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Numerical Solution of Differential Difference Equations With Two Boundary Layers

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Abstract:

In this paper, the numerical solution of differential-difference equation with two boundary layers is discussed. Using Taylor's series, the given second order differential-difference equation is replaced by an asymptotically equivalent first order differential equation and solved by suitable choice of integrating factor and finite difference approximations. Numerical results for several test examples are presented to demonstrate the applicability of the method.

Keywords: Differential difference equations; Boundary layers; Taylor series; Integrating factor; Finite differences

MSC(2010) No: 65L11, 65Q10

1. Introduction

Singularly perturbed differential difference equations having two boundary layers occur in fluid mechanics, skin layer problems in electrical application, edge layer problems in solid mechanics, WKB problems, the modelling of steady and unsteady viscous flow problems with large Reynolds numbers, convective heat transport problems with large Peclet numbers and by different names in other branches of science and engineering. If we apply existing numerical methods to these problems, we get oscillatory results or bad results, because of boundary layers. Lange and Miura (1994a,b) were the first to present numerical methods to solve singularly perturbed differential difference equations. Kadalbajoo and Sharma (2005) have presented numerical treatment of boundary value problems for second order singularly perturbed delay differential equations. Recently, Adilaxmi et al. (2019) proposed a numerical method to solve singularly perturbed differential difference equations by numerical integration using non-polynomial interpolating function and an initial value technique using exponential fitted method. Derya Arslan (2019) has presented a novel hybrid method for singularly perturbed delay differential equations. Derya Arslan (2020) has discussed the numerical solution for singularly perturbed multi-point boundary value problems with the Numerical Integration Method. Mesfin Mekuria Woldaregay and Gemechis File Duressa (2020) have presented a higher order uniformly convergent numerical scheme for singularly perturbed differential difference equations with mixed small shifts. For more details of singular perturbation theory and problems one can refer to popular books by Bellman and Cooke (1963) Bender and Orsag (1978), Doolan et al (1977), Driver (1977), Elsgolts and Norkin (1973), Hale (1977), Miller et al. (1996), Nayfeh (1979), O'Malley (1974), Reddy and Awoke (2013) and Van Dyke (1964). In this paper, numerical solution of singularly perturbed differential-difference equation with two boundary layers is considered. Using Taylor's series, the given second order singularly perturbed differential-difference equation is replaced by an asymptotically equivalent first order differential equation and solved by suitable choice of integrating factor and finite difference approximations. Numerical results for several test examples are presented to demonstrate the applicability of the method.

2. Description of the Method

Consider singularly perturbed differential-difference equation with small shifts of mixed type

$$\varepsilon y''(x) + a(x)y(x-\delta) + c(x)y(x) + b(x)y(x+\eta) = f(x), \quad 0 \le x \le 1 , \tag{1}$$

under the boundary conditions

$$y(x) = \alpha(x), \quad -\delta \le x \le 0, \tag{2}$$

$$y(x) = \beta(x), \ 1 \le x \le 1+\eta, \tag{3}$$

where $0 < \varepsilon \ll 1$, $0 < \delta = O(\varepsilon)$ and $0 < \eta = O(\varepsilon)$ are the perturbation parameter, the delay parameter and the advance parameter respectively. $a(x), b(x),\ c(x), f(x), \alpha(x)$ and $\beta(x)$ are sufficiently differentiable in $(0,1)$. Assume $a(x) + b(x) + c(x) \le 0$ on interval [0, 1], then the solution of Equation (1) - (3) exhibits two boundary layers in the interval [0, 1], for details, refer to Lange and Miura (1994 a, b). Taylor series expansion, in the neighbourhood of x, gives

$$y(x-\delta) \approx y(x) - \delta y'(x) + \frac{\delta^2}{2} y''(x), \tag{4}$$

$$y(x+\eta) \approx y(x) + \eta y'(x) + \frac{\eta^2}{2} y''(x). \tag{5}$$

Substituting (4) and (5) into (1), we get

$$\varepsilon' y''(x) + A(x)y'(x) + B(x)y(x) = f(x), \tag{6}$$

under boundary conditions

$$y(0) = \alpha(0) = \varphi_0, \tag{7}$$

$$y(1) = \beta(1) = \gamma_{1,} \tag{8}$$

where $A(x) = b(x)\eta - a(x)\delta$, $B(x) = a(x) + b(x) + c(x)$, $\varepsilon' = \varepsilon + a(x)\frac{\delta^2}{2} + b(x)\frac{\eta^2}{2}$, φ_0 and γ_1 are constants. Since $0 < \delta \ll 1$ and $0 < \eta \ll 1$, the transformation from Equation (1) to Equation (6) is permitted. For more details on the validity of this transformation one can refer to El'sgolt's and Norkin (1973).

We divide the interval [0, 1] into two subintervals $\left[0, \frac{1}{2}\right]$ and $\left[\frac{1}{2}, 1\right]$. Combine the solution of both subintervals to get the total solution of Equation (1).

2.1. Problem with left end boundary layer in $\left[0, \frac{1}{2}\right]$

Taylor's series expansion about the deviating argument $\sqrt{\varepsilon'}$ in the neighbourhood of point x, is

$$y\left(x - \sqrt{\varepsilon'}\right) \approx y(x) - \sqrt{\varepsilon'}y'(x) + \frac{\varepsilon'}{2}y''(x). \quad (9)$$

Substituting Equation (9) in Equation (6), we get

$$y'(x) = p(x)y\left(x - \sqrt{\varepsilon'}\right) + q(x)y(x) + r(x), \quad (10)$$

where

$$p(x) = \frac{-2}{2\sqrt{\varepsilon'} + A(x)}, \quad (11)$$

$$q(x) = \frac{2 - B(x)}{2\sqrt{\varepsilon'} + A(x)}, \quad (12)$$

$$r(x) = \frac{f(x)}{2\sqrt{\varepsilon'} + A(x)}. \quad (13)$$

The transition from Equation (6) to Equation (10) is valid, because of the condition that $\sqrt{\varepsilon'}$ is small. For more details on the validity of this transition, one can refer El'sgolt's and Norkin (1973). Now, we divide the interval [0, 1] into n equal parts with constant mesh length h. Let $0 = x_0, x_1, \ldots, x_n = 1$ be the mesh points, then we have $x_i = ih$, $i = 0, 1, 2, \ldots, n$. We choose N such that $x_N = \frac{1}{2}$. Equation (10) can be written as

$$y'(x) - qy(x) = py\left(x - \sqrt{\varepsilon'}\right) + r(x). \quad (14)$$

By taking an integrating factor e^{-qx} for Equation (14) proceeding as in McCartin (2001), we obtain

$$\frac{d}{dx}\left[e^{-qx}y(x)\right] = e^{-qx}\left[py\left(x - \sqrt{\varepsilon'}\right) + r(x)\right]. \quad (15)$$

On integrating Equation (15) from x_i to x_{i+1}, we get

$$e^{-qx_{i+1}}y_{i+1} - e^{-qx_i}y_i = \int_{x_i}^{x_{i+1}} e^{-qx}\, py\left(x - \sqrt{\varepsilon'}\right)dx + \int_{x_i}^{x_{i+1}} e^{-qx}\, r(x)dx. \quad (16)$$

Using Newton's forward interpolation for y term, and taking p as constant in this paper for simplicity, we get

$$e^{-qx_{i+1}}y_{i+1} = e^{-qx_i}y_i + p\int_{x_i}^{x_{i+1}} e^{-qx}\left[y\left(x_i-\sqrt{\varepsilon'}\right)+\frac{(x-x_i)}{h}\left\{y\left(x_{i+1}-\sqrt{\varepsilon'}\right)-y\left(x_i-\sqrt{\varepsilon'}\right)\right\}\right]dx + \int_{x_i}^{x_{i+1}} e^{-qx}\left[r_i+\frac{(x-x_i)}{h}\{r_{i+1}-r_i\}\right]dx, \tag{17}$$

$$\begin{aligned} y_{i+1} &= e^{qh}y_i + py\left(x_i-\sqrt{\varepsilon'}\right)\int_{x_i}^{x_{i+1}} e^{q(x_{i+1}-x)}\,dx \\ &+ \frac{py\left(x_{i+1}-\sqrt{\varepsilon'}\right)}{h}\int_{x_i}^{x_{i+1}} e^{q(x_{i+1}-x)}\,(x-x_i)dx \\ &+ \frac{py\left(x_i-\sqrt{\varepsilon'}\right)}{h}\int_{x_i}^{x_{i+1}} e^{q(x_{i+1}-x)}\,(x_i-x)dx + r_i\int_{x_i}^{x_{i+1}} e^{q(x_{i+1}-x)}\,dx \\ &+ \frac{r_{i+1}}{h}\int_{x_i}^{x_{i+1}} e^{q(x_{i+1}-x)}\,(x-x_i)dx \\ &+ \frac{r_i}{h}\int_{x_i}^{x_{i+1}} e^{q(x_{i+1}-x)}\,(x_i-x)dx. \end{aligned} \tag{18}$$

After evaluating the integrals involves in Equation (18), we get

$$y_{i+1} = e^{qh}y_i + J\left[py\left(x_i-\sqrt{\varepsilon'}\right)+r_i\right] + K\left[\frac{py\left(x_{i+1}-\sqrt{\varepsilon'}\right)}{h}+\frac{r_{i+1}}{h}\right] + L\left[\frac{py\left(x_i-\sqrt{\varepsilon'}\right)}{h}+\frac{r_i}{h}\right], \tag{19}$$

where

$$J = \frac{e^{qh}}{q} - \frac{1}{q}, \tag{20}$$

$$K = -\frac{h}{q} - \frac{1}{q^2} + \frac{e^{qh}}{q^2}, \tag{21}$$

$$L = \frac{h}{q} + \frac{1}{q^2} - \frac{e^{qh}}{q^2}. \tag{22}$$

From finite difference approximation, we have

$$y\left(x_i-\sqrt{\varepsilon'}\right) \approx \left(1-\frac{\sqrt{\varepsilon'}}{h}\right)y_i + \frac{\sqrt{\varepsilon'}}{h}y_{i-1}, \tag{23}$$

$$y\left(x_{i+1}-\sqrt{\varepsilon'}\right) \approx \left(1-\frac{\sqrt{\varepsilon'}}{h}\right)y_{i+1} + \frac{\sqrt{\varepsilon'}}{h}y_i. \tag{24}$$

Substituting Equations (23)-(24) in Equation (19), we get after rearranging/simplification:

$$E_i y_{i-1} - F_i y_i + G_i y_{i+1} = H_i, \quad i = 1,2,\dots,N-1. \tag{25}$$

where

$$E_i = -\frac{Jp\sqrt{\varepsilon'}}{h} - \frac{Lp\sqrt{\varepsilon'}}{h^2},$$
$$F_i = Jp\left(1 - \frac{\sqrt{\varepsilon'}}{h}\right) + \frac{Kp\sqrt{\varepsilon'}}{h^2} + \frac{Lp}{h}\left(1 - \frac{\sqrt{\varepsilon'}}{h}\right) + e^{qh},$$
$$G_i = 1 - \frac{Kp}{h}\left(1 - \frac{\sqrt{\varepsilon'}}{h}\right),$$
$$H_i = Jr_i + \frac{Kr_{i+1}}{h} + \frac{Lr_i}{h}.$$

2.2. Problem with right end boundary layer in $\left[\frac{1}{2}, 1\right]$

Taylor's series expansion about the deviating argument $\sqrt{\varepsilon'}$ in the neighbourhood of point x, is

$$y\left(x + \sqrt{\varepsilon'}\right) \approx y(x) + \sqrt{\varepsilon'}y'(x) + \frac{\varepsilon'}{2}y''(x). \tag{26}$$

Substituting Equation (26) in Equation (6), we get

$$y'(x) = p(x)y\left(x + \sqrt{\varepsilon'}\right) + q(x)y(x) + r(x), \tag{27}$$

where

$$p(x) = \frac{-2}{-2\sqrt{\varepsilon'} + A(x)}, \tag{28}$$
$$q(x) = \frac{2 - B(x)}{-2\sqrt{\varepsilon'} + A(x)}, \tag{29}$$
$$r(x) = \frac{f(x)}{-2\sqrt{\varepsilon'} + A(x)}. \tag{30}$$

Equation (27) can be written as

$$y'(x) - qy(x) = py\left(x + \sqrt{\varepsilon'}\right) + r(x). \tag{31}$$

By taking an integrating factor e^{-qx} for equation (31) and proceeding as in McCartin (2001), we get.

$$\frac{d}{dx}[e^{-qx}y(x)] = e^{-qx}\left[py\left(x + \sqrt{\varepsilon'}\right) + r(x)\right]. \tag{32}$$

On integrating Equation (32) from x_{i-1} to x_i, we get

$$e^{-qx_i}y_i - e^{-qx_{i-1}}y_{i-1} = \int_{x_{i-1}}^{x_i} e^{-qx}\, py\left(x + \sqrt{\varepsilon'}\right)dx + \int_{x_{i-1}}^{x_i} e^{-qx}\, r(x)dx, \tag{33}$$

Using Newton's forward interpolation for y term, and taking *p* as constant in this paper for simplicity, we get

$$e^{-qx_i}y_i =$$

$$e^{-qx_{i-1}}y_{i-1} + p\int_{x_{i-1}}^{x_i} e^{-qx}\left[y\left(x_i+\sqrt{\varepsilon'}\right) + \frac{(x-x_i)}{h}\left\{y\left(x_i+\sqrt{\varepsilon'}\right) - y\left(x_{i-1}+\sqrt{\varepsilon'}\right)\right\}\right]dx + \int_{x_{i-1}}^{x_i} e^{-qx}\left[r_i + \frac{(x-x_i)}{h}\{r_i - r_{i-1}\}\right]dx, \quad (34)$$

$$y_i = e^{qh}y_{i-1} + py\left(x_i+\sqrt{\varepsilon'}\right)\int_{x_{i-1}}^{x_i} e^{q(x_i-x)}\,dx + \frac{py\left(x_i+\sqrt{\varepsilon'}\right)}{h}\int_{x_{i-1}}^{x_i} e^{q(x_i-x)}\,(x-x_i)dx + \frac{py\left(x_{i-1}+\sqrt{\varepsilon'}\right)}{h}\int_{x_{i-1}}^{x_i} e^{q(x_i-x)}\,(x_i-x)dx + r_i\int_{x_{i-1}}^{x_i} e^{q(x_i-x)}\,dx + \frac{r_i}{h}\int_{x_{i-1}}^{x_i} e^{q(x_i-x)}\,(x-x_i) + \frac{r_{i-1}}{h}\int_{x_{i-1}}^{x_i} e^{q(x_i-x)}\,(x_i-x)\,dx. \quad (35)$$

After evaluating the integrals involves in Equation (35), we get

$$y_i = e^{qh}y_{i-1} + J\left[py\left(x_i+\sqrt{\varepsilon'}\right) + r_i\right] + K\left[\frac{py\left(x_i+\sqrt{\varepsilon'}\right)}{h} + \frac{r_i}{h}\right] + L\left[\frac{py\left(x_{i-1}+\sqrt{\varepsilon'}\right)}{h} + \frac{r_{i-1}}{h}\right], \quad (36)$$

where

$$J = \frac{e^{qh}}{q} - \frac{1}{q}, \quad (37)$$

$$K = -\frac{he^{qh}}{q} - \frac{1}{q^2} + \frac{e^{qh}}{q^2}, \quad (38)$$

$$L = \frac{he^{qh}}{q} + \frac{1}{q^2} - \frac{e^{qh}}{q^2}\,. \quad (39)$$

From finite difference approximation, we have

$$y\left(x_i+\sqrt{\varepsilon'}\right) \approx \left(1-\frac{\sqrt{\varepsilon'}}{h}\right)y_i + \frac{\sqrt{\varepsilon'}}{h}y_{i+1}\,, \quad (40)$$

$$y\left(x_{i-1}+\sqrt{\varepsilon'}\right) \approx \left(1-\frac{\sqrt{\varepsilon'}}{h}\right)y_{i-1} + \frac{\sqrt{\varepsilon'}}{h}y_i. \quad (41)$$

Substituting Equations (40)-(41) in Equation (36), we get

$$E_iy_{i-1} - F_iy_i + G_iy_{i+1} = H_i, \quad i = N+1, N+2, \dots, n-1, \quad (42)$$

where

$$E_i = -e^{qh} - \frac{Lp}{h}\left(1 - \frac{\sqrt{\varepsilon'}}{h}\right),$$
$$F_i = -1 + Jp\left(1 - \frac{\sqrt{\varepsilon'}}{h}\right) + \frac{Lp\sqrt{\varepsilon'}}{h^2} + \frac{Kp}{h}\left(1 - \frac{\sqrt{\varepsilon'}}{h}\right),$$
$$G_i = -\frac{Jp\sqrt{\varepsilon'}}{h} - \frac{Kp\sqrt{\varepsilon'}}{h^2},$$
$$H_i = Jr_i + \frac{Kr_i}{h} + \frac{Lr_{i-1}}{h}.$$

We have a system of $n-2$ equations from both left and right end boundary layer problem with $n+1$ unknowns. From the given boundary conditions, Equation (7) and Equation (8), we get two equations i.e.

$$y(0) = \alpha(0) = \varphi_0,$$
$$y(1) = \beta(1) = \gamma_1.$$

We need one more equation to solve for the unknowns $(y_0, y_1, \ldots, y_n)$. For this, we consider the Equation. (6) at $\varepsilon = 0$ and the point $x = x_N$, we get

$$A(x_N)y'(x_N) + B(x_N)y(x_N) = f(x_N).$$

Using second order central finite difference approximation for derivative, we get

$$\frac{A_N}{2h}y_{N-1} - B_N y_N + \left(-\frac{A_N}{2h}\right)y_{N+1} = -f_N.$$

With the above equations, we now have $n+1$ equations to solve $n+1$: $(y_0, y_1, \ldots, y_n)$. Using invariant imbedding algorithm given in the book Bellman and Cooke (1963) we get the solution.

3. Numerical Experiments

The exact solution of the differential -difference equation

$$\varepsilon y''(x) + a(x)y(x-\delta) + c(x)y(x) + b(x)y(x+\eta) = f(x), \quad 0 \le x \le 1,$$

with the boundary conditions $y(x) = \alpha(x), \quad -\delta \le x \le 0$ and $y(x) = \beta(x), 1 \le x \le 1+\eta$ with constant coefficients ($a(x) = a, b(x) = b, c(x) = c, f(x) = f, \alpha(x) = \alpha, \ \beta(x) = \beta$) is given by

$$y(x) = \frac{[\{(1-a-b-c)\exp(m_2)-1\}\exp(m_1x) - \{(1-a-b-c)\exp(m_1)-1\}\exp(m_2x)]}{[(a+b+c)(\exp(m_1)-\exp(m_2))]} + \frac{1}{(a+b+c)}, \tag{43}$$

where

$$m_1 = \frac{\left[(a\delta - b\eta) + \sqrt{(b\eta - a\delta)^2 - 4\varepsilon(a+b+c)}\right]}{2\varepsilon},$$

$$m_2 = \frac{\left[(a\delta - b\eta) - \sqrt{(b\eta - a\delta)^2 - 4\varepsilon(a+b+c)}\right]}{2\varepsilon}.$$

Example 1. Consider the differential-difference equation having layer at both ends

$$\varepsilon y''(x) - 2y(x-\delta) - y(x) - 2y(x+\eta) = 1, \quad 0 \le x \le 1,$$

with the boundary conditions $y(0) = 1$ and $y(1) = 0$. The exact solution is given by Equation (43). Numerical results are shown in Table 1 and 2 and the layer behaviour in Figure 1 and 2 for different values of parameters.

Example 2. Consider the differential-difference equation having layer at both ends

$$\varepsilon y''(x) + 0.25y(x-\delta) - y(x) + 0.25y(x+\eta) = 1, \quad 0 \le x \le 1,$$

with the boundary conditions $y(0) = 1$ and $y(1) = 0$. The exact solution is given by Equation (43). Numerical results are shown in Table 3 and 4 and the layer behaviour in Figure 3 and 4 for different values of parameters.

Example 3. Consider the differential-difference equation having layer at both ends

$$\varepsilon y''(x) - y(x-\delta) - y(x) - 3y(x+\eta) = 1, \quad 0 \le x \le 1,$$

with the boundary conditions $y(0) = 1$ and $y(1) = 0$. The exact solution is given by Equation (43). Numerical results are shown in Table 5 and 6 and the layer behaviour in Figure 5 and 6 for different values of Parameters.

4. Discussion and Conclusions

In this paper, numerical solution of differential-difference equation with two boundary layers is discussed. Using Taylor's series, the given second order differential-difference equation is replaced by an asymptotically equivalent first order differential equation and solved by suitable choice of integrating factor and finite difference approximations. Finite difference approximation is taken on equidistant mesh to discretize the continuous problem. Scheme is simple and easy to implement on considered problems also applicable on wide range of problems. The numerical results for several test examples are presented to demonstrate the applicability of the method. We can observe that by decreasing the perturbation parameter absolute error is decreased. It shows that whenever perturbation parameter is very small or tending towards zero, the scheme gives good results.

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APPENDIX

Table 1: Results for Example-1 with $h = 0.01, \varepsilon = 0.001, \delta = 0.001$ and $\eta = 0.003$

x	Our Solution	Exact Solution	Result by Kadalbajoo and Sharma (2005)
0.00	1	1	1
0.02	-0.01518168	0.10347453	0.10808822
0.04	-0.17153515	-0.12325267	-0.12090137
0.06	-0.19561598	-0.18059095	-0.17969220
0.08	-0.19932479	-0.19509153	-0.19478617
0.1	-0.19989600	-0.19875867	-0.19866140
0.2	-0.19999999	-0.19999871	-0.19999850
0.3	-0.19999999	-0.19999999	-0.19999999
0.4	-0.20000000	-0.19999999	-0.19999999
0.5	-0.20000000	-0.19999999	-0.19999999
0.6	-0.20000000	-0.19999999	-0.19999999
0.7	-0.19999999	-0.19999999	-0.19999999
0.8	-0.19999999	-0.19999990	-0.19999988
0.9	-0.19999031	-0.19986131	-0.19985167
1.0	0	0	0

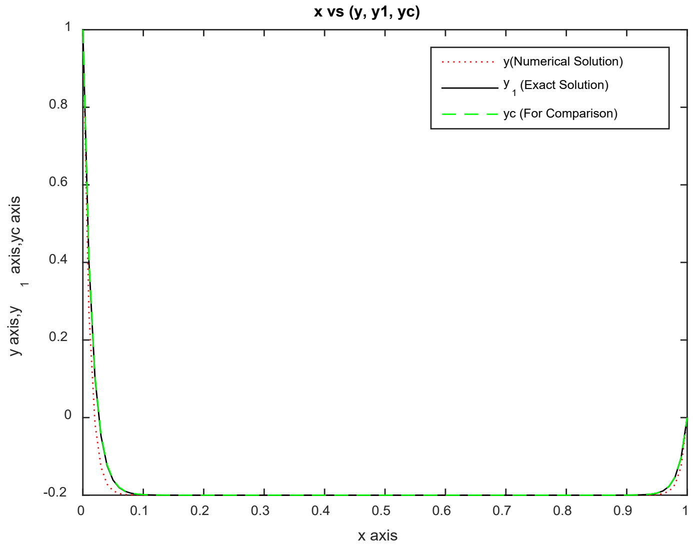

Figure 1: Example-1 with $h = 0.01, \varepsilon = 0.001, \delta = 0.001$ and $\eta = 0.003$

Table 2: Results for Example-1 with $h = 0.01, \varepsilon = 0.0001, \delta = 0.002$ and $\eta = 0.003$

x	Our Solution	Exact Solution	Result by Kadalbajoo and Sharma (2005)
0.0	1	1	1
0.02	-0.14130475	-0.18333232	-0.17310964
0.04	-0.19712905	-0.19976849	-0.19939742
0.06	-0.19985957	-0.19999678	-0.19998649
0.08	-0.19999313	-0.19999995	-0.19999969
0.1	-0.19999966	-0.19999999	-0.19999999
0.2	-0.19999999	-0.20000000	-0.20000000
0.3	-0.20000000	-0.20000000	-0.20000000
0.4	-0.20000000	-0.20000000	-0.20000000
0.5	-0.20000000	-0.20000000	-0.20000000
0.6	-0.20000000	-0.20000000	-0.20000000
0.7	-0.20000000	-0.20000000	-0.20000000
0.8	-0.19999999	-0.20000000	-0.20000000
0.9	-0.19999996	-0.19999999	-0.19999999
1.0	0	0	0

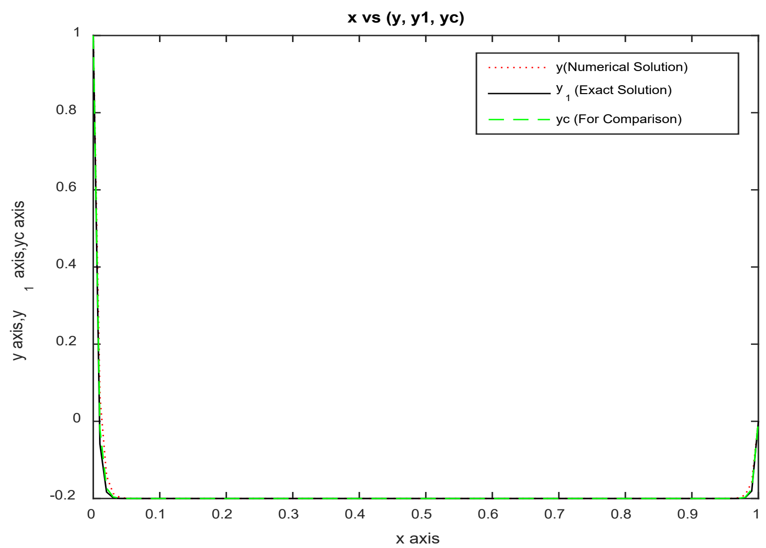

Figure 2: Example-1 with $h = 0.01, \varepsilon = 0.0001, \delta = 0.002$ and $\eta = 0.003$

Table 3: Results for Example- 2 with $h = 0.01, \varepsilon = 0.0001,\ \delta = 0.003$ and $\eta = 0.001$

x	Our Solution	Exact Solution	Result by Kadalbajoo and Sharma (2005)
0.0	1	1	1
0.02	-1.02446681	-1.23393241	-1.22465608
0.04	-1.68277833	-1.80438015	-1.79961393
0.06	-1.89684658	-1.95004732	-1.94821062
0.08	-1.96645680	-1.98724429	-1.98661514
0.1	-1.98909250	-1.99674275	-1.99654071
0.2	-1.99996034	-1.99999646	-1.99999601
0.3	-1.99999985	-1.99999999	-1.99999999
0.4	-1.99999999	-1.99999999	-1.99999999
0.5	-1.99999999	-1.99999999	-1.99999999
0.6	-1.99999999	-1.99999999	-1.99999999
0.7	-1.99999995	-1.99999999	-1.99999999
0.8	-1.99998521	-1.99999913	-1.99999904
0.9	-1.99456154	-1.99868292	-1.99861919
1.0	0	0	0

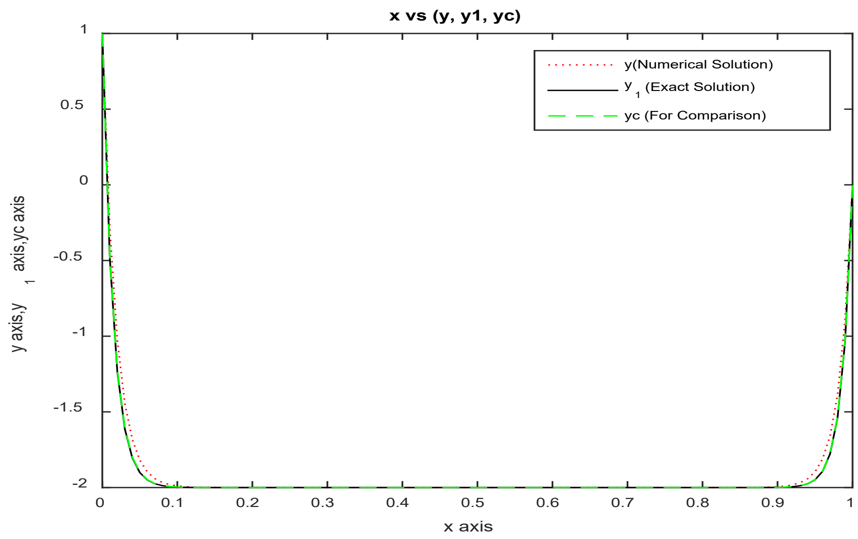

Figure 3: Example- 2 with $h = 0.01, \varepsilon = 0.0001,\ \delta = 0.003$ and $\eta = 0.001$

Table 4: Results for Example-2 with $h = 0.01, \varepsilon = 0.0001,\ \delta = 0.00001$ and $\eta = 0.0003$

x	Our Solution	Exact Solution	Result by Kadalbajoo and Sharma (2005)
0.0	1	1	1
0.02	-1.05889666	-1.27593191	-1.25361769
0.04	-1.70477483	-1.82524180	-1.81430448
0.06	-1.90738753	-1.95782105	-1.95380005
0.08	-1.97094736	-1.98981985	-1.98850572
0.1	-1.99088615	-1.99754296	-1.99714029
0.2	-1.99997231	-1.99999798	-1.99999727
0.3	-1.99999991	-1.99999999	-1.99999999
0.4	-1.99999999	-1.99999999	-1.99999999
0.5	-2.00000000	-1.99999999	-1.99999999
0.6	-1.99999999	-1.99999999	-1.99999999
0.7	-1.99999993	-1.99999999	-1.99999999
0.8	-1.99997991	-1.99999844	-1.99999790
0.9	-1.99366142	-1.99823880	-1.99795071
1.0	0	0	0

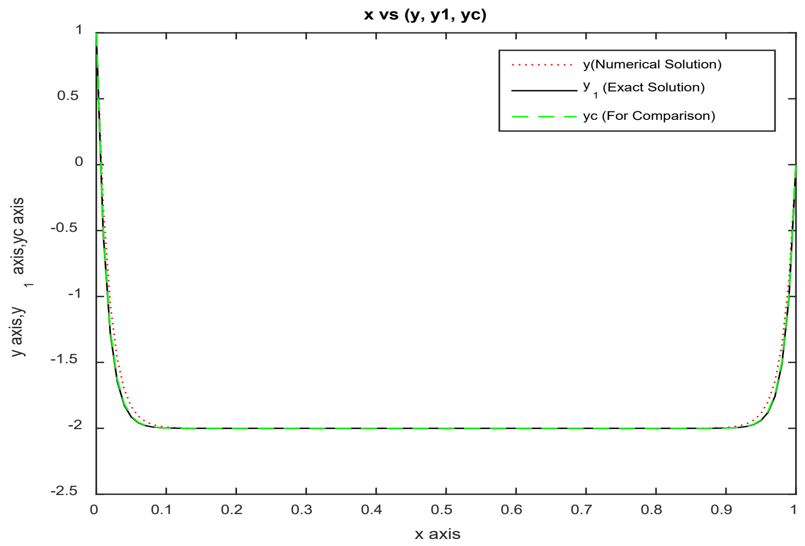

Figure 4: Example-2 with $h = 0.01, \varepsilon = 0.0001,\ \delta = 0.00001$ and $\eta = 0.0003$

Table 5: Results for Example-3 with $h = 0.01, \varepsilon = 0.0001, \ \delta = 0.007$ and $\eta = 0.003$

x	Our Solution	Exact Solution	Result by Kadalbajoo and Sharma (2005)
0.0	1	1	1
0.02	-0.15197485	-0.18333232	-0.1731096
0.04	-0.19807798	-0.19976849	-0.19939742
0.06	-0.19992307	-0.19999678	-0.19998649
0.08	-0.19999692	-0.19999995	-0.19999969
0.1	-0.19999987	-0.19999999	-0.19999999
0.2	-0.19999999	-0.20000000	-0.20000000
0.3	-0.20000000	-0.20000000	-0.20000000
0.4	-0.20000000	-0.20000000	-0.20000000
0.5	-0.20000000	-0.20000000	-0.20000000
0.6	-0.20000000	-0.20000000	-0.20000000
0.7	-0.20000000	-0.20000000	-0.20000000
0.8	-0.19999999	-0.20000000	-0.20000000
0.9	-0.19999998	-0.19999999	-0.19999999
1.0	0	0	0

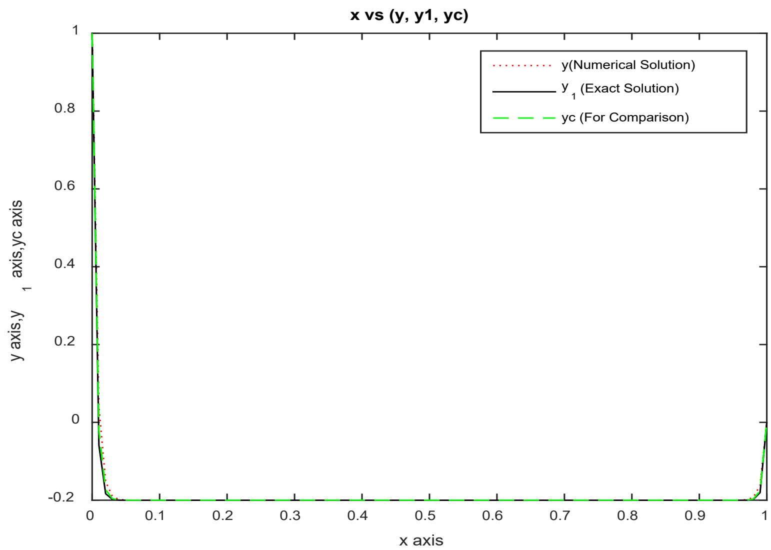

Figure 5: Example-3 with $h = 0.01, \varepsilon = 0.0001, \ \delta = 0.007$ and $\eta = 0.003$

Table 6: Results for Example-3 with $h = 0.01, \varepsilon = 0.0001,\ \delta = 0.007$ and $\eta = 0.005$

x	Our Solution	Exact Solution	Result by Kadalbajoo and Sharma (2005)
0.0	1	1	1
0.02	-0.15817225	-0.17158406	-0.16845338
0.04	-0.19854203	-0.19932711	-0.19917067
0.06	-0.19994918	-0.19998406	-0.19997819
0.08	-0.19999822	-0.19999962	-0.19999942
0.1	-0.19999993	-0.19999999	-0.19999998
0.2	-0.19999999	-0.20000000	-0.20000000
0.3	-0.20000000	-0.20000000	-0.20000000
0.4	-0.20000000	-0.20000000	-0.20000000
0.5	-0.20000000	-0.20000000	-0.20000000
0.6	-0.20000000	-0.20000000	-0.20000000
0.7	-0.20000000	-0.20000000	-0.20000000
0.8	-0.20000000	-0.20000000	-0.20000000
0.9	-0.19999999	-0.19999999	-0.20000000
1.0	0	0	0

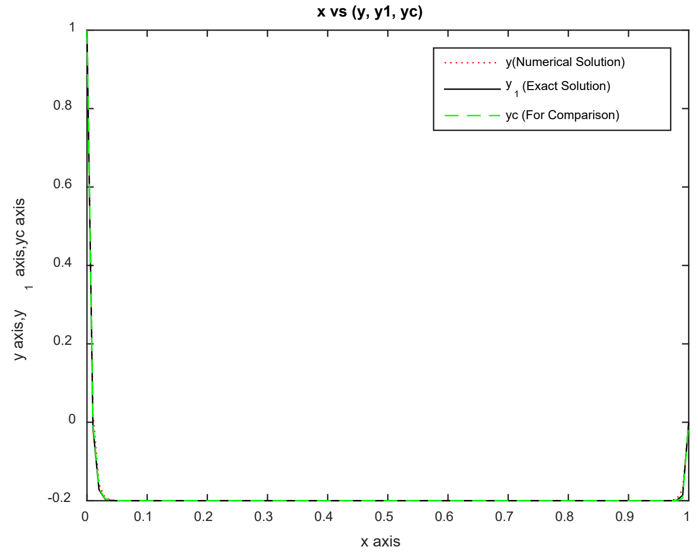

Figure 6: Example-3 with $h = 0.01, \varepsilon = 0.0001,\ \delta = 0.007$ and $\eta = 0.005$