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Asymptotical Stability of Riemann-Liouville Fractional Neutral System with Multiple Time-varying Delays

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Abstract

In this manuscript, we investigate the asymptotical stability of solutions of Riemann-Liouville fractional neutral systems associated to multiple time-varying delays. Then, we use the linear matrix inequality (LMI) and the Lyapunov-Krasovskii method to obtain sufficient conditions for the asymptotical stability of solutions of the system when the given delays are time dependent and one of them is unbounded. Finally, we present some examples to indicate the efficacy of the consequences obtained.

Keywords: Asymptotical stability; Fractional neutral systems; Riemann-Liouville derivative; Lyapunov functional; LMI

MSC 2010 No.: 34K20, 34K37, 34K40, 46B20

1. Introduction

The systems of fractional differential equations have become a widely studied topic for so many authors, which have recently found application in fields such as control theory, engineering, physics and biology. However, most of the recent studies are on the Caputo fractional derivative; very little work has been done on the Riemann-Liouville fractional derivative. In particular, delayed neutral equation systems remain an open problem. We aim to obtain sufficient conditions for the asymptotic stability of the system by taking variable delay instead of fixed delays in the study of Liu et al. (2017).

Although the history of fractional derivatives is as old as ordinary derivatives, it has been the subject of research by many scientists recently such as Podlubny (1999), Kilbas et al. (2006), Zhou and Jiao (2010), Tunç et al. (2020), Graef et al. (2016), Tarasov (2013), Liu et al. (2014), Shahri et al. (2015), Bohner et al. (2021), Hristova and Tunc (2019), Tunç et al. (2021), Zafar et al. (2021) and Deng and Deng (2014). In particular, resources related to solutions of fractional differential equations and their qualitative behavior can be listed as Matignon (1996), Lu and Chen (2009), Deng et al. (2007) and Li et al. (2010). Unlike the ordinary derivative, the fractional derivative has several different descriptions. These descriptions are mostly not equal with their counterparts, see, for example, Podlubny (1999) and Kilbas et al. (2006). Among these definitions, the Caputo derivative and the Riemman-Liouville derivative are very popular. Some of the studies on the Caputo derivative are Duarte-Mermoud et al. (2015), Yang et al. (2017), Chen et al. (2014), Liu et al. (2016a), Liu et al. (2016c), Chen et al. (2016), Brzdek and Eghbali (2016) and Aguila-Camacho et al. (2010). However, studies on the Riemman-Liouville derivative are fewer.

The fractional differential equations having initial conditions, which are defined by the Caputo derivative, are of integer order as in ordinary differential equations. However, the fractional differential equations having initial conditions, which are defined by the Riemann-Liouville derivative, are of fractional order. Most researchers consider that it is not easy to measure the initial conditions given in fractional order, even though this idea can not be proven easily most of the time. Heymans and Podlubny (2006) investigate geometric and physical interpretations for Riemman-Liouville differentiation.

If entire roots of the characteristic equation possess negative real parts, Qian et al. (2010), show the asymptotically stablity of solutions of the fractional linear system. Altun and Tunç (2020) constructed a Lyapunov-Krasovskii functional for asymptotic stability of solutions of nonlinear fractional equations. Liu et al. (2016b) and Liu et al. (2016d) explore fractional singular systems and nonlinear systems by expanding the inequality to Riemann-Liouville derivatives. Moreover, they determine various conditions on delay-independent stability. Li et al. (2015) remind the current conditions connected with the asymptotical stability of fractional neutral systems in terms of matrix norm matrices and matrix measure of the system.

2. Main Results

In this section, we introduce some fundamental definitons of fractional calculus together with improtant lemmas.

Note that n-dimensional Euclidean space is denoted by R^n. The set of entire $n \times n$ real matrices indicated by $R^{n \times n}$. The Euclidean norm of a real vector x is denoted by $\|x\|$. The spectral norm of matrix A is indicated by $\|A\|$. When $A < 0$ (or $A > 0$), the symmetric matrix A is negative definite (or positive definite). The Riemman-Liouville fractional derivative and integral are described in Podlubny (1999).

Now, we present stability of solutions of fractional linear neutral systems having time-varying delays. We further consider linear matrix inequality to determine sufficient conditions on asymptotical stability of solutions of these systems.

Let the fractional neutral system be given by the following,

$$ {}_{t_0}D_t^{\alpha}x(t) = Ax(t) + Bx(t-\tau_1(t)) + C\,{}_{t_0}D_t^{\alpha}x(t-\tau_2(t)), \tag{1} $$

where $0 < \alpha < 1$, $x(t) \in R^n$ is the state vector, $A, B, C \in R^{n\times n}$ are constant matrices, for all $t > t_0$, $\tau_1(t), \tau_2(t) > 0$ are time-varying delays.

We also indicate an operator $\Phi : \Phi(x_t) = x(t) - Cx(t-\tau_2(t))$. Note that the operator Φ is stable if $\|C\| < 1$. If the zero solution of the homogeneous difference equation $\Phi(x_t) = 0$, $t \geq 0$ is uniformly asymptotically stable then the operator Φ is called to be stable (see Hale (1977)).

Theorem 2.1.

The trival solution of system (1) is asymptotically stable, if $\|C\| < 1$, for all $t > t_0$, $\tau_i'(t) \leq d_i < 1 \quad (i = 1, 2)$, $\tau_2(t)$ bounded function. There exist positive and symmetric define matrices P, Q, R_1, R_2 such that the following LMI satisfies:

$$ M = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{12}^T & M_{22} & M_{23} \\ M_{13}^T & M_{23}^T & M_{33} \end{pmatrix} < 0, \tag{2} $$

where

$$ \begin{aligned} M_{11} &= PA + A^TP + Q + A^T(R_1 + mR_2)A, \\ M_{12} &= PB + A^T(R_1 + mR_2)B, \\ M_{13} &= PC + A^T(R_1 + mR_2)C, \\ M_{22} &= B^T(R_1 + mR_2)B - (1-d_1)Q, \\ M_{23} &= B^T(R_1 + mR_2)C, \\ M_{33} &= C^T(R_1 + mR_2)C - (1-d_2)R_1, \end{aligned} $$

and m is a constant such that $|\tau_2(t)| \leq m$.

Proof:

Let the Lyapunov-Krasovskii functional be defined by:

$$ V(t) = {}_{t_0}D_t^{\alpha-1}(x^T(t)Px(t)) + \int_{t-\tau_1(t)}^{t} x^T(s)Qx(s)ds + \int_{-\tau_2(t)}^{0} ({}_{t_0}D_t^{\alpha}x(t+s))^T R_1 ({}_{t_0}D_t^{\alpha}x(t+s))ds $$
$$ + \int_{t-\tau_2(t)}^{t}\int_{\theta}^{t} ({}_{t_0}D_s^{\alpha}x(s))^T R_2 ({}_{t_0}D_s^{\alpha}x(s))ds d\theta. \tag{3} $$

From Property 2.8 in Kilbas et al. (2006) and Lemma 2.2 in Liu et al. (2016a), the derivative of $V(t)$ is obtained along the trajectories of system (1) as follows:

$$\begin{aligned}\dot{V}(t) =&{}_{t_0}D_t^{\alpha}(x^T(t)Px(t)) + x^T(t)Qx(t) - (1-\tau_1'(t))x^T(t-\tau_1(t))Qx(t-\tau_1(t))\\&+({}_{t_0}D_t^{\alpha}x(t))^T R_1({}_{t_0}D_t^{\alpha}x(t)) - (1-\tau_2'(t))({}_{t_0}D_t^{\alpha}x(t-\tau_2(t)))^T R_1({}_{t_0}D_t^{\alpha}x(t-\tau_2(t)))\\&+\tau_2(t)({}_{t_0}D_t^{\alpha}x(t))^T R_2({}_{t_0}D_t^{\alpha}x(t)) - (1-\tau_2'(t))\int_{t-\tau_2(t)}^{t}({}_{t_0}D_s^{\alpha}x(s))^T R_2({}_{t_0}D_s^{\alpha}x(s))ds\\\le&2x^T(t)P{}_{t_0}D_t^{\alpha}x(t) + x^T(t)Qx(t) - (1-d_1)x^T(t-\tau_1(t))Qx(t-\tau_1(t))\\&+({}_{t_0}D_t^{\alpha}x(t))^T R_1({}_{t_0}D_t^{\alpha}x(t)) - (1-d_2)({}_{t_0}D_t^{\alpha}x(t-\tau_2(t)))^T R_1({}_{t_0}D_t^{\alpha}x(t-\tau_2(t)))\\&+m({}_{t_0}D_t^{\alpha}x(t))^T R_2({}_{t_0}D_t^{\alpha}x(t)).\end{aligned} \tag{4}$$

Now considering Equation (1), we write as

$$\begin{aligned}2x^T(t)P{}_{t_0}D_t^{\alpha}x(t) =&2x^T(t)P\left[Ax(t)+Bx(t-\tau_1(t))+C{}_{t_0}D_t^{\alpha}x(t-\tau_2(t))\right]\\=&x^T(t)(PA+A^TP)x(t)+2x^T(t)PBx(t-\tau_1(t))\\&+2x^T(t)PC{}_{t_0}D_t^{\alpha}x(t-\tau_2(t)),\end{aligned} \tag{5}$$

and

$$\begin{aligned}&({}_{t_0}D_t^{\alpha}x(t))^T R_1({}_{t_0}D_t^{\alpha}x(t)) + m({}_{t_0}D_t^{\alpha}x(t))^T R_2({}_{t_0}D_t^{\alpha}x(t))\\&=\left[Ax(t)+Bx(t-\tau_1(t))+C{}_{t_0}D_t^{\alpha}x(t-\tau_2(t))\right]^T(R_1+mR_2)\\&\quad\times\left[Ax(t)+Bx(t-\tau_1(t))+C{}_{t_0}D_t^{\alpha}x(t-\tau_2(t))\right]\\&=x^T(t)A^T(R_1+mR_2)Ax(t)+x^T(t)A^T(R_1+mR_2)Bx(t-\tau_1(t))\\&\quad+x^T(t)A^T(R_1+mR_2)C{}_{t_0}D_t^{\alpha}x(t-\tau_2(t))+x^T(t-\tau_1(t))B^T(R_1+mR_2)Ax(t)\\&\quad+x^T(t-\tau_1(t))B^T(R_1+mR_2)Bx(t-\tau_1(t))\\&\quad+x^T(t-\tau_1(t))B^T(R_1+mR_2)C{}_{t_0}D_t^{\alpha}x(t-\tau_2(t))\\&\quad+({}_{t_0}D_t^{\alpha}x(t-\tau_2(t)))^TC^T(R_1+mR_2)Ax(t)\\&\quad+({}_{t_0}D_t^{\alpha}x(t-\tau_2(t)))^TC^T(R_1+mR_2)Bx(t-\tau_1(t))\\&\quad+({}_{t_0}D_t^{\alpha}x(t-\tau_2(t)))^TC^T(R_1+mR_2)C{}_{t_0}D_t^{\alpha}x(t-\tau_2(t)).\end{aligned} \tag{6}$$

From (4), (5) and (6), one has

$$\dot{V}(t) \le \xi^T M \xi, \tag{7}$$

where

$$M = \begin{pmatrix} M_{11} & M_{12} & M_{13}\\ M_{12}^T & M_{22} & M_{23}\\ M_{13}^T & M_{23}^T & M_{33}\end{pmatrix} < 0,$$

$$\begin{aligned}M_{11} &= PA + A^TP + Q + A^T(R_1+mR_2)A,\\M_{12} &= PB + A^T(R_1+mR_2)B,\\M_{13} &= PC + A^T(R_1+mR_2)C,\\M_{22} &= B^T(R_1+mR_2)B-(1-d_1)Q,\\M_{23} &= B^T(R_1+mR_2)C,\\M_{33} &= C^T(R_1+mR_2)C-(1-d_2)R_1,\end{aligned}$$

and

$$\xi = (x^T(t), x^T(t-\tau_1(t)), ({}_{t_0}D_t^{\alpha} x(t-\tau_2(t)))^T)^T.$$

From inequality (2), since $\dot{V}(t)$ is negative definite the trivial solution of system (1) is asymptotically stable. ■

Theorem 2.2.

The trivial solution of system (1) is asymptotically stable, if $\|C\| < 1$, for all $t > t_0$, $\tau_i'(t) \leq d_i < 1 \quad (i=1,2)$, $\tau_2(t)$ bounded function. There exist positive and symmetric define matrices P, Q_1, Q_2, R such that the following LMI statisfies:

$$N = \begin{pmatrix} N_{11} & N_{12} & N_{13} \\ N_{12}^T & N_{22} & N_{23} \\ N_{13}^T & N_{23}^T & N_{33} \end{pmatrix} < 0, \tag{8}$$

where

$$\begin{aligned} N_{11} &= PA + A^T P + Q_1 + Q_2 + mA^T RA, \\ N_{12} &= PB + mA^T RB, \\ N_{13} &= -A^T PC, \\ N_{22} &= mB^T RB - (1-d_1)Q_1, \\ N_{23} &= -B^T PC, \\ N_{33} &= -(1-d_2)Q_2, \end{aligned}$$

and m is a constant such that $|\tau_2(t)| \leq m$.

Proof:

Let the Lyapunov-Krasovskii functional be defined by:

$$\begin{aligned} V(t) =& {}_{t_0}D_t^{\alpha-1}((x(t) - Cx(t-\tau_2(t)))^T P(x(t) - Cx(t-\tau_2(t)))) \\ &+ \int_{t-\tau_1(t)}^{t} x^T(s) Q_1 x(s) ds + \int_{t-\tau_2(t)}^{t} x^T(s) Q_2 x(s) ds \\ &+ \int_{t-\tau_2(t)}^{t} \int_{\theta}^{t} ({}_{t_0}D_s^{\alpha}(x(s) - Cx(s-\tau_2(s))))^T R({}_{t_0}D_s^{\alpha}(x(s) - Cx(s-\tau_2(s)))) ds d\theta. \end{aligned} \tag{9}$$

From Property 2.8 in Kilbas et al. (2006) and Lemma 2.2 in Liu et al. (2016a), the derivative of

$V(t)$ is obtained along the trajectories of system (1) as follows:

$$\begin{aligned}\dot{V}(t) =&{}_{t_0}D_t^{\alpha}((x(t)-Cx(t-\tau_2(t)))^T P(x(t)-Cx(t-\tau_2(t))))\\ &+x^T(t)Q_1x(t)-(1-\tau_1'(t))x^T(t-\tau_1(t))Q_1x(t-\tau_1(t))\\ &+x^T(t)Q_2x(t)-(1-\tau_2'(t))x^T(t-\tau_2(t))Q_2x(t-\tau_2(t))\\ &+\tau_2(t)({}_{t_0}D_t^{\alpha}(x(t)-Cx(t-\tau_2(t))))^T R({}_{t_0}D_t^{\alpha}(x(t)-Cx(t-\tau_2(t))))\\ &-(1-\tau_2'(t))\int_{t-\tau_2(t)}^{t}({}_{t_0}D_s^{\alpha}(x(s)-Cx(s-\tau_2(s))))^T R({}_{t_0}D_s^{\alpha}(x(s)-Cx(s-\tau_2(s))))ds\\ \le& 2(x(t)-Cx(t-\tau_2(t)))^T P {}_{t_0}D_t^{\alpha}(x(t)-Cx(t-\tau_2(t)))\\ &+x^T(t)Q_1x(t)-(1-d_1)x^T(t-\tau_1(t))Q_1x(t-\tau_1(t))\\ &+x^T(t)Q_2x(t)-(1-d_2)x^T(t-\tau_2(t))Q_2x(t-\tau_2(t))\\ &+m({}_{t_0}D_t^{\alpha}(x(t)-Cx(t-\tau_2(t))))^T R({}_{t_0}D_t^{\alpha}(x(t)-Cx(t-\tau_2(t)))).\end{aligned} \tag{10}$$

Thus, we obtain the following equalities:

$$\begin{aligned}2(x(t)-&Cx(t-\tau_2(t)))^T P {}_{t_0}D_t^{\alpha}(x(t)-Cx(t-\tau_2(t)))\\ &=2(x(t)-Cx(t-\tau_2(t)))^T P(Ax(t)+Bx(t-\tau_1(t)))\\ &=x^T(t)(PA+A^TP)x(t)-2x^T(t-\tau_2(t))C^TPAx(t)\\ &\quad+2x^T(t)PBx(t-\tau_1(t))-2x^T(t-\tau_2(t))C^TPBx(t-\tau_1(t))\end{aligned} \tag{11}$$

and

$$\begin{aligned}m({}_{t_0}&D_t^{\alpha}(x(t)-Cx(t-\tau_2(t))))^T R({}_{t_0}D_t^{\alpha}(x(t)-Cx(t-\tau_2(t))))\\ &=m\left[Ax(t)+Bx(t-\tau_1(t))\right]^T R\left[Ax(t)+Bx(t-\tau_1(t))\right]\\ &=mx^T(t)A^TRAx(t)+mx^T(t)A^TRBx(t-\tau_1(t))+mx^T(t-\tau_1(t))B^TRAx(t)\\ &\quad+mx^T(t-\tau_1(t))B^TRBx(t-\tau_1(t)).\end{aligned} \tag{12}$$

From (10), (11) and (12), one has

$$\dot{V}(t)\le \xi^T N\xi, \tag{13}$$

where

$$N=\begin{pmatrix} N_{11} & N_{12} & N_{13}\\ N_{12}^T & N_{22} & N_{23}\\ N_{13}^T & N_{23}^T & N_{33}\end{pmatrix}<0,$$

$$\begin{aligned}&N_{11}=PA+A^TP+Q_1+Q_2+mA^TRA,\\ &N_{12}=PB+mA^TRB,\\ &N_{13}=-A^TPC,\\ &N_{22}=mB^TRB-(1-d_1)Q_1,\\ &N_{23}=-B^TPC,\\ &N_{33}=-(1-d_2)Q_2,\end{aligned}$$

and

$$\xi=(x^T(t),x^T(t-\tau_1(t)),x^T(t-\tau_2(t)))^T.$$

From inequality (8), since $\dot{V}(t)$ is negative definite the trivial solution of system (1) is asymptotically stable. ■

3. Numerical Examples

We give the following two examples to demonstrate the proposed technique.

Example 3.1.

Let the fractional system is given by:

$$_{t_0}D_t^{\alpha}x(t) = Ax(t) + Bx(t-\tau_1(t)) + C\,_{t_0}D_t^{\alpha}x(t-\tau_2(t)), \tag{14}$$

where $\alpha \in (0,1)$, $\tau_1(t) = 0.7t$, $\tau_2(t) = 6 + 0.4\sin(t)$

$$A = \begin{bmatrix} -10 & 2 \\ 0 & -6 \end{bmatrix}, \quad B = \begin{bmatrix} 0.3 & 0.2 \\ 0 & 0.1 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.1 \end{bmatrix}.$$

Let us choose $d_1 = 0.75,\ d_2 = 0.5$, $m = 6.5$,

$$P = \begin{bmatrix} 272.6339 & 0 \\ 0 & 272.6339 \end{bmatrix}, \quad Q = \begin{bmatrix} 607.2883 & 0 \\ 0 & 607.2883 \end{bmatrix},$$
$$R_1 = \begin{bmatrix} 30.0517 & 0 \\ 0 & 30.0517 \end{bmatrix}, \quad R_2 = \begin{bmatrix} 2.0625 & 0 \\ 0 & 2.0625 \end{bmatrix}.$$

A straightforward calculation shows that

$$M = \begin{pmatrix} -499.5947 & -323.8912 & -48.5837 & -32.3891 & -32.3891 & 0 \\ -323.8912 & -926.0005 & 26.0748 & 18.5718 & 17.3832 & 1.1886 \\ -48.5837 & 26.0748 & -147.9109 & 2.6075 & 2.6075 & 0 \\ -32.3891 & 18.5718 & 2.6075 & -149.6492 & 1.7383 & 0.4346 \\ -32.3891 & 17.3832 & 2.6075 & 1.7383 & -13.2875 & 0 \\ 0 & 1.1886 & 0 & 0.4346 & 0 & -14.5913 \end{pmatrix} < 0.$$

Due to condition (2) and Theorem 2.1, the trivial solution of system (14) is asymptotically stable.

Example 3.2.

Let the fractional system be given by:

$$_{t_0}D_t^{\alpha}x(t) = Ax(t) + Bx(t-\tau_1(t)) + C\,_{t_0}D_t^{\alpha}x(t-\tau_2(t)), \tag{15}$$

where $\alpha \in (0,1)$, $\tau_1(t) = 0.6t$, $\tau_2(t) = 4 + 0.2\sin(t)$,

$$A = \begin{bmatrix} -5 & 1 \\ 0 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 0.5 & 0.2 \\ 0 & 0.5 \end{bmatrix}, \quad C = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.2 \end{bmatrix}.$$

Let us choose $d_1 = 0.65,\ d_2 = 0.3$ and $m = 5$,

$$P = \begin{bmatrix} 0.6183 & 0 \\ 0 & 0.6183 \end{bmatrix}, \quad Q_1 = \begin{bmatrix} 0.5014 & 0 \\ 0 & 0.5014 \end{bmatrix},$$
$$Q_2 = \begin{bmatrix} 0.8418 & 0 \\ 0 & 0.8418 \end{bmatrix}, \quad R = \begin{bmatrix} 0.0295 & 0 \\ 0 & 0.0295 \end{bmatrix}.$$

A straightforward calculation shows that

$$N = \begin{pmatrix} -1.1523 & -0.1192 & -0.0596 & -0.0238 & 0.3092 & 0 \\ -0.1192 & -0.8916 & 0.0737 & 0.1174 & -0.0618 & 0.3710 \\ -0.0596 & 0.0737 & -0.1386 & 0.0147 & -0.0309 & 0 \\ -0.0238 & 0.1174 & 0.0147 & -0.1327 & -0.0124 & -0.0618 \\ 0.3092 & -0.0618 & -0.0309 & -0.0124 & -0.5893 & 0 \\ 0 & 0.3710 & 0 & -0.0618 & 0 & -0.5893 \end{pmatrix} < 0.$$

Due to condition (8) and Theorem 2.2, the trivial solution of system (15) is asymptotically stable.

4. Conclusion

In this paper, we use Lyapunov-Krasovskii method to analyze Riemann-Liouville fractional neutral systems with multiple time-varying delays, sufficient conditions on asymptotical stability are obtained by using linear matrix equation inequality. The obtained sufficient conditions are expressed in terms of LMI to find the less conservative criteria and can be easily solved. The most important advantage of the method used is that we can take integer order derivatives of Lyapunov functional to comparing to the fractional Lyapunov stability theorem. It is showed that it is convenient and efficient to check stability of practical fractional systems with the given two examples by using the Matlab-LMI Toolbox.

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