\left(\sum_{i=1}^{n}\Omega_i\right) = O\left(\frac{1}{n\varphi_a\left(x_n\right)^d}\right).$$

Now, we apply Bernstein's inequality for empirical processes,

$$\mathbb{P}\left\{\sup_{a_n\leq h_L\leq b_0}\sqrt{\frac{n^{1-\gamma}\varphi_a\left(h\right)}{\log n}}\left|\hat{\Lambda}_N^*(a_\ell,t) - E\left[\hat{\Lambda}_N^*(a_\ell,t)\right]\right| \geq \eta_0\right\} \leq \log(n)n^{-C'\eta_0^2}.$$

After this, by using the fact that $\wp_n z_n \leq n^{\zeta}$, where $\zeta = \gamma + \frac{1}{2}$, and with a suitable range of η_0 helps us to conclude that for every $h_L \in (a_n, b_0)$, that

$$\mathbb{P}\left(\max_{1\leq k\leq \wp_n} \max_{1\leq j\leq z_n} \max_{\phi\in\{\phi_j - g_n, \phi_j + g_n\}} \left|\hat{\Lambda}_N^*(a,z) - \mathbb{E}\left[\hat{\Lambda}_N^*(a,z)\right]\right| = O\sqrt{\frac{\log n}{n^{1-\gamma}\varphi\left(x_n\right)^d}}\right).$$

So the proof of (9) is finished.

- Second, with respect to Z_1 and Z_3, the Lipschitz condition on the kernel L in (H2) allows to write directly, for all $a \in S$, and $\forall \phi \in [-\beta_0, \beta_0]$ that

$$\begin{aligned}\left|\hat{\Lambda}_N^*(a,t) - \hat{\Lambda}_N^*\left(a_{\ell(a)}, t\right)\right| &= \frac{\theta_n}{n\mathbb{E}\left[L_1\right]^d}\left|\sum_{i=1}^n L_i(a) - \sum_{i=1}^n L_i\left(a_{\ell(a)}\right)\right| \\ &\leq \frac{\theta_n}{\mathbb{E}\left[L_1\right]^{d+1}}\left\|a - a_{\ell(a)}\right\| \\ &\leq \frac{C\theta_n\tau_n}{\mathbb{E}\left[L_1\right]^{d+1}}.\end{aligned}$$

Then, we have since $\tau_n = O\left(n^{-\gamma}\right)$, that

$$\frac{\tau_n}{\mathbb{E}\left[L_1\right]^{d+1}} = O_{a.co}\left(\sqrt{\frac{\log n}{n\varphi_a\left(x_n\right)^d}}\right). \tag{25}$$

Hence,

$$\sup_{x_n\leq h_L\leq y_n} \sup_{a\in S} \sup_{\phi\in[-\beta_0,\beta_0]} \left|\hat{\Lambda}_N^*(a,\phi) - \hat{\Lambda}_N^*\left(a_{\ell(a)},\phi\right)\right| = O_{a.co}\left(\sqrt{\frac{\log n}{n\varphi_a\left(x_n\right)^d}}\right), \tag{26}$$

and

$$\sup_{x_n\leq h_L\leq y_n} \sup_{a\in S} \sup_{\phi\in[-\beta_0,\beta_0]} \left|\mathbb{E}\left[\hat{\Lambda}_N^*\left(a_{\ell(a)},\phi\right)\right] - \mathbb{E}\left[\hat{\Lambda}_N^*(a,\phi)\right]\right| = O\left(\sqrt{\frac{\log n}{n\varphi_a\left(x_n\right)^d}}\right). \tag{27}$$

We now demonstrate (7). For all $a \in S$ and all $\phi \in [-\beta_0, \beta_0]$, with $\sup_{x_n\leq h_L\leq y_n}$ we get

$$\begin{aligned}\left|\mathbb{E}\left[\hat{\Lambda}_N(a,\phi)\right] - \mathbb{E}\left[\hat{\Lambda}_N^*(a,\phi)\right]\right| &= \frac{1}{n\mathbb{E}\left[L_1\right]^d}\left|\mathbb{E}\left[\sum_{i=1}^n \rho\left(B_i,\phi\right)\mathbb{I}_{\{|\rho(B_i,\phi)|>\theta_n\}} L_i\right]\right| \\ &\leq \mathbb{E}\left[L_1\right]^{-d}\mathbb{E}\left[\left|\rho\left(B_1,\phi\right)\right|\mathbb{I}_{\{|\rho(B_i,\phi)|>\theta_n\}} L_1\right] \\ &\leq \mathbb{E}\left[L_1\right]^{-d}\mathbb{E}\left[\exp\left(\left|\rho\left(B_1,\phi\right)\right|/4\right)\mathbb{I}_{\{|\rho(B_i,\phi)|>\theta_n\}} L_1\right].\end{aligned}$$

We utilize the Holder's inequality, and we obtain

$$\begin{aligned}&\sup_{x_n\leq h_L\leq y_n} \sup_{a\in S} \sup_{\phi\in[-\beta_0,\beta_0]} \left|\mathbb{E}\left[\hat{\Lambda}_N(a,\phi)\right] - \mathbb{E}\left[\hat{\Lambda}_N^*(a,\phi)\right]\right| \\ &\leq \mathbb{E}\left[L_1\right]^{-d}\left(\mathbb{E}\left[\exp\left(\left|\rho\left(B_1,\phi\right)\right|/2\right)\mathbb{I}_{\{|\rho(B_i,\phi)|>\theta_n\}}\right]\right)^{\frac{1}{2}}\left(\mathbb{E}\left(L_1^2\right)\right)^{\frac{1}{2}} \\ &\leq \mathbb{E}\left[L_1\right]^{-d}\exp\left(-\theta_n/4\right)\left(\mathbb{E}\left[\exp\left(\left|\Lambda\left(B_1,\phi\right)\right|\right)\right]\right)^{\frac{1}{2}}\left(\mathbb{E}\left(L_1^2\right)\right)^{\frac{1}{2}} \\ &\leq C\varphi_a\left(x_n\right)^{\frac{-d}{2}}\exp\left(-\theta_n/4\right).\end{aligned} \tag{28}$$

Then, since $\theta_n = n^{\gamma/6}$, we can then write

$$\sup_{x_n \le h_L \le y_n} \sup_{a\in S} \sup_{\phi\in[-\beta_0,\beta_0]} \left| \mathbb{E}\left[\hat{\Lambda}_N(a,\phi)\right] - \mathbb{E}\left[\hat{\Lambda}^*_N(a,\phi)\right]\right| = O\left(\left(\frac{\log n}{n^{1-\gamma}\varphi_a(x_n)^d}\right)^{1/2}\right).$$

Markov's inequality is used to demonstrate the last stated result (8). In addition, for all ℓ, for all $\phi \in \mathcal{N}_n$ and for all $\epsilon > 0$, and for all $\sup_{x_n \le h_L \le y_n}$, we observe that

$$\begin{aligned} &\sum \mathbb{P}\left(\sup_{x_n \le h_L \le y_n} \sup_{n\ge 1} \sup_{a\in S} \left|\hat{\Lambda}_N(a,\phi) - \hat{\Lambda}^*_N(a,\phi)\right| > \epsilon_0 \left(\sqrt{\frac{\log n}{n^{1-\gamma}\varphi_a(x_n)^d}}\right)\right) \\ &\le C \sum_{n\ge 1} n \exp(-\theta_n). \end{aligned} \tag{29}$$

The proof of (8) is completed while using the definition of θ_n, which in turn completes the demonstration of this Lemma. ∎

Lemma 4.3.

Under hypotheses of Lemma 4.2 we have

$$\sup_{x_n \le h_L \le y_n} \sup_{a\in S} \left|\mathbb{E}\left[\hat{\Lambda}_D(a)\right] - \hat{\Lambda}_D(a)\right| = O\left(\sqrt{\frac{\log n}{n\varphi_a(x_n)^d}}\right).$$

Proof:

We then use the compactness of S with respect to the notations of Lemma 4.2, so we have

$$\begin{aligned} \sup_{x_n \le h_L \le y_n} \sup_{a\in S} \left|\hat{\Lambda}_D(a) - \mathbb{E}\left[\hat{\Lambda}_D(a)\right]\right| \le \underbrace{\sup_{x_n \le h_L \le y_n} \sup_{a\in S} \left|\hat{\Lambda}_D(a) - \hat{\Lambda}_D(a_\ell)\right|}_{Z'_1} \\ + \underbrace{\sup_{x_n \le h_L \le y_n} \sup_{a\in S} \left|\hat{\Lambda}_D(a_\ell) - \mathbb{E}\left[\hat{\Lambda}_D(a_\ell)\right]\right|}_{Z'_2} + \underbrace{\sup_{x_n \le h_L \le y_n} \sup_{a\in S} \left|\mathbb{E}\left[\hat{\Lambda}_D(a_\ell)\right] - \mathbb{E}\left[\hat{\Lambda}_D(a)\right]\right|}_{Z'_3}. \end{aligned}$$

- For Z'_1 and Z'_3,for all $a \in S$ and for all $\sup_{x_n \le h_L \le y_n}$, the Lipschitz condition on the kernel L permit to write, for all $a \in S$

$$\begin{aligned} \left|\hat{\Lambda}_D(a) - \hat{\Lambda}_D(a_\ell)\right| &= \frac{1}{n\mathbb{E}[L_1]^d} \mid \sum_{i=1}^n L_i(a) - \sum_{i=1}^n L_i(a_\ell) \\ &\le \frac{C}{\varphi_a(x_n)^{d+1}} \|a - a_\ell\| \\ &\le \frac{C\tau_n}{\varphi_a(x_n)^{d+1}}. \end{aligned}$$

By the fact that $\frac{\tau_n}{\varphi_a(x_n)^{d+1}} = O\left(\sqrt{\frac{\log n}{n\varphi_a(x_n)^d}}\right)$, the result follows directly,

$$\sup_{x_n \leq h_L \leq y_n} \sup_{a \in S} \left| \hat{\Lambda}_D(a) - \hat{\Lambda}_D(a_\ell) \right| = O\left(\sqrt{\frac{\log n}{n\varphi_a(x_n)^d}}\right), \quad \text{a.co.}, \tag{30}$$

and

$$\sup_{x_n \leq h_L \leq y_n} \sup_{a \in S} \left| \mathbb{E}\left[\hat{\Lambda}_D(a_\ell)\right] - \mathbb{E}\left[\hat{\Lambda}_D(a)\right] \right| = O\left(\sqrt{\frac{\log n}{n\varphi_a(x_n)^d}}\right), \quad \text{a.co.} \tag{31}$$

- For all real $\eta > 0$, we have for Z_2', that

$$\begin{aligned} &\mathbb{P}\left(Z_2' > \eta\sqrt{\frac{\log n}{n\varphi_a(x_n)^d}}\right) \\ &\leq \mathbb{P}\left(\sup_{x_n \leq h_L \leq y_n} \max_{\ell \in \{1,\dots\rho_n\}} \left| \hat{\Lambda}_D(a_\ell) - \mathbb{E}\left[\hat{\Lambda}_D(a_\ell)\right] \right| > \eta\sqrt{\frac{\log n}{n\varphi_a(x_n)^d}}\right) \\ &\leq \wp_n \sup_{x_n \leq h_L \leq y_n} \max_{\ell \in \{1,\dots\wp_n\}} \mathbb{P}\left(\left| \hat{\Lambda}_D(a_\ell) - \mathbb{E}\left[\hat{\Lambda}_D(a_\ell)\right] \right| > \eta\sqrt{\frac{\log n}{n\varphi_a(x_n)^d}}\right). \end{aligned} \tag{32}$$

Using the same procedure as in Lemma 4.2 so we get, by changing Λ with 1,

$$\sup_{x_n \leq h_L \leq y_n} \mathbb{P}\left(\sup_{a \in S} \left| \hat{\Lambda}_D(a) - \mathbb{E}\left[\hat{\Lambda}_D(a)\right] \right| > \eta\sqrt{\frac{\log n}{n\varphi_a(x_n)^d}}\right) \leq n^{\gamma - C\eta^2}.$$

With the right choice of η, we can obtain

$$\sup_{x_n \leq h_L \leq y_n} \sum_{n=1}^{\infty} \mathbb{P}\left(\sup_{a \in S} \left| \mathbb{E}\left[\hat{\Lambda}_D(a)\right] - \hat{\Lambda}_D(a) \right| > \eta\sqrt{\frac{\log n}{n\varphi_a(x_n)^d}}\right) < \infty. \qquad \blacksquare$$

Lemma 4.4.

Under the hypotheses of Lemma 4.1 we have

$$\sup_{x_n \leq h_L \leq y_n} \sup_{a \in S} \left| \mathbb{E}\left[\hat{\Lambda}_D(a) - \Upsilon(a)\right] \right| = O(y_n^{\varpi}).$$

Proof:

It is sufficient just to use the same reasoning as in Lemma 4.1,

$$\begin{aligned} \mathbb{E}\left[\hat{\Lambda}_D(a)\right] &= \frac{1}{\mathbb{E}\left[L_1\right]^d} \int_{\mathbb{R}^d} L\left(\frac{a-u}{h_L}\right) \Upsilon(u) du \\ &= \frac{1}{\mathbb{E}\left[L_1\right]} \mathbb{E}\left[L\left(\frac{d(a, A_1)}{h_L}\right) \mathbb{I}_{B(a,h_L)} \Upsilon(A)\right]. \end{aligned}$$

Then, using analytical reasoning, we arrive at the following conclusion,

$$\sup_{x_n \leq h_L \leq y_n} \sup_{a \in S} \left| \mathbb{E}\left[\hat{\Lambda}_D(a)\right] - \Upsilon(a) \right| = \frac{1}{\mathbb{E}\left[L_1\right]} \mathbb{E}\left[L\left(\frac{d\left(a, A_1\right)}{h_L}\right) \mathbb{I}_{B(a,h_L)} |\Upsilon(A) - \Upsilon(a)| \right].$$

So let us implement Lipschitz's (H10) condition to obtain

$$\mathbb{I}_{\mathcal{B}(a,h_L/2)} \left|\Upsilon(A) - \Upsilon(a)\right| \leq C(h_L/2)^{\varpi}.$$

We obtain what we require by changing

$$\sup_{x_n \leq h_L \leq y_n} \sup_{a \in S} \left| \mathbb{E}\left[\hat{\Lambda}_D(a)\right] - \Upsilon(a) \right| \leq C y_n^{\varpi}.$$

The verification of this Lemma is then finalized. ∎

Corollary 4.1.

Under the hypotheses of Lemma 4.3, we have

$$\exists C > 0 \sum_{n=1}^{\infty} \mathbb{P}\left(\inf_{a \in S} \inf_{x_n \leq h_L \leq a_n} \hat{\Lambda}_2(a) < C \right) < \infty.$$

Proof:

Under Condition (H10) we have

$$\mathbb{E}\left[\hat{\Lambda}_D(a)\right] \rightarrow \Lambda_D(a) > 0.$$

As a result, there exists a constant $C > 0$ for n large enough,

$$\mathbb{E}\left[\hat{\Lambda}_D(a)\right] \geq C \text{ for all } h_L \in \left(x_n, y_n\right).$$

Therefore,

$$\inf_{a \in S} \inf_{h_L \in (x_n, y_n)} \hat{\Lambda}_1(a) \leq \frac{C}{2} \Rightarrow \exists h_L \in (x_n, y_n) \text{ such that } \left| \mathbb{E}\left[\hat{\Lambda}_D(a)\right] - \hat{\Lambda}_D(a) \right| \geq \frac{C}{2},$$

which enables us to write

$$\sup_{a \in S} \sup_{h_L \in (a_n, b_n)} \left| \mathbb{E}\left[\hat{\Lambda}_D(a)\right] - \hat{\Lambda}_D(a) \right| \geq \frac{C}{2},$$

$$\mathbb{P}\left(\inf_{a \in S} \inf_{h_L \in (x_n, y_n)} \hat{\Lambda}_D(a) \leq \frac{C}{2} \right) \leq \mathbb{P}\left(\sup_{a \in S} \sup_{h_L \in (x_n, y_n)} \left| \mathbb{E}\left[\hat{\Lambda}_D(a)\right] - \hat{\Lambda}_D(a) \right| \geq \frac{C}{2} \right).$$

As a result of Lemma 4.3 and Lemma 4.4, we are able to obtain the desired result. ∎

Lemma 4.5.

Under the hypotheses of Theorem 4.1, $\hat{\vartheta}_a$ for n large enough, it occurs and is almost surely unique.

Proof:

The Λ strict monotony of involves:

$$\sup_{k_n^1 \leq k \leq k_n^2} \Lambda\left(a, \vartheta_a - \epsilon\right) \leq \sup_{k_n^1 \leq k \leq k_n^2} \Lambda\left(a, \vartheta_a\right) \leq \sup_{k_n^1 \leq k \leq k_n^2} \Lambda\left(a, \vartheta_a + \epsilon\right).$$

Lemmas 4.1 and 4.3 and Corollary 4.1 show that for all real fixed ϕ, we have

$$\sup_{k_n^1 \leq k \leq k_n^2} \hat{\Lambda}(a, \phi) - \sup_{k_n^1 \leq k \leq k_n^2} \Lambda(a, \phi) \rightarrow 0, \quad \text{a.co.}$$

So we obtain, for a sufficiently large n, that

$$\sup_{k_n^1 \leq k \leq k_n^2} \hat{\Lambda}\left(a, \vartheta_a - \epsilon\right) \leq 0 \leq \sup_{k_n^1 \leq k \leq k_n^2} \hat{\Lambda}\left(a, \vartheta_a + \epsilon\right), \quad \text{a.co.}$$

$\hat{\Lambda}(i, \phi)$ is continuous for all ϕ since ρ and L are continuous functions. There exists a $\phi_0 = \hat{\vartheta}_a$ in certain range $[\vartheta_a - \epsilon, \vartheta_a + \epsilon]$ for all $\sup_{k_n^1 \leq k \leq k_n^2} \hat{\Lambda}\left(a, \hat{\vartheta}_a\right) = 0$. Finally, the unicity of $\hat{\vartheta}_a$ is a direct result of Λ's strict monotony and L's positivity. ∎

4.2. Asymptotic Normality

Now we study the asymptotic normality of $\widehat{\vartheta}_a$ for a fixed $a \in S$.

Theorem 4.2.

Assume that (H4)-(H5) hold, then we have that

$$\left(\frac{n\varphi_a\left(x_n\right)^d}{\sigma^2\left(a, \vartheta_a\right)}\right)^{1/2}\left(\widehat{\vartheta}_a - \vartheta_a\right) \xrightarrow{\mathscr{D}} \mathscr{N}(0,1) \quad \text{as} \quad n \rightarrow \infty,$$

where,

$$\sigma^2\left(a, \vartheta_a\right) = \frac{\mathbb{E}\left[\Lambda_a^2\left(B\vartheta_a\right) \mid A = a\right]}{\left(\frac{\partial}{\partial\phi}\Lambda\left(a, \vartheta_a\right)\right)^2} \int_{\mathbb{R}'} L^2(z) dz,$$

$$\mathscr{A} = \left\{a \in S, \mathbb{E}\left[\Lambda_a^2\left(B, \vartheta_a\right) \mid A = a\right] \frac{\partial}{\partial\phi}\Lambda\left(a, \vartheta_a\right) \neq 0\right\},$$

and $\xrightarrow{\mathscr{D}}$ denotes the convergence in distribution.

Proof:

We present the justification for the case of an increasing ρ, identical to Theorem 4.1, with the decreasing situation achieved by including $-\rho$.

We define the case of all $u \in \mathbb{R}$ as follows, $\phi = \vartheta_a + u\left[\hat{n}\varphi_a\left(x_n\right)^d\right]^{-1/2} \sigma\left(a, \vartheta_a\right)$.

So we'll be able to write if $\hat{\Lambda}_D(a) \neq 0$ that

$$\begin{aligned}\mathbb{P}&\left\{\left(\frac{n\varphi_a\left(x_n\right)^d}{\sigma^2\left(a,\vartheta_a\right)}\right)^{1/2}\left(\widehat{\vartheta}_a-\vartheta_a\right)<u\right\}\\&=\mathbb{P}\left\{\widehat{\vartheta}_a<\vartheta_a+u\left[\hat{n}\varphi_a\left(x_n\right)^d\right]^{-1/2}\sigma\left(a,\vartheta_a\right)\right\}\\&=\mathbb{P}\left\{0<\hat{\Lambda}_N(a,\phi)\right\}\\&=\mathbb{P}\left\{\mathbb{E}\left[\hat{\Lambda}_N(a,\phi)\right]-\hat{\Lambda}_N(a,\phi)<\mathbb{E}\left[\hat{\Lambda}_N(a,\phi)\right]\right\}.\end{aligned}$$

As a result, Theorem 4.2 follows from the intermediate results. ∎

Lemma 4.6.

Under Hypotheses (H4), (H5), (H6)-(H9), we have

$$\mathbb{P}\left\{\left(\hat{\Lambda}_D(a)=0\right)\right\}\longrightarrow 0 \quad \text{as} \quad n\longrightarrow\infty.$$

Proof:

We have for all $\varepsilon<1$, that

$$\mathbb{P}\left\{\hat{\Lambda}_D(a)=0\right\}\leq\mathbb{P}\left\{\hat{\Lambda}_D(a)\leq 1-\varepsilon\right\}\leq\mathbb{P}\left\{\left|\hat{\Lambda}_D(a)-1\right|\geq\varepsilon\right\}.$$

Lemma 4.3 and Lemma 4.4 are sufficient to demonstrate that

$$\hat{\Lambda}_D(a)-1\to 0 \quad \text{in probability.} \tag{33}$$

∎

Lemma 4.7.

Under the hypotheses of Theorem 4.2, we have for all $\sup\limits_{x_n\leq h_L\leq y_n}$, that

$$\left(\left(\frac{n\varphi_a\left(x_n\right)^d}{\left(\frac{\partial}{\partial\phi}\Lambda\left(a,\vartheta_a\right)\right)^2\sigma^2\left(a,\vartheta_a\right)}\right)^{1/2}\left(\hat{\Lambda}_N(a,\phi)-\mathbb{E}\left[\hat{\Lambda}_N(a,\phi)\right)\right]\right)\xrightarrow{\mathscr{D}}\mathscr{N}(0,1), \quad \text{as } n\to\infty.$$

Proof:

For a fixed $a\in S$, identical to Lemma 4.2, we write

$$\begin{aligned}\hat{\Lambda}_N(a,\phi)-\mathbb{E}\left[\hat{\Lambda}_N(a,\phi)\right]=&\hat{\Lambda}_N(a,\phi)-\hat{\Lambda}_N^*(a,\phi)+\hat{\Lambda}_N^*(a,\phi)\\&-\mathbb{E}\left[\hat{\Lambda}_N^*(a,\phi)\right]+\mathbb{E}\left[\hat{\Lambda}_N^*(a,\phi)\right]-\mathbb{E}\left[\hat{\Lambda}_N(a,\phi)\right],\end{aligned}$$

where $\hat{\Lambda}_N^*(a,\phi)$ is already given in Lemma 4.2. Once more we utilize the same justifications like in Lemma 4.2, then we have

$$\left(\frac{n\varphi_a\left(x_n\right)^d}{\left(\frac{\partial}{\partial\phi}\Lambda\left(a,\vartheta_a\right)\right)^2\sigma^2\left(a,\vartheta_a\right)}\right)^{1/2} \mid \hat{\Lambda}_N(a,\phi)-\hat{\Lambda}_N^*(a,\phi)\mid = o_p(1),$$

and

$$\left(\frac{n\varphi_a\left(x_n\right)^d}{\left(\frac{\partial}{\partial\phi}\Lambda\left(a,\vartheta_a\right)\right)^2\sigma^2\left(a,\vartheta_a\right)}\right)^{1/2} \mid \mathbb{E}\left[\hat{\Lambda}_N^*(a,\phi)\right]-\mathbb{E}\left[\hat{\Lambda}_N(a,\phi)\right]\mid = o(1).$$

Then it's only a matter of demonstrating the asymptotic normality of

$$\left(\frac{n\varphi_a\left(x_n\right)^d}{\left(\frac{\partial}{\partial\phi}\Lambda\left(a,\vartheta_a\right)\right)^2\sigma^2\left(a,\vartheta_a\right)}\right)^{1/2} \left|\hat{\Lambda}_N^*(a,\phi)-\mathbb{E}\left[\hat{\Lambda}_N^*(a,\phi)\right]\right|.$$

This is what we've done:

$$\rho^*\left(B_i,\phi\right)=\rho_a\left(B_i,\phi\right)\mathbb{I}_{|\rho(B_i,\phi)|<\theta_n}, \qquad \Omega_i=\frac{1}{n\mathbb{E}\left[L_1\right]^d}\left(L_i\rho^*\left(B_i,\phi\right)-\mathbb{E}\left[L_i\rho^*\left(B_i,\phi\right)\right]\right),$$

$$W_{ni}=\sqrt{n\varphi_a\left(h_L\right)^d}\,\Omega_i, \qquad \text{and} \qquad Y_n=\sum_{i=1}^n W_{ni},$$

then,

$$Y_n=\sqrt{n\varphi_a\left(h_L\right)^d}\left(\hat{\Lambda}_N^*(a,\phi)-\mathbb{E}\left[\hat{\Lambda}_N^*(a,\phi)\right]\right).$$

As a consequence, our claimed result is now

$$Y_n \to \mathscr{N}\left(0,\sigma_1(a)\right), \tag{34}$$

where $\sigma_1^2(a)=\left(\frac{\partial}{\partial\phi}\Lambda\left(a,\vartheta_a\right)\right)^2\sigma^2\left(a,\vartheta_a\right)$.

To do that, we use the simple methodology of Doob (1953). After all, we assume two series of natural numbers tending to ∞, $\zeta=\zeta_n$, and $\eta=\eta_n$, such that $\zeta=o\left(n^{1/2}\theta_n^{-1}\varphi_a\left(x_n\right)^{d/2}\right)$ and $\eta=O\left(\zeta^{1-\varsigma}\right)$ for a certain $\varsigma\in(0,1)$ and we divided Y_n into

$$Y_n=Z_n+Z_n'+D_\ell, \text{ with } Z_n=\sum_{j=1}^{\ell}U_j, \text{ and } Z_n'=\sum_{j=1}^{\ell}\xi_j,$$

where

$$U_j := \sum_{i \in I_j} W_{ni}, \quad \xi_j := \sum_{i \in J_j} W_{ni}, \quad D_\ell := \sum_{i=\ell(\zeta+\eta)+1}^{n} W_{ni},$$

with

$$I_j = \{(j-1)(\zeta+\eta)+1, \dots, (j-1)(\zeta+\eta)+\zeta\},$$
$$J_j = \{(j-1)(\zeta+\eta)+\zeta+1, \dots, j(\zeta+\eta)\}.$$

Observe that, for $\ell = \left[\frac{n}{\zeta+\eta}\right]$, (where [.] stands for the integer part), we have $\frac{\ell\eta}{n} \to 0$, and $\frac{\ell\zeta}{n} \to 1$, $\frac{\eta}{n} \to 0$, which imply that $\frac{\zeta}{n} \to 0$ as $n \to \infty$. Now, our asymptotic result is based on

$$\mathbb{E}\left(Z_n'\right)^2 + \mathbb{E}\left(D_\ell\right)^2 \to 0, \tag{35}$$

$$Z_n \to \mathscr{N}\left(0, \sigma_1^2(a)\right). \tag{36}$$

For the proof of (35), the stationarity of variables is used to obtain

$$\mathbb{E}\left(Z_n'\right)^2 = \ell \operatorname{Var}\left(\xi_1\right) + 2 \sum_{1 \le i < j \le \ell} \left|\operatorname{Cov}\left(\xi_i, \xi_j\right)\right|, \tag{37}$$

$$\ell \operatorname{Var}\left(\xi_1\right) \le \eta\ell \operatorname{Var}\left(W_{n1}\right) + 2\ell \sum_{1 \le i < j \le \eta} \left|\operatorname{Cov}\left(W_{ni}, W_{nj}\right)\right|. \tag{38}$$

A first expression in (38) in the upper right-hand hand can be deduced from (21) and the fact that $\frac{\ell\eta}{n} \to 0$. In reality,

$$\eta\ell \operatorname{Var}\left(W_{n1}\right) = \varphi_a\left(h_L\right)^d n\ell\eta \operatorname{Var}\left(\Omega_1\right) = O\left(\frac{\ell\eta}{n}\right) \to 0, \text{ as } n \to \infty. \tag{39}$$

Then, the second term is as follows

$$\ell \sum_{1 \le i < j \le \eta} \left|\operatorname{Cov}\left(W_{ni}, W_{nj}\right)\right| = \ell n \varphi_a\left(h_L\right)^d \sum_{1 \le i < j \le \eta} \left|\operatorname{Cov}\left(\Omega_i, \Omega_j\right)\right|,$$

after that, according to (22), we demonstrate that

$$\sup_{x_n \le h_L \le y_n} \sum_{1 \le i < j \le \eta} \left|\operatorname{Cov}\left(\Omega_i, \Omega_j\right)\right| = o\left(\frac{\eta}{n^2 \varphi_a\left(x_n\right)^d}\right).$$

Therefore,

$$\ell \sum_{1 \le i < j \le \eta} \left|\operatorname{Cov}\left(W_{ni}, W_{nj}\right)\right| = o\left(\frac{\ell\eta}{n}\right) \to 0, \text{as } n \to \infty. \tag{40}$$

We use the stationarity to evaluate the second term in the right-hand side of (37),

$$\begin{aligned}\sum_{1\le i<j\le \ell}\left|\mathrm{Cov}\left(\xi_i,\xi_j\right)\right| &= \sum_{l=1}^{\ell-1}(\ell-l)\left|\mathrm{Cov}\left(\xi_1,\xi_{l+1}\right)\right|\\ &\le \ell\sum_{l=1}^{\ell-1}\left|\mathrm{Cov}\left(\xi_1,\xi_{l+1}\right)\right|\\ &\le \ell\sum_{l=1}^{\ell-1}\sum_{(i,j)\in J_1\times J_{l+1}}\mathrm{Cov}\left(W_{ni},W_{nj}\right).\end{aligned}$$

So, for all $(i,j)\in J_1\times J_j$, we have $|i-j|\ge\zeta+1>\zeta$, then for all $\sup_{x_n\le h_L\le y_n}$, we have that

$$\begin{aligned}\sum_{1\le i<j\le \ell}\left|\mathrm{Cov}\left(\xi_i,\xi_j\right)\right| &\le \ell\frac{C\theta_n^2}{n\varphi_a\left(x_n\right)^{d+2}}\sum_{i=1}^{\zeta}\sum_{j=2\zeta+\eta+1,|i-j|>\zeta}^{\ell(\zeta+\eta)}\Delta_{i,j}\\ &\le \frac{C\ell\zeta\theta_n^2}{n\varphi_a\left(x_n\right)^{d+2}}\Delta_\zeta\\ &\le \frac{C\ell\zeta\theta_n^2}{n\varphi_a\left(x_n\right)^{d+2}}e^{-a\zeta}\to 0.\end{aligned}$$

Finally, when this last finding is combined with (38) and (40), we can write

$$\mathbb{E}\left(Z_1'\right)^2\to 0 \text{ as } n\to\infty.$$

When $(n-\ell(\zeta+\eta))\le\zeta$, we get

$$\begin{aligned}\mathbb{E}\left(D_\ell\right)^2 &\le (n-\ell(\zeta+\eta))\,\mathrm{Var}\left(W_{n1}\right)+2\sum_{1\le i<j\le n}\left|\mathrm{Cov}\left(W_{ni},W_{nj}\right)\right|\\ &\le \zeta\,\mathrm{Var}\left(W_{n1}\right)+2\sum_{1\le i<j\le n}\left|\mathrm{Cov}\left(W_{ni},W_{nj}\right)\right|\\ &\le \zeta n\varphi_a\left(x_n\right)^d\mathrm{Var}\left(\Omega_1\right)+n\varphi_a\left(x_n\right)^d\sum_{1\le i<j\le n}\left|\mathrm{Cov}\left(\Omega_i,\Omega_j\right)\right|\\ &\le \frac{C\zeta}{n}+o(1).\end{aligned}$$

Then,

$$\mathbb{E}\left(D_\ell\right)^2\to 0,\ \text{as } n\to\infty.$$

Proof of (36): it is focused on

$$\left|\mathbb{E}\left(e^{i\phi\sum_{j=1}^{\ell}U_j}\right)-\prod_{j=1}^{\ell}\mathbb{E}\left(e^{i\phi U_j}\right)\right|\to 0, \tag{41}$$

and

$$\ell\,\mathrm{Var}\left(U_1\right)\to\sigma_1^2(a),\quad \ell\mathbb{E}\left(U_1^2\mathbb{I}_{\{U_1>\epsilon\sigma_1(a)\}}\right)\to 0. \tag{42}$$

Proof of (41): we have that

$$\begin{aligned}&\left|\mathbb{E}\left(e^{i\phi\sum_{j=1}^{\ell}U_j}\right)-\prod_{j=1}^{\ell}\mathbb{E}\left(e^{i\phi Uj}\right)\right|\\&\quad\leq\left|\mathbb{E}\left(e^{i\phi\sum_{j=1}^{\ell}U_j}\right)-\mathbb{E}\left(e^{i\phi\sum_{j=1}^{\ell-1}U_j}\right)\mathbb{E}\left(e^{i\phi U_\ell}\right)\right|\\&\qquad+\left|\mathbb{E}\left(e^{i\phi\sum_{j=1}^{\ell-1}U_j}\right)-\prod_{j=1}^{\ell-1}\mathbb{E}\left(e^{i\phi U_j}\right)\right|,\end{aligned}\tag{43}$$

$$\left|\mathrm{Cov}\left(e^{i\phi\sum_{j=1}^{\ell-1}U_j},e^{i\phi U_\ell}\right)\right|+\left|\mathbb{E}\left(e^{i\phi\sum_{j=1}^{\ell-1}U_j}\right)-\prod_{j=1}^{\ell-1}\mathbb{E}\left(e^{i\phi U_j}\right)\right|,\tag{44}$$

and after that, we've had

$$\begin{aligned}\left|\mathbb{E}\left(e^{i\phi\sum_{j=1}^{\ell}U_j}\right)-\prod_{j=1}^{\ell}\mathbb{E}\left(e^{i\phi U_j}\right)\right|&\leq\left|\mathrm{Cov}\left(e^{i\phi\sum_{j=1}^{\ell-1}U_j},e^{i\phi U_\ell}\right)\right|\\&\quad+\left|\mathrm{Cov}\left(e^{i\phi\sum_{j=1}^{\ell-2}U_j},e^{i\phi U_{\ell-1}}\right)\right|\\&\quad+\cdots+\left|\mathrm{Cov}\left(e^{i\phi U_2},e^{i\phi U_1}\right)\right|.\end{aligned}\tag{45}$$

Utilizing the property of the quasi-association to write

$$\left|\mathrm{Cov}\left(e^{i\phi U_2},e^{i\phi U_1}\right)\right|\leq\frac{C\phi^2\theta_n^2}{n\varphi_a\left(h_L\right)^{d+2}}\sum_{i\in I_1}\sum_{j\in I_2}\Delta_{i,j},$$

by applying this inequality to each term on the right-hand side of (45), we get

$$\begin{aligned}&\left|\mathbb{E}\left(e^{i\phi\sum_{j=1}^{\ell}U_j}\right)-\prod_{j=1}^{\ell}\mathbb{E}\left(e^{i\phi U_j}\right)\right|\\&\quad\leq\frac{C\phi^2\theta_n^2}{n\varphi_a\left(h_L\right)^{d+2}}\left[\sum_{i\in I_1}\sum_{j\in I_2}\Delta_{i,j}+\sum_{i\in I_1\cup I_2}\sum_{j\in I_3}\Delta_{i,j}+\cdots+\sum_{i\in I_1\cup\cdots\cup I_{\ell-1}}\sum_{j\in I_\ell}\Delta_{i,j}\right].\end{aligned}$$

Observe that for each and every $\ell-1\geq l\geq 2,(i,j)\in I_l\times I_{l+1}$, we have $\eta<\eta+1\leq|i-j|$, then

$$\sum_{i\in I_1\cup\ldots\cup I_{l-1}}\sum_{j\in I_l}\Delta_{i,j}\leq p\Delta_\eta.$$

For all $\sup\limits_{x_n\leq h_L\leq y_n}$, inequality (44) becomes

$$\begin{aligned}\left|\mathbb{E}\left(e^{i\phi\sum_{j=1}^{\ell}U_j}\right)-\prod_{j=1}^{\ell}\mathbb{E}\left(e^{i\phi Uj}\right)\right|&\leq\frac{C\phi^2\theta_n^2}{n\varphi_a\left(x_n\right)^{d+2}}\ell\zeta\Delta_\eta\\&\leq\frac{C\phi^2\theta_n^2}{n\varphi_a\left(x_n\right)^{d+2}}\ell\zeta e^{-a\eta}\to 0.\end{aligned}$$

Then, for (42), we utilize similar arguments as those in (37), to obtain

$$\begin{aligned}\lim_{n\to\infty} \ell \operatorname{Var}(U_1) &= \lim_{n\to\infty} \ell\zeta \operatorname{Var}(W_{n1})\\ &= \lim_{n\to\infty} \ell\, \zeta\, n\varphi_a(h_L)^d \operatorname{Var}(\Omega_1).\end{aligned}$$

On the other hand,

$$\begin{aligned}\operatorname{Var}(\Omega_1) =& \frac{1}{n^2\varphi_a(h_L)^{2d}}\left\{\mathbb{E}\left[L^2\left(h_L^{-1}(a-A_i)\right)\rho_a^2(B_i,\phi)\right]\right.\\ &\left.-\mathbb{E}\left[L^2\left(\varphi_a(h_L)^{-1}(a-A_i)\right)\rho_a^2(B_i,\phi)\,\mathbb{1}_{|\rho(B_i,\phi)>\theta_n}\right]\right\}\\ &-\frac{1}{n^2}\left(\frac{1}{\varphi_a(h_L)^d}\mathbb{E}\left[L\left(\varphi_a(h_L)^{-1}(a-A_i)\right)\rho_a(B_i,\phi)\,\mathbb{1}_{|\rho(B_i,\phi)|<\theta_n}\right]\right)^2.\end{aligned}$$

In the same way of (8) and Lemma 4.1, for all $\sup\limits_{x_n\le h_L\le y_n}$, we show that

$$\operatorname{Var}(\Omega_1) = \frac{\sigma_1^2(a)}{n^2\varphi_a(x_n)^d} + o\left(\frac{1}{n^2\varphi_a(x_n)^d}\right).$$

Hence,

$$\ell \operatorname{Var}(U_1) = \frac{\ell\zeta\sigma_1^2(a)}{n} + o\left(\frac{\ell\zeta}{n}\right) \to \sigma_1^2(a).$$

Now we are able to use Tchebychev's inequality with the second part of (42), and utilising the fact that $|U_1| \le C\zeta\,|W_{n1}| \le \dfrac{C\theta_n\zeta}{\sqrt{n\varphi_a(h_L)^d}}$, to obtain

$$\begin{aligned}\ell\mathbb{E}\left(U_1^2\mathbb{1}_{\{U_1>\epsilon\sigma_1(a)\}}\right) &\le \frac{C\theta_n^2\zeta^2\ell}{nh^d}\mathbb{P}\left(U_1>\epsilon\sigma_1(a)\right)\\ &\le \frac{C\theta_n^2\zeta^2\ell}{n\varphi_a(h_L)^d}\frac{\operatorname{Var}(U_1)}{\epsilon^2\sigma_1^2(a)}\\ &= O\left(\frac{\theta_n^2\zeta^2}{n\varphi_a(x_n)^d}\right).\end{aligned}$$

which completes the proof. ■

Lemma 4.8.

Under Hypotheses (H4), (H5) and (H9) and if the bandwidth parameter h_L satisfies as $\sup\limits_{x_n\le h_L\le y_n}$, we have

$$\left(\frac{n\varphi_a(x_n)^d}{\left(\frac{\partial}{\partial\phi}\Lambda(a,\vartheta_a)\right)^2\sigma^2(a,\vartheta_a)}\right)^{1/2}\mathbb{E}\left[\hat{\Lambda}_N(a,\phi)\right] = u + o(1), \text{ as } \quad n\to+\infty.$$

Proof:

By simple analytical arguments we write

$$\mathbb{E}\left[\hat{\Lambda}_N(a,\phi)\right] = \int_{\mathbb{R}^d} H(a - h_L z, \phi) L(z) dz.$$

We utilize a Taylor expansion of $H\left(a - h_L z, \vartheta_a + u\left[n\varphi_a\left(x_n\right)^d\right]^{-1/2}\sigma\left(a,\vartheta_a\right)\right)$, with $\sup\limits_{x_n \le h_L \le y_n}$ to write

$$\mathbb{E}\left[\hat{\Lambda}_N(a,\phi)\right] = u\left[n\varphi_a\left(x_n\right)^d\right]^{-1/2}\sigma\left(a,\vartheta_a\right)\Lambda'(a,\vartheta(a)) + o(y_n).$$

The result is then a consequence of (H4). ■

5. Simulation study

In this section, we evaluate the behavior of our results over finite sample data. More precisely our main aim is to show the easy implementation of the kNN M-Regression estimator against the CV (Cross Validation) kernel estimator given by Ferraty and Vieu (2006) and to examine the influence of the degree of dependency on this asymptotic property. For this purpose, we generate functional observations by considering the following functional nonparametric model

$$B_i = \Lambda\left(A_i\right) + \epsilon_i \text{ for } i = 1, \ldots, n,$$

where the ϵ_i 's are generated according to a normal distribution $\mathcal{N}(0, 0.5)$. It is well documented that the linear process is quasi-associated variables satisfies condition (H5). Thus, we generate the quasi-associates functional regressor as follow

$$A_i(t) = \sum_{k=i+1}^{i+m} Z_k(t) \quad \text{where, } Z_k(t) = \sin\left(W_k * t\right) + V_k * t \quad t \in \left[0, \frac{\pi}{2}\right],$$

and $\left(W_k\right)_k$ (respectively $\left(V_k\right)_k$) are independent and identically distributed as $\mathcal{N}(1, 0.5)$ (respectively $\mathcal{N}(0,1)$). The A_i 's curves are discretized in the same grid which is composed of 100 points in $[0, \pi/2]$ and are plotted in Figure 1 for three values of $m = 1$ (independent case), 5 and 10.

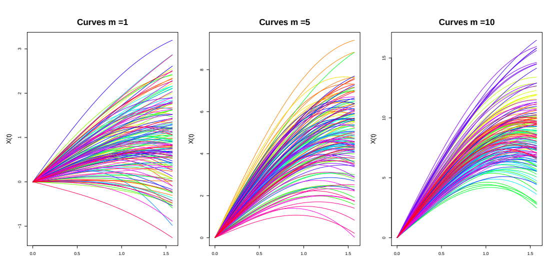

Figure 1. The curves $A_i(t), t \in [0, \pi/2]$ for $i = 1, \ldots, 150$ and $m = 1, 5, 10$

Furthermore, the scalar variable B_i is computed by the regression operator

$$\Lambda(a) = 5 \int_0^{\pi/2} \frac{1}{1 + |a(t)|} dt.$$

Figure 1 displays the curves of the sample sizes $n = 150$. Second, we need to select a suitable semi-metric $d(.,.)$, kernel $L(.)$, smoothing parameter k_{opt} for functional kNN estimator and h_{opt} for CV kernel estimator. For that purpose, we choose the asymmetrical quadratic kernel defined as $L(u) = \frac{3}{4}\left(1 - u^2\right) 1_{[0,1]}(u)$. Meanwhile, because of the smoothness of curves $A_i(t)$, we consider the following semimetric based on the first derivative

$$d^{deriv}\left(A_i, A_j\right) = \sqrt{\int_0^{\pi/2} \left(A_i'(t) - A_j'(t)\right)^2 \mathrm{d}t}, \quad \forall A_i, A_j \in \mathcal{G}.$$

In what follows, we randomly split the 150-sample into two parts: one is a training sample $(A_i, B_i)_{i=1}^{100}$ which is used to model, and the other is a testing sample $(A_j, B_j)_{j=101}^{150}$ which is used to verify the prediction effect. On the one hand, by the training sample, we can select the optimal parameter k_{opt} for kNN kernel and robust estimator, and the optimal parameter h_{opt} for CV classical kernel and robust estimator by the following cross-validation procedures, respectively. Concretely, we select $k_{opt} = \arg\min_k CV_1(k)$, where $CV_1(k) = \sum_{i=1}^{n} \left(B_i - \widehat{m}_{(-i)}^{kNN}(A)\right)^2$ and

$$\hat{m}_{(-i)}^{kNN}(A) = \frac{\sum_{j=1, j\neq i}^{n} B_j L\left(\frac{d^{deriv}\left(A_j, A\right)}{E_k(A)}\right)}{\sum_{j=1, j\neq i}^{n} L\left(\frac{d^{deriv}(A_j; A)}{E_k(A)}\right)}.$$

And the robust kNN one by $k_{opt} = \arg\min_k CV_2(k)$, where $CV_2(k) = \sum_{i=1}^{n} \left(B_i - \widehat{\Lambda}_{(-i)}^{kNN}(X)\right)^2$ and

$$\hat{\Lambda}_{(-i)}^{kNN}(A) = \arg\min_t \frac{\sum_{j=1, j\neq i}^{n} \rho(B_j, t) L\left(\frac{d^{deriv}\left(A_j, A\right)}{E_k(A)}\right)}{\sum_{j=1, j\neq i}^{n} L\left(\frac{d^{deriv}(A_j, A)}{E_k(A)}\right)}.$$

Similarly, we choose $h_{opt} = \arg\min_{h_L} CV(h_L)$ for the CV classic and robust methods, where $CV_3(h_L) = \sum_{i=1}^{n} \left(B_i - \widehat{m}_{(-i)}^{CV}(A)\right)^2$ and $CV_4(h_L) = \sum_{i=1}^{n} \left(B_i - \widehat{\Lambda}_{(-i)}^{CV}(A)\right)^2$,

where $\hat{m}_{(-i)}^{CV}(A) = \frac{\sum_{j=1, j\neq i}^{n} B_j L\left(\frac{d^{deriv}\left(A_j, A\right)}{E_k(A)}\right)}{\sum_{j=1, j\neq i}^{n} L\left(\frac{d^{deriv}(A_j; A)}{h_L(A)}\right)}$,

$$\text{and } \hat{\Lambda}^{CV}_{(-i)}(A) = \arg\min_t \frac{\displaystyle\sum_{j=1, j\neq i}^{n} \rho(B_j, t) L\left(\frac{d^{deriv}\,(A_j, A)}{E_k(A)}\right)}{\displaystyle\sum_{j=1, j\neq i}^{n} L\left(\frac{d^{deriv}(A_j, A)}{h_L(A)}\right)}.$$

Theoretical support for such cross-validation procedure has been given as well for dependent data in Härdle and Vieu (1992) as for functional data in Rachdi and Vieu (2007). On the other hand, by the testing sample, we can calculate the prediction values of the response variables denoted by $(B_j)_{j=101}^{150}$. Thus, the predicted responses for the four methods are illustrated in Figure 2 where we see that the forecasting of kNN estimator is more accurate than that of CV kernel one under the quasi-associates functional dependent sample.

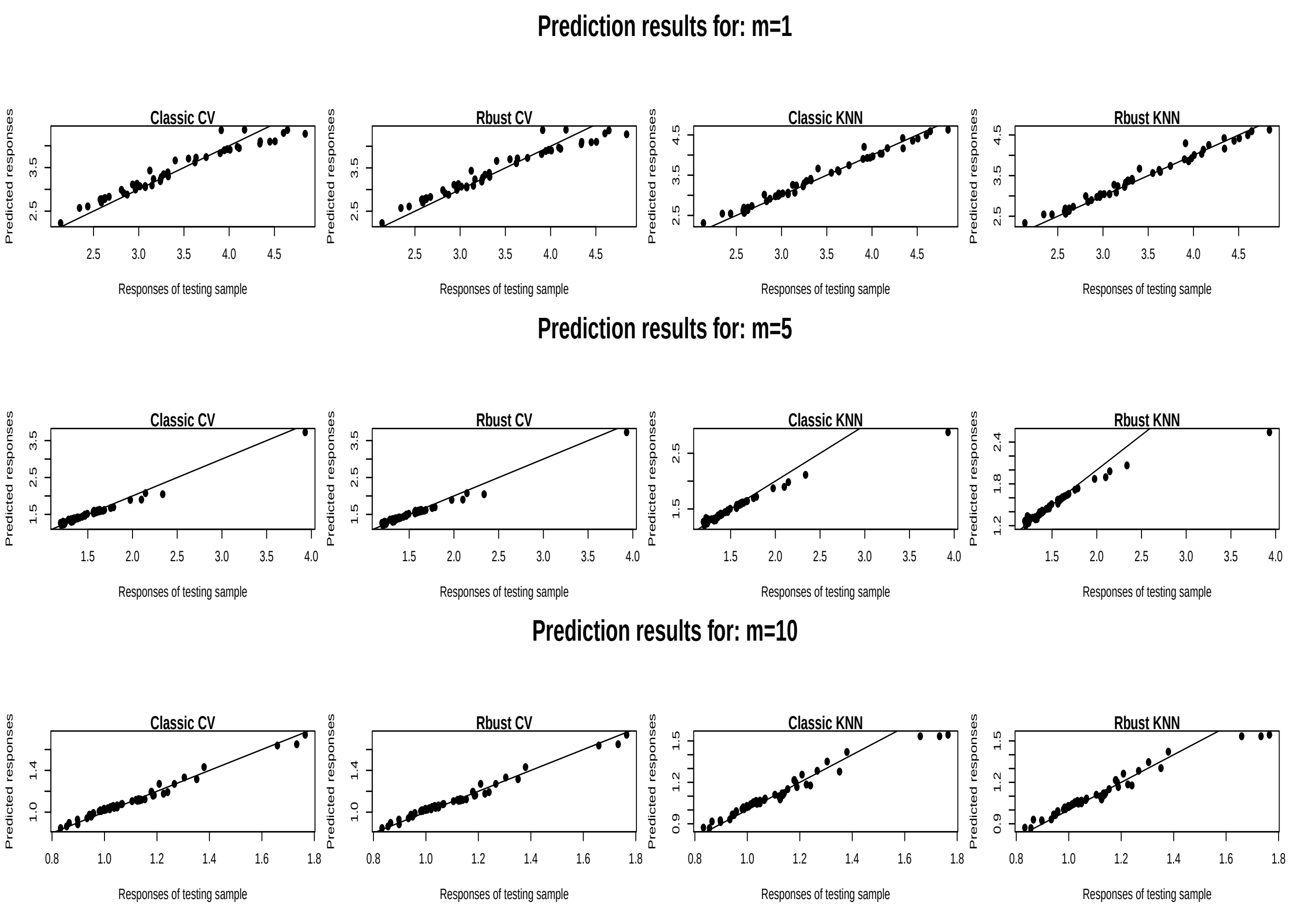

Figure 2. Prediction of the classical and robust estimator with CV and kNN methods, respectively

Now, we carry out 100 independent replications which allows to compute 100 values for MSE and to display their distribution by means of a boxplot. Figure 3 shows the boxplots of the MSE of the prediction values.

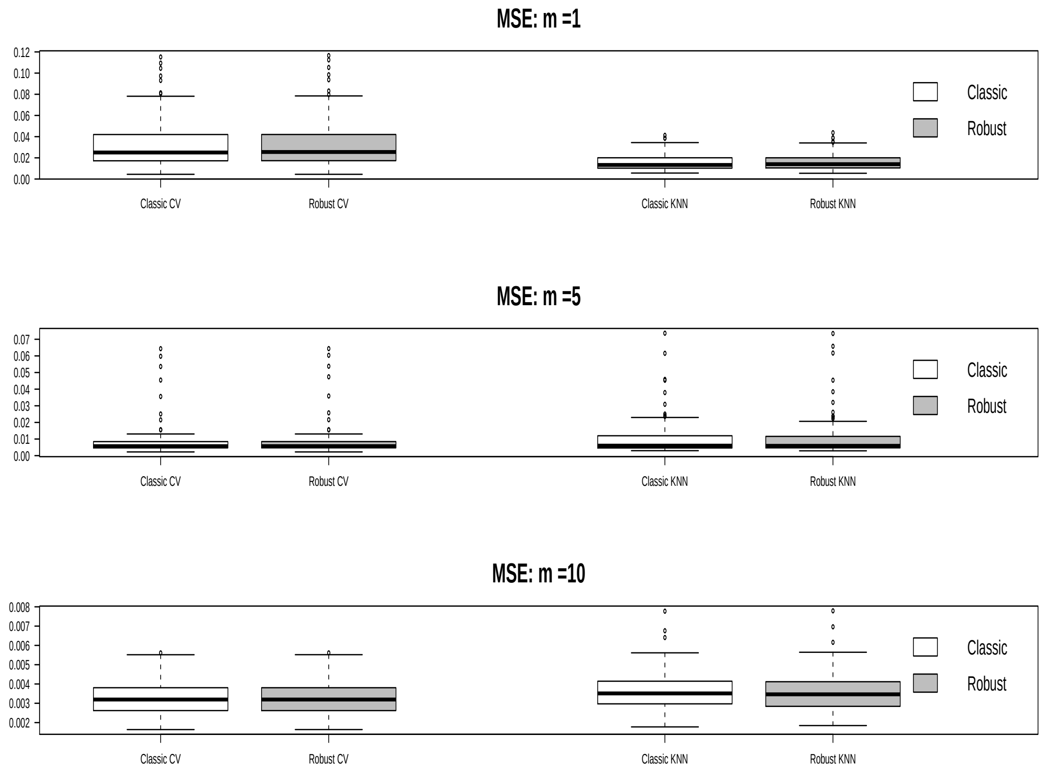

Figure 3. The boxplots of the MSE of the prediction values by the four methods for the different degree of dependency without outliers

Meanwhile, let us calculate the average of MSE of kNN estimator and CV kernel estimator. The results are reported in Table 1.

Table 1. Comparison between the four methods in the presence of outliers with different degree of dependency

m	Number of Outliers	Classic CV	Robust CV	Classic kNN	Robust kNN
	0	0.037514	0.037881	0.015731	0.016489
1	6	502.040275	62.369308	558.648462	0.051654
	12	1772.638790	101.099854	1993.288220	0.161468
	18	3800.416704	180.703892	4157.818408	0.418262
	0	0.008346	0.008406	0.010926	0.011399
5	6	111.125504	11.549593	119.765797	0.035228
	12	400.821394	29.412965	420.740074	0.099072
	18	864.387731	61.053536	929.041219	0.261310
	0	0.003400	0.003400	0.003780	0.003784
10	6	43.848807	1.388947	46.273798	0.026864
	12	164.544865	10.976110	169.472108	0.080355
	18	360.790930	26.130611	376.862336	0.183755

We observe in Table 1 that in the presence of outliers, the kNN robust regression gives better results than the other models, in a sense that, even if the MSE value of all methods increases substantially relative to the number of the perturbed points, it remaining very low for the kNN robust one.

6. Conclusion

The uniform kNN reliability approach is a smoothing alternative that allows for the development of an adaptive estimator for a variety of statistical problems, including bandwidth choice.

In our situation, furthermore, uniform consistency is not a straightforward extension of the pointwise approach, as it necessitates the use of additional methods and techniques. The assumption that the bandwidth parameter in the kNN method is a random variable adds to the complexity of this problem.

In the situation of quasi-associated results, the key innovation of this approach is to estimate the regression function by mixing two essential statistical techniques: the M-regression method and the kNN procedures. This strategy allowed for the development of a new estimator that combines the benefits of both methods.

To summarize, the behavior of the developed estimator is unaffected by the number of outliers in the data collection. In comparison to the classical kernel method, the mixture of the kNN algorithm and the robust method allows for a reduction in the impact of outliers in results.

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