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Results on Fekete-Szegö Problem for Some New Subclasses of Univalent Analytic Functions with Fractional-Order Operators

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Abstract

We introduce some new subclasses of analytic functions which are univalent in an open unit disk by means of fractional calculus. The elemental interest is to explore the significance of fractional-order operators while formulating a few distinct subclasses of univalent analytic functions. Present work establishes the Fekete-Szegö inequality for the proposed subclasses. In addition, some classical Fekete-Szegö problems have also been retrieved and discussed as particular cases of the presented work. To make the suggested work more evident, an extremal function is also provided for which a sharp upper bound is attained.

Keywords: Fekete-Szegö inequality; Fractional-order operators; Analytic functions; Univalent functions; Spirallike functions; Starlike functions; Convex functions

MSC 2010 No.: 26A33, 26D07, 30A10

1. Introduction

This work aims to construct some general subclasses of univalent analytic functions with the assistance of fractional-order derivatives. The notion of derivative/integral is an extensively relevant concept in all fields of science and engineering. With the origin of fractional calculus, the order of derivative/integral no longer needs to be an integer but any real or complex number. This noticeable innovation of describing a non-integer order derivative has successfully achieved the attention of researchers to look for the admissible applications and physical properties. And thus, researchers are efficiently looking for a possible extension of existing problems in the sense of fractional calculus. The field of fractional calculus has already sustained its significance by finding applications not only in pure mathematics but in various branches of science and engineering. For instance, Osler (1971) gave the generalized Taylor's series expansion for fractional derivatives. Some of the potential problems include Cauchy's integral formula for an analytic function of fractional-order presented by Jumarie (2010), a numerical scheme for generalized fractional optimal control problems given by Singha and Nahak (2016), fractional derivatives in complex plane discussed by Li et al. (2009), generalized fractional derivative operator by Gaboury et al. (2013), fractionally convex functions introduced by Singha and Nahak (2020), etc.

This modernization of classical problems and properties motivates us to impose a more general structure on univalent analytic functions. We introduce some subclasses of locally univalent analytic functions by employing fractional-order operators. For definitions and basic properties of fractional order derivative in the complex plane, we refer the reader to Srivastava and Owa (1989).

To achieve the motive, we replace the classical derivatives with fractional derivatives in the geometric property of the functions and thus obtain the Fekete-Szegö inequality. This improvisation makes an effort to extend the results on Fekete-Szegö problem already discussed in the literature by Darus and Thomas (1996) and Darus and Thomas (2000). The Fekete-Szegö problem introduced by Fekete and Szegö (1933) is to find a bound for a relationship between the second and third coefficients of analytic functions having a specified geometric property. For related results on already existing subclasses of univalent analytic functions, one may look at Obradovic et al. (2013) and references therein.

2. Motivation

We start with a class $\mathcal{A}$ of analytic functions, having a power series representation, defined over an open unit disk $\mathbb{D}$. And, let $\mathcal{S}$ denote the subclass of functions in $\mathcal{A}$, which are univalent in $\mathbb{D}$. Further, we denote $f^{\lambda}(z)$ as the λ times derivative of the function $f(z) \in \mathcal{S}$, that is, the fractional derivative of the complex-valued function $f(z)$. The globally accepted notation of fractional derivative of complex-valued function f is ${}_0D_z^{\lambda} f(z)$. But to reduce the complexity of notation in computations, we adopt $f^{\lambda}(z)$ as the fractional-order derivative of f.

To provide a relevant motivation for the proposed work, we examine the geometric properties of the function $g(z) = z^{\lambda} f^{\lambda}(z)$ for $0 < \lambda < 1$. We now proceed to check the conditions for the function g to be starlike (for definition and properties of a starlike function, we refer the reader to Obradovic et al. (2013)). For the function g to be a starlike function of order α, we must have

$$\mathrm{Re}\left(\lambda + z\frac{f^{\lambda+1}(z)}{f^{\lambda}(z)}\right) > \alpha, \quad z \in \mathbb{D}, \quad \lambda \in (0,1). \tag{1}$$

In particular, by substituting $\lambda = 1$ in Equation (1), the function f turns out to be a convex function of order α (see Ponnusamy and Singh (1996) and Ponnusamy (2005)). Futher, by taking $\lambda = 0$ in Equation (1), the function f is a starlike function of order α (see Obradovic et al. (2013)). Thus, one may note that the investigation of geometric properties of functions involving fractional-order derivatives of univalent analytic functions f (like $g(z) = z^{\lambda} f^{\lambda}(z)$) assists in retrieving the geometric properties of f as particular cases. The following sections of the paper are focused on introducing certain subclasses of $\mathcal{S}$ and deriving the results for Fekete-Szegö problem.

3. Mathematical Formulation

For $0 < \alpha \leq 1$ and $0 < \lambda < 1$, we introduce the subclasses $\mathcal{F}^{\lambda}(\alpha)$, $\mathcal{G}^{\lambda}(\alpha)$ and $\mathcal{H}^{\lambda}(\alpha)$ of normalized analytic functions $f \in \mathcal{S}$ as explained below.

The class $\mathcal{F}^{\lambda}(\alpha)$ is defined as:

$$\mathcal{F}^{\lambda}(\alpha) = \left\{ f \in \mathcal{S} : a_0 = 0, a_1 = 1, \text{ and } \mathrm{Re}\left(\lambda + \frac{\mathrm{z\,f^{\lambda+1}(z)}}{\mathrm{f^{\lambda}(z)}}\right) > \frac{\alpha}{2} - 1 \right\}. \tag{2}$$

Note:

- Let $g(z) = z^{\lambda} f^{\lambda}(z)$, where $f \in \mathcal{F}^{\lambda}(\alpha)$. Thus,

$$\begin{aligned}\mathrm{Re}\left(\frac{zg'(z)}{g(z)}\right) &= \mathrm{Re}\left(\mathrm{z} \cdot \frac{\lambda\,\mathrm{z^{\lambda-1}\,f^{\lambda}(z) + z^{\lambda}\,f^{\lambda+1}(z)}}{\mathrm{z^{\lambda}\,f^{\lambda}(z)}}\right)\\ &= \mathrm{Re}\left(\frac{\lambda\, z^{\lambda}\, f^{\lambda}(z) + z^{\lambda+1}\, f^{\lambda+1}(z)}{z^{\lambda}\, f^{\lambda}(z)}\right)\\ &= \mathrm{Re}\left(\lambda + \frac{z\, f^{\lambda+1}(z)}{f^{\lambda}(z)}\right)\\ &> \frac{\alpha}{2} - 1, \quad \text{since } \ f \in \mathcal{F}^{\lambda}(\alpha),\end{aligned}$$

  and the function g turns out to be a starlike function of order γ, $\gamma = \frac{\alpha}{2} - 1$.

- By substituting $\lambda = 0$ in (2), the class $\mathcal{F}^{\lambda}(\alpha)$ reduces to a class of starlike functions of order $\gamma = \frac{\alpha}{2} - 1$, since $\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \frac{\alpha}{2} - 1$.

- On taking $\lambda = 1$ in (2), the class $\mathcal{F}^{\lambda}(\alpha)$ reduces to a class $\mathcal{F}(\alpha)$ of normalized analytic convex functions of order $\gamma = \frac{\alpha}{2} - 1$, since $\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \frac{\alpha}{2} - 1$. Further, if we take $\alpha = 1$, then $\mathcal{F}(\alpha)$ reduces to the class $\mathcal{F}(1)$ of normalized analytic convex functions $f \in \mathcal{S}$ of order $-\frac{1}{2}$, since $\operatorname{Re}\left(1 + z\frac{f''(z)}{f'(z)}\right) > -\frac{1}{2}$. The extremal function for this class $\mathcal{F}(1)$ has already been explored by Muhanna et al. (2014).

The class $\mathcal{G}^{\lambda}(\alpha)$ is defined as:

$$\mathcal{G}^{\lambda}(\alpha) = \left\{ f \in \mathcal{S} : a_0 = 0, a_1 = 1, \text{ and } \operatorname{Re}\left(\lambda + \frac{zf^{\lambda+1}(z)}{f^{\lambda}(z)}\right) < 1 + \frac{\alpha}{2} \right\}. \tag{3}$$

Note:

- For $\lambda = 1$, the class $\mathcal{G}^{\lambda}(\alpha)$ in (3) reduces to the class $\mathcal{G}(\alpha)$, as $\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) < 1 + \frac{\alpha}{2}$. The coefficient characterization for this class $\mathcal{G}(\alpha)$ has been discussed by Obradovic et al. (2013).

- On substituting $\lambda = 1 = \alpha$ in (3), $\operatorname{Re}\left(1 + z\frac{f''(z)}{f'(z)}\right) < \frac{3}{2}$. That is, the class reduces to the class $\mathcal{G}^{\lambda}(\alpha)$ reduces to the class $\mathcal{G}(1)$ of locally univalent normalized analytic function f in the unit disk $\mathbb{D}$, presented by Ozaki (1941).

For $|\alpha| < \frac{\pi}{2}$, the class $\mathcal{H}^{\lambda}(\alpha)$ is defined as:

$$\mathcal{H}^{\lambda}(\alpha) = \left\{ f \in \mathcal{S} : a_0 = 0, a_1 = 1, \text{ and } \operatorname{Re}\left(e^{i\alpha}\left(\lambda + \frac{z\, f^{\lambda+1}(z)}{f^{\lambda}(z)}\right)\right) > 0 \right\}. \tag{4}$$

Note:

- Let us consider $g(z) = z^{\lambda}\, f^{\lambda}(z)$, where $f \in \mathcal{H}^{\lambda}(\alpha)$. Then, we can write

$$\begin{aligned} \operatorname{Re}\left(e^{i\alpha}\left(\frac{z\, g'(z)}{g(z)}\right)\right) &= \operatorname{Re}\left(e^{i\alpha}\left(z \cdot \frac{\lambda\, z^{\lambda-1}\, f^{\lambda}(z) + z^{\lambda}\, f^{\lambda+1}(z)}{z^{\lambda}\, f^{\lambda}(z)}\right)\right) \\ &= \operatorname{Re}\left(e^{i\alpha}\left(\lambda + \frac{z\, f^{\lambda+1}(z)}{f^{\lambda}(z)}\right)\right) \\ &> 0, \quad \text{since } f \in \mathcal{H}^{\lambda}(\alpha), \end{aligned}$$

  that is, $\mathcal{H}^{\lambda}(\alpha)$ is the class of all normalized analytic functions $f \in \mathcal{S}$ for which $z^{\lambda} f^{\lambda}(z)$ is α-spirallike.

- By taking $\lambda = 0$ in (4),
$$\mathcal{H}^0(\alpha) = \left\{ f \in \mathcal{S} : a_0 = 0, a_1 = 1, \text{ and } \operatorname{Re}\left(e^{i\alpha} \left(\frac{z\,f'(z)}{f(z)} \right) \right) > 0 \right\},$$
which represents a class of all normalized analytic functions $f \in \mathcal{S}$ for which f is α-spirallike.

- By taking $\lambda = 1$ in (4), $\operatorname{Re}\left(e^{i\alpha} \left(1 + \frac{z\,f''(z)}{f'(z)} \right) \right) > 0$, which implies that the class $\mathcal{H}^\lambda(\alpha)$ reduces to the class $\mathcal{H}(\alpha)$ for which the function $g = zf'(z)$ is α-spirallike. For a detailed review on the class $\mathcal{H}(\alpha)$, we refer the reader to Robertson (1969).

4. Results and Discussions

We shall now determine the bound for the Fekete-Szegö functional $(a_3 - \mu a_2^2)$; $\mu \in \mathbb{C}$ for the classes $\mathcal{F}^\lambda(\alpha), \mathcal{G}^\lambda(\alpha)$ and $\mathcal{H}^\lambda(\alpha)$. The sharp bound is obtained for the class $\mathcal{F}^\lambda(\alpha)$ by giving an extremal function.

Theorem 4.1.

Let $f \in \mathcal{F}^\lambda(\alpha)$ for some $0 < \alpha \leq 1$ and $0 < \lambda < 1$. For $\mu \in \mathbb{C}$,
$$|a_3 - \mu {a_2}^2| \leq \begin{cases} \left(\frac{3+\lambda-\alpha}{6}\right)(2-\lambda), & \text{if } \left|\mu - \frac{2}{3}\left(\frac{4+\lambda-\alpha}{3+\lambda-\alpha}\right)\right| \leq \frac{2}{3}\left(\frac{2-\lambda}{3+\lambda-\alpha}\right), \\ \left(\frac{3+\lambda-\alpha}{6}\right)\left|1+\left(1-\frac{3}{2}\mu\right)(3+\lambda-\alpha)\right|, & \text{if } \left|\mu - \frac{2}{3}\left(\frac{4+\lambda-\alpha}{3+\lambda-\alpha}\right)\right| > \frac{2}{3}\left(\frac{2-\lambda}{3+\lambda-\alpha}\right). \end{cases}$$

The above inequality is sharp for the following function,
$$f(z) = \frac{1}{(2+\lambda-\alpha)}(1-z)^{-(2+\lambda-\alpha)} - \frac{1}{(2+\lambda-\alpha)}\,.$$

Proof:

Let $f \in \mathcal{F}^\lambda(\alpha)$. Then, we can write
$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \; a_0 = 0, \; a_1 = 1, \tag{5}$$
and
$$\operatorname{Re}\left(\lambda + \frac{zf^{\lambda+1}(z)}{f^\lambda(z)} \right) > \gamma, \quad \gamma = \frac{\alpha}{2} - 1. \tag{6}$$

By taking $g(z) = z^\lambda f^\lambda(z)$, we have
$$\operatorname{Re}\left(\frac{zg'(z)}{g(z)} \right) > \gamma, \qquad z \in \mathbb{D},$$
that is,
$$\operatorname{Re}\left(\frac{\frac{zg'(z)}{g(z)} - \gamma}{\lambda - \gamma} \right) > 0, \quad z \in \mathbb{D}.$$

Therefore, there exists an analytic function $w : \mathbb{D} \to \overline{\mathbb{D}}$, such that

$$\frac{\frac{zg'(z)}{g(z)} - \gamma}{\lambda - \gamma} = \frac{1 + zw(z)}{1 - zw(z)},$$

or

$$zg'(z) - \lambda g(z) = [zg'(z) + (\lambda - 2\gamma)g(z)]zw(z), \quad z \in \mathbb{D}.$$

By using $g(z) = z^{\lambda} f^{\lambda}(z)$, we arrive at

$$z^{\lambda+1} f^{\lambda+1}(z) = [z^{\lambda+1} f^{\lambda+1}(z) + 2(\lambda - \gamma) z^{\lambda} f^{\lambda}(z)]zw(z), \quad z \in \mathbb{D}.$$

Since $\gamma = \frac{\alpha}{2} - 1$, we can write

$$z^{\lambda+1} f^{\lambda+1}(z) = [z^{\lambda+1} f^{\lambda+1}(z) + (2\lambda - \alpha + 2) z^{\lambda} f^{\lambda}(z)]zw(z), \quad z \in \mathbb{D}.$$

Let us assume

$$w(z) = \sum_{n=0}^{\infty} c_n z^n. \tag{7}$$

By using Equations (5) and (7), we have

$$z^{\lambda} f^{\lambda}(z) = \sum_{n=0}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n-\lambda+1)} a_n z^n = \sum_{n=0}^{\infty} A_n z^n; \quad A_n = \frac{\Gamma(n+1)}{\Gamma(n-\lambda+1)} a_n,$$

$$z^{\lambda+1} f^{\lambda+1}(z) = \sum_{n=0}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n-\lambda)} a_n z^n = \sum_{n=0}^{\infty} B_n z^n; \quad B_n = \frac{\Gamma(n+1)}{\Gamma(n-\lambda)} a_n,$$

$$z^{\lambda} f^{\lambda}(z) w(z) = \sum_{n=0}^{\infty} A_n z^n \sum_{n=0}^{\infty} c_n z^n = \sum_{n=0}^{\infty} D_n z^n; \quad D_n = \sum_{k=0}^{n} A_k \, c_{n-k},$$

$$z^{\lambda+1} f^{\lambda+1}(z) w(z) = \sum_{n=0}^{\infty} B_n \, z^n \sum_{n=0}^{\infty} c_n z^n = \sum_{n=0}^{\infty} E_n z^n; \quad E_n = \sum_{k=0}^{n} B_k \, c_{n-k}.$$

Thus, Equation (7) can be rewritten as

$$\sum_{n=0}^{\infty} B_n z^n = \sum_{n=0}^{\infty} E_n z^{n+1} + (2\lambda - \alpha + 2) \sum_{n=0}^{\infty} D_n z^{n+1}.$$

On comparing the coefficients of z^2 and z^3 in Equation (8), we get

$$a_2 = \left(\frac{3+\lambda-\alpha}{2}\right) c_0,$$
$$a_3 = \left(\frac{3+\lambda-\alpha}{6}\right) [(2-\lambda)c_1 + (4+\lambda-\alpha)c_0^2].$$

For $\mu \in \mathbb{C}$ (using the values of a_2 and a_3),

$$\left|a_3 - \mu a_2^2\right| = \left(\frac{3+\lambda-\alpha}{6}\right) \left|(2-\lambda)c_1 + \mathcal{K}c_0^2\right|,$$

where

$$\mathcal{K} = (4+\lambda-\alpha) - \frac{3}{2}\mu(3+\lambda-\alpha).$$

By using $|c_0| \leq 1$ and $|c_1| \leq 1 - |c_0^2|$, we arrive at

$$\begin{aligned} \left|a_3 - \mu a_2^2\right| &\leq \left(\frac{3+\lambda-\alpha}{6}\right) \left[(2-\lambda) + (|\mathcal{K}| - (2-\lambda))\, |c_0|^2\right] \\ &\leq \left(\frac{3+\lambda-\alpha}{6}\right) (2-\lambda), \quad \text{if } \left|\frac{\mathcal{K}}{2-\lambda}\right| \leq 1 \\ &\leq \left(\frac{3+\lambda-\alpha}{6}\right) |\mathcal{K}|, \quad \text{if } \left|\frac{\mathcal{K}}{2-\lambda}\right| > 1. \end{aligned}$$

Finally, we see that

$$\left|\frac{\mathcal{K}}{2-\lambda}\right| \leq 1,$$
$$\Leftrightarrow \left|\mu - \frac{2}{3}\left(\frac{4+\lambda-\alpha}{3+\lambda-\alpha}\right)\right| \leq \frac{2}{3}\left(\frac{2-\lambda}{3+\lambda-\alpha}\right).$$

At last, we arrive at the following inequality,

$$|a_3 - \mu {a_2}^2| \leq \begin{cases} \left(\frac{3+\lambda-\alpha}{6}\right)(2-\lambda), & \text{if } \left|\mu - \frac{2}{3}\left(\frac{4+\lambda-\alpha}{3+\lambda-\alpha}\right)\right| \leq \frac{2}{3}\left(\frac{2-\lambda}{3+\lambda-\alpha}\right), \\ \left(\frac{3+\lambda-\alpha}{6}\right)\left|1+\left(1-\frac{3}{2}\mu\right)(3+\lambda-\alpha)\right|, & \text{if } \left|\mu - \frac{2}{3}\left(\frac{4+\lambda-\alpha}{3+\lambda-\alpha}\right)\right| > \frac{2}{3}\left(\frac{2-\lambda}{3+\lambda-\alpha}\right). \end{cases} \tag{8}$$

We shall now obtain the sharp inequality in Equation (8) by defining the function f as follows,

$$f(z) = \frac{1}{(2+\lambda-\alpha)}(1-z)^{-(2+\lambda-\alpha)} - \frac{1}{(2+\lambda-\alpha)}. \tag{9}$$

Here, $f(0)=0,\ f'(0)=1$, and

$$\text{Re}\left(\lambda + z\frac{f^{\lambda+1}(z)}{f^{\lambda}(z)}\right) > \frac{\alpha}{2} - 1. \tag{10}$$

Since the calculation consists of various complex terms involving hypergeometric functions, we use Mathematica Software to visualize the property in Equation (10). We get

$$a_2 = \frac{f''(0)}{2} = \frac{(3+\lambda-\alpha)}{2},$$
$$a_3 = \frac{f'''(0)}{6} = \frac{(3+\lambda-\alpha)(4+\lambda-\alpha)}{6}.$$

For $\mu \in \mathbb{C}$,

$$\begin{aligned} |a_3 - \mu a_2^2| &= \left|\frac{(3+\lambda-\alpha)(4+\lambda-\alpha)}{6} - \mu\frac{(3+\lambda-\alpha)^2}{4}\right|, \\ &= \frac{(3+\lambda-\alpha)}{6}\left|(4+\lambda-\alpha) - \frac{3}{2}\mu(3+\lambda-\alpha)\right|, \\ &= \frac{(3+\lambda-\alpha)}{6}|\mathcal{K}|, \end{aligned}$$

where $\mathcal{K} = (4+\lambda-\alpha) - \frac{3}{2}\mu(3+\lambda-\alpha)$. ■

Corollary 4.1.

If $\lambda = 1$ in Theorem 4.1, $f \in \mathcal{F}(\alpha)$ for some $0 < \alpha \le 1$. And, for $\mu \in \mathbb{C}$,

$$|a_3 - \mu {a_2}^2| \le \begin{cases} \left(\frac{4-\alpha}{6}\right), & \text{if } \left|\mu - \frac{2}{3}\left(\frac{5-\alpha}{4-\alpha}\right)\right| \le \frac{2}{3}\left(\frac{1}{4-\alpha}\right), \\ \left(\frac{4-\alpha}{6}\right)\left|1 + \left(1-\frac{3}{2}\mu\right)(4-\alpha)\right|, & \text{if } \left|\mu - \frac{2}{3}\left(\frac{5-\alpha}{4-\alpha}\right)\right| > \frac{2}{3}\left(\frac{1}{4-\alpha}\right). \end{cases}$$

The sharp inequality is attained by the function

$$f(z) = \frac{1}{(3-\alpha)}(1-z)^{-(3-\alpha)} - \frac{1}{(3-\alpha)}.$$

Proof:

On substituting $\lambda = 1$ in Equations (8)–(9), we obtain the required result. ■

Corollary 4.2.

If $\lambda = 1 = \alpha$, $f \in \mathcal{F}(1)$. For $\mu \in \mathbb{C}$,

$$|a_3 - \mu {a_2}^2| \le \begin{cases} \frac{1}{2}, & \text{if } \left|\mu - \frac{8}{9}\right| \le \frac{2}{9}, \\ \left|2 - \frac{9}{4}\mu\right|, & \text{if } \left|\mu - \frac{8}{9}\right| > \frac{2}{9}. \end{cases}$$

The sharp inequality is attained by the function

$$f(z) = \frac{1}{2}(1-z)^{-2} - \frac{1}{2}.$$

Proof:

The proof follows immediately by substituting $\lambda = 1 = \alpha$ in equations (8)–(9). ∎

Theorem 4.2.

Let $f \in \mathcal{G}^{\lambda}(\alpha)$ for some $0 < \alpha \leq 1$ and $0 < \lambda < 1$. For $\mu \in \mathbb{C}$,

$$|a_3 - \mu {a_2}^2| \leq \begin{cases} \left(\frac{1-\lambda+\alpha}{6}\right)(2-\lambda), & \text{if } \left|\mu + \frac{2}{3}\left(\frac{\lambda-\alpha}{1-\lambda+\alpha}\right)\right| \leq \frac{2}{3}\left(\frac{2-\lambda}{1-\lambda+\alpha}\right), \\ \left(\frac{1-\lambda+\alpha}{6}\right)|(\lambda-\alpha) + \frac{3}{2}\mu(1-\lambda+\alpha)|, & \text{if } \left|\mu + \frac{2}{3}\left(\frac{\lambda-\alpha}{1-\lambda+\alpha}\right)\right| > \frac{2}{3}\left(\frac{2-\lambda}{1-\lambda+\alpha}\right). \end{cases}$$

Proof:

Let $f \in \mathcal{G}^{\lambda}(\alpha) \Rightarrow f(z) = \sum_{n=0}^{\infty} a_n z^n$ with $a_0 = 0,\ a_1 = 1$, and

$$\operatorname{Re}\left(\lambda + \frac{zf^{\lambda+1}(z)}{f^{\lambda}(z)}\right) < \gamma = 1 + \frac{\alpha}{2}.$$

Take $g(z) = z^{\lambda} f^{\lambda}(z)$. Thus,

$$\operatorname{Re}\left(\frac{\gamma - \frac{zg'(z)}{g(z)}}{\gamma - \lambda}\right) > 0, \quad z \in \mathbb{D}.$$

Therefore, there exists an analytic function $w : \mathbb{D} \to \overline{\mathbb{D}}$, such that

$$\frac{\gamma - \frac{zg'(z)}{g(z)}}{\gamma - \lambda} = \frac{1 + zw(z)}{1 - zw(z)},$$
$$\Rightarrow zg'(z) - \lambda g(z) = [zg'(z) + (\lambda - 2\gamma)g(z)]zw(z), \quad z \in \mathbb{D}.$$

Using $g(z) = z^{\lambda} f^{\lambda}(z)$, we get

$$z^{\lambda+1} f^{\lambda+1}(z) = [z^{\lambda+1} f^{\lambda+1}(z) + 2(\lambda - \gamma) z^{\lambda} f^{\lambda}(z)]zw(z), \quad z \in \mathbb{D}.$$

Since $\gamma = 1 + \frac{\alpha}{2}$, we can write

$$z^{\lambda+1} f^{\lambda+1}(z) = [z^{\lambda+1} f^{\lambda+1}(z) + (2\lambda - \alpha - 2) z^{\lambda} f^{\lambda}(z)]zw(z), \quad z \in \mathbb{D}. \tag{11}$$

Let us assume

$$w(z) = \sum_{n=0}^{\infty} c_n z^n.$$

(Proceeding as in Theorem 1) On comparing the coefficients of z^2 and z^3 in Equation (11), we get

$$a_2 = \left(\frac{\lambda - \alpha - 1}{2}\right) c_0,$$

and

$$a_3 = \left(\frac{\lambda - \alpha - 1}{6}\right) [(2-\lambda)c_1 + (\lambda - \alpha)c_0^2].$$

Thus, we arrive at

$$\left|a_3 - \mu a_2^2\right| = \left(\frac{1 - \lambda + \alpha}{6}\right) \left|(2-\lambda)c_1 + \mathcal{K}c_0^2\right|,$$

where

$$\mathcal{K} = (\lambda - \alpha) + \frac{3}{2}\mu(1 - \lambda + \alpha).$$

By using $|c_0| \leq 1$ and $|c_1| \leq 1 - |c_0^2|$, we arrive at

$$\begin{aligned}\left|a_3 - \mu a_2^2\right| &\leq \left(\frac{1 - \lambda + \alpha}{6}\right) \left[(2-\lambda) + (|\mathcal{K}| - (2-\lambda))\,|c_0|^2\right] \\ &\leq \left(\frac{1 - \lambda + \alpha}{6}\right)(2-\lambda), \quad \text{if } \left|\frac{\mathcal{K}}{2-\lambda}\right| \leq 1,\end{aligned}$$

and

$$\left|a_3 - \mu a_2^2\right| \leq \left(\frac{1 - \lambda + \alpha}{6}\right)|\mathcal{K}|, \quad \text{if } \left|\frac{\mathcal{K}}{2-\lambda}\right| > 1.$$

Finally, we see that

$$\left|\frac{\mathcal{K}}{2-\lambda}\right| \leq 1,$$

$$\Leftrightarrow \left|\mu + \frac{2}{3}\left(\frac{\lambda - \alpha}{1 - \lambda + \alpha}\right)\right| \leq \frac{2}{3}\left(\frac{2-\lambda}{1-\lambda+\alpha}\right).$$

Hence, we obtain

$$|a_3 - \mu {a_2}^2| \leq \begin{cases} \left(\frac{1-\lambda+\alpha}{6}\right)(2-\lambda), & \text{if } \left|\mu + \frac{2}{3}\left(\frac{\lambda-\alpha}{1-\lambda+\alpha}\right)\right| \leq \frac{2}{3}\left(\frac{2-\lambda}{1-\lambda+\alpha}\right), \\ \left(\frac{1-\lambda+\alpha}{6}\right)\left|(\lambda-\alpha) + \frac{3}{2}\mu(1-\lambda+\alpha)\right|, & \text{if } \left|\mu + \frac{2}{3}\left(\frac{\lambda-\alpha}{1-\lambda+\alpha}\right)\right| > \frac{2}{3}\left(\frac{2-\lambda}{1-\lambda+\alpha}\right). \end{cases}$$

■

Corollary 4.3.

If $\lambda = 1$ in Theorem 4.2, $f \in \mathcal{G}(\alpha)$ for some $0 < \alpha \leq 1$. Then, we have

$$|a_3 - \mu {a_2}^2| \leq \begin{cases} \frac{\alpha}{6}, & \text{if } \left|\mu + \frac{2(1-\alpha)}{3\alpha}\right| \leq \frac{2}{3\alpha}, \\ \frac{\alpha}{6}\left|(1-\alpha) + \frac{3}{2}\mu\alpha\right|, & \text{if } \left|\mu + \frac{2(1-\alpha)}{3\alpha}\right| > \frac{2}{3\alpha}. \end{cases}$$

Proof:

The proof follows immediately by substituting $\lambda = 1$ in Theorem 4.2. ∎

Corollary 4.4.

If $\lambda = 1 = \alpha$, $f \in \mathcal{G}(1)$ for some $0 < \alpha \leq 1$. Then, we have

$$|a_3 - \mu {a_2}^2| \leq \begin{cases} \frac{1}{6}, & \text{if } \left|\mu + \frac{2}{3}\right| \leq \frac{2}{3}, \\ \frac{1}{4}\left|\mu\right|, & \text{if } \left|\mu + \frac{2}{3}\right| > \frac{2}{3}. \end{cases}$$

Proof:

The proof follows trivially by substituting $\alpha = 1$ in Corollary 4.3. ∎

Theorem 4.3.

Let $f \in \mathcal{H}^{\lambda}(\alpha)$ for some $|\alpha| \leq \frac{\pi}{2}$ and $0 < \lambda < 1$. Then, we have

$$|a_3 - \mu {a_2}^2| \leq \begin{cases} \frac{(2-\lambda)(3-\lambda)}{6}, & \text{if } \left|\mu - \frac{2}{3}\left(\frac{2+e^{-2i\alpha}}{1+e^{-2i\alpha}}\right)\right| \leq \frac{3-\lambda}{6}, \\ \frac{(2-\lambda)}{3}\left|\frac{(3-\lambda)(2+e^{-2i\alpha})}{2} - \frac{3}{2}\mu(1+e^{-2i\alpha})\right|, & \text{if } \left|\mu - \frac{2}{3}\left(\frac{2+e^{-2i\alpha}}{1+e^{-2i\alpha}}\right)\right| > \frac{3-\lambda}{6}. \end{cases}$$

Proof:

Let $f \in \mathcal{H}^{\lambda}(\alpha) \Rightarrow f(z) = \sum_{n=0}^{\infty} a_n z^n$ with $a_0 = 0,\ a_1 = 1$, and

$$\text{Re}\left(e^{i\alpha}\left(\lambda + \frac{zf^{\lambda+1}(z)}{f^{\lambda}(z)}\right)\right) > 0.$$

Therefore, there exists an analytic function $w : \mathbb{D} \to \overline{\mathbb{D}}$ such that

$$e^{i\alpha}\left(\lambda + \frac{zf^{\lambda+1}(z)}{f^{\lambda}(z)}\right) = \cos\alpha\left[\frac{1+zw(z)}{1-zw(z)}\right] + i\,\sin\alpha,$$
$$\Rightarrow \lambda + \frac{zf^{\lambda+1}(z)}{f^{\lambda}(z)} = e^{-i\alpha}\cos\alpha\left[\frac{1+zw(z)}{1-zw(z)}\right] + ie^{-i\alpha}\,\sin\alpha.$$

On simplifying above expression

$$z^{\lambda+1}f^{\lambda+1}(z) - (1-\lambda)z^{\lambda}f^{\lambda}(z) = \left[(\lambda + e^{-2i\alpha})z^{\lambda}f^{\lambda}(z) + z^{\lambda+1}f^{\lambda+1}(z)\right]\,zw(z). \tag{12}$$

Here, we take

$$w(z) = \sum_{n=0}^{\infty} c_n z^n.$$

(As done previously in Theorem 4.1 and Theorem 4.2) On comparing the coefficients of z^2 and z^3 in (12), we obtain

$$a_2 = \frac{(2-\lambda)\left(1+e^{-2i\alpha}\right)}{2} c_0,$$

and

$$a_3 = \frac{(2-\lambda)(3-\lambda)\left(1+e^{-2i\alpha}\right)}{12}\left[c_1 + (2+e^{-2i\alpha})c_0^2\right].$$

Thus, we can write

$$\left|a_3 - \mu a_2^2\right| = \frac{(2-\lambda)\left(1+e^{-2i\alpha}\right)}{6}\left[\frac{(3-\lambda)}{2}c_1 + \mathcal{K}c_0^2\right],$$

where

$$\mathcal{K} = \frac{(3-\lambda)\left(2+e^{-2i\alpha}\right)}{2} - \frac{3}{2}\mu\left(1+e^{-2i\alpha}\right).$$

Using $|c_0| \leq 1$ and $|c_1| \leq 1 - |c_0^2|$, we obtain

$$\left|a_3 - \mu a_2^2\right| \leq \tfrac{(2-\lambda)}{3}\left[\tfrac{(3-\lambda)}{2} + \left(|\mathcal{K}| - \tfrac{(3-\lambda)}{2}\right)|c_0|^2\right]$$

$$\leq \frac{(2-\lambda)(3-\lambda)}{6}, \quad \text{if } |\mathcal{K}| \leq \frac{3-\lambda}{2},$$

and

$$\leq \frac{(2-\lambda)}{3}|\mathcal{K}|, \qquad \text{if } |\mathcal{K}| > \frac{3-\lambda}{2}.$$

Finally, we see that

$$|\mathcal{K}| \leq \frac{3-\lambda}{2},$$

$$\Leftrightarrow \left|\mu - \frac{2}{3}\left(\frac{2+e^{-2i\alpha}}{1+e^{-2i\alpha}}\right)\right| \leq \frac{3-\lambda}{6}.$$

Finally, we arrive at

$$|a_3 - \mu {a_2}^2| \leq \begin{cases} \frac{(2-\lambda)(3-\lambda)}{6}, & \text{if } \left|\mu - \frac{2}{3}\left(\frac{2+e^{-2i\alpha}}{1+e^{-2i\alpha}}\right)\right| \leq \frac{3-\lambda}{6}, \\ \frac{(2-\lambda)}{3}\left|\frac{(3-\lambda)(2+e^{-2i\alpha})}{2} - \frac{3}{2}\mu(1+e^{-2i\alpha})\right|, & \text{if } \left|\mu - \frac{2}{3}\left(\frac{2+e^{-2i\alpha}}{1+e^{-2i\alpha}}\right)\right| > \frac{3-\lambda}{6}. \end{cases}$$ ■

Corollary 4.5.

If $\lambda = 1$, $f \in \mathcal{H}(\alpha)$ for some $|\alpha| \leq \frac{\pi}{2}$. Then, we have

$$|a_3 - \mu {a_2}^2| \leq \begin{cases} \frac{1}{3}, & \text{if } \left|\mu - \frac{2}{3}\left(\frac{2+e^{-2i\alpha}}{1+e^{-2i\alpha}}\right)\right| \leq \frac{1}{3}, \\ \frac{1}{3}\left|1 + (1+e^{-2i\alpha})(1-\frac{3}{2}\mu)\right|, & \text{if } \left|\mu - \frac{2}{3}\left(\frac{2+e^{-2i\alpha}}{1+e^{-2i\alpha}}\right)\right| > \frac{1}{3}. \end{cases}$$

Proof:

The proof follows immediately by substituting $\lambda = 1$ in Theorem 4.3. ∎

5. Conclusion

The Fekete-Szegö inequality is obtained for the proposed classes of univalent analytic functions which are equipped with specified geometric properties with fractional-order derivatives. The benefit of utilizing fractional order derivatives is inspected while giving a more general structure to the posed subclasses. Some particular consequences are presented together with an extremal function of Fekete-Szegö inequality. In future, extremal functions for all the presented subclasses can also be discussed. Present work encourages us to look for several classes of univalent functions for which Fekete-Szegö inequality is not known. One may employ fractional derivatives in order to produce analogous properties and construct distinct subclasses of univalent analytic functions.

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REFERENCES

Darus, M. and Ibrahim, R.W. (2010). On generalisation of polynomials in complex plane, Adv. Decis. Sci., 9 pp.

Darus, M. and Thomas, D.K. (1996). On the Fekete-Szegö Theorem for close-to-convex functions, Math. Japonica, pp. 507–511.

Darus, M. and Thomas, D.K. (2000). The Fekete-Szegö Theorem for strongly close-to-convex functions, Scientiae Mathematicae, pp. 201–212.

Das, S. (2011). *Functional Fractional Calculus*, Second edition, Springer-Verlag, Berlin.

Fekete, M. and Szegö, G. (1933). Eine Bemerkung suber ungerade schlichte Funktionen, J. London Math. Soc., Vol. 8, pp. 85–89.

Gaboury, S., Tremblay, R. and Fugere, B. (2013). Some relations involving a generalized fractional derivative operator, J. Inequal. Appl., Vol. 167, 9 pp.

Jumarie, G. (2010). Cauchy's integral formula via the modified Riemann-Liouville derivative for analytic functions of fractional order, J. Appl. Math. Lett., Vol. 23, pp. 1444-1450.

Keogh, F.R. and Merkes, E.P. (1969). A coefficient inequality for certain classes of analytic functions, Proc. Amer. Math. Soc., Vol. 20, pp 8–12.

Koepf, W. (1987). On the Fekete-Szegö problem for close-to-convex functions, Proc. Amer. Math. Soc., pp. 89–95.

Li, C., Dao, X. and Guo, P. (2009). Fractional derivatives in complex planes, Nonlinear Analysis: Theory, Methods & Applications, Vol. 71, pp. 1857–1869.

London, R.R. (1993). Fekete-Szegö inequalities for close-to-convex functions, Proc. Amer. Math. Soc., Vol. 117, pp. 947–950.

Muhanna, Y.A., Li, L. and Ponnusamy, S. (2014). Extremal problems on the class of convex functions of order -1/2, Arch. Maths. (Basel), Vol. 103, pp. 421–471.

Obradovic, M., Ponnusamy, S. and Wirths, K.J. (2013). Coefficient characterizations and sections for some univalent functions, Siberian Mathematical Journal, pp. 679–696.

Osler, T.J. (1971). Taylor's series generalized for fractional derivatives and applications, SIAM J. Math. Anal., Vol. 1, pp. 37–48.

Ozaki, S. (1941). On the theory of multivalent functions II, Sci. Rep. Tokyo Bunrika Daigaku. Sect. A, Vol. 4, pp. 45–87.

Pommerenke, C. (1975). *Univalent Functions*, Vandenhoeck and Ruprecht, Göttingen.

Ponnusamy, S. and Singh, V. (1996). Univalence of certain integral transforms, Glas. Mat. Ser. III, Vol. 31, pp. 253–261.

Ponnusamy, S. (2005). *Foundations of Complex Analysis*, Alpha Science International Publishers, UK.

Robertson, M.S. (1969). Univalent functions $f(z)$ for which $zf'(z)$ is spirallike, Michigan Math. J., Vol. 16, pp. 97–101.

Singha, N. (2020). Implementation of fractional optimal control problems in real-world applications, Fract. Calc. Appl. Anal., Vol. 23, pp. 1783–1796.

Singha, N. and Nahak, C. (2016). A numerical scheme for generalized fractional optimal control problems, Appl. Appl. Math., Vol. 11, pp. 798–814.

Singha, N. and Nahak, C. (2017). An efficient approximation technique for solving a class of fractional optimal control problems, J. Optim. Theory Appl., Vol. 174, pp. 785–802.

Singha, N. and Nahak, C. (2019). Solutions of the generalized Abel's integral equation using Laguerre orthogonal approximation, Appl. Appl. Math., Vol. 14, pp. 1051–1066.

Singha, N. and Nahak, C. (2020). α-fractionally convex functions, Fract. Calc. Appl. Anal., Vol. 23, pp. 534–552.

Srivastava, H. and Owa, S. (1989). *Univalent Functions, Fractional Calculus and Their Applications*, Ellis Horwood Series: Mathematics and Its Applications, John Wiley & Sons, New York, USA.