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Abstract

Probability of ultimate survival is one of the central problems in insurance because it is a management tool that may be used to check on the solvency levels of the insurer. In this article, we numerically compute this probability for an insurer whose portfolio is compounded by investments arising from a risky asset. The uncertainty in the celebrated Cramér-Lundberg model is provided by a standard Brownian motion that is independent of the standard Brownian motion in the model for the risky asset. We apply an order four block-by-block method in conjunction with the Simpson rule to solve the resulting Volterra integral equation of the second kind. The ultimate survival probability is arrived at by taking a linear combination of some two solutions to the Volterra equations. The several numerical examples show that the results are accurate and reliable even when the net profit condition is violated.

Keywords: Risk theory; Survival/Ruin probability; Volterra integral equations; Block-by-block method; Singularity

MSC 2010 No.: 45D05, 62P05, 91B30

1. Introduction

Classical collective risk model was first introduced by Lundberg (1903) and since then, a variety of risk models appear in the literature in many forms and dimensions. The model describes the free surplus process of an insurer's portfolio. The probability that such a portfolio becomes negative for the first time was among the prime quantities of interest in risk theory. This probability measures the safely aspect of the portfolio. Lots of mathematical machinery has since then been devoted to finding the probability of survival (equivalently, probability of ruin) of an insurance portfolio, thus making risk theory a rich and challenging field of research especially under more general and more realistic assumptions. There are two approaches: one is to describe the classical collective risk model based on a compound Poisson process, and the other is to describe it using a diffusion model. The former is Markovian, and hence, dynamic programming principles come into play. We chose to use the memoryless Poisson process to describe the development of the jumps because we require that the probability of a jump is independent of the history of the process. The total of the jumps then constitute a compound Poisson process.

Historically, the survival probability was approximated by the initial insurer's capital and the insolvency coefficient or Lundberg's constant. However, with improved computing power, we can now calculate the exact survival probability. The goal of this paper is to compute the ultimate survival probabilities exactly in a more realistic risk process model based on the special case model by Kasozi and Paulsen (2005). The paper derives its novelty here, additionally using a linear combination of two solutions from the Volterra integral equations. Insurers use this probability to make important decisions, for example, paying dividends only when the survival probability is at a certain predetermined value.

In the available literature on risk theory, the insurance models studied are rooted on the famous Cramér-Lundberg model or the surplus generating model that describes the free reserve. The model contains the initial capital, usually set by government regulatory authorities. The premiums are assumed to be collected by the insurance company continuously over time with constant intensity. This is considered as the drift in the process. The total claim amount is then modeled by the compound Poisson process. The number of claim occurrences is a homogeneous Poisson process with known intensity. The claims are non-negative independent and identically distributed (iid) random variables with continuous distribution. The distribution has finite expectation and finite variance and is independent of the Poisson process. Additionally, the inter-occurrence times are iid with exponential distribution or otherwise. Several authors have dealt with iid claims and inter arrival times (e.g., see Liu and Qingwu (2016), Li and Sendova (2015), Kievinaité and Šiaulys (2018), and with some dependence Huang et al. (2017) and Yang et al. (2014)). The reserve process in this paper is in the family of spectrally negative Levy processes and the family of piece-wise deterministic Markov processes, thus a strong Markov process.

One of the problems in risk theory is to study versions and realistic extensions of Cramér-Lundberg model with an objective of finding the first time that the reserve becomes non positive (e.g., see Kasumo et al. (2018)). The very first time that this happens is termed ruin time and the associated probability is the probability of ultimate ruin, whose complement is survival probability. Ruin time does not imply that the insurance company is bankrupt, rather, it helps the company to take action in order to keep the business afloat. Insurance companies have other measures in place to keep the probability of survival high like reinsurance arrangements, capital injections, premium adjustments and investments. The ultimate survival probability is also termed infinite survival probability. This probability is related to finite survival probability.

In Kizinevic and Šiaulys (2018), a non-homogeneous risk model was studied in which claims and inter-arrival times were independent but possibly non-identically distributed. Conditions were established such that the ultimate ruin probability of the model satisfied a certain exponential estimate. In Kasozi and Paulsen (2005), the issue of an insurer whose portfolio was exposed to insurance risk arising from the basic reserve process and compounded by investments from a safe investment was studied. Linear Volterra integral equations of the second kind were derived and solved numerically using the block-by-block method in conjunction with Simpson rule to calculate ruin probability exactly. The probability was arrived at by taking a linear combination of some two solutions to the integral equation. Unlike the basic classical model considered, in this paper, we incorporate a diffusion in the Cramér-Lundberg model to make the reserve process more realistic. Secondly, we consider investment into a risky asset so that the models in Kasozi and Paulsen (2005) are special cases of the models in this article. In Section 2, we formulate the risk models and present the numerical method. Section 3 will contain the results and the associated discussion before we give concluding remarks in Section 4.

2. The Risk Models, Materials and Numerical Method

The models and variables in this paper are random. Accordingly, we start with a stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}^+}, \mathbb{P})$, onto which all stochastic quantities studied herein are defined. The basis satisfies the usual conditions, i.e. $\mathcal{F}_t$ is right continuous and $\mathbb{P}$-complete. To expound on the variables in the stochastic basis: Ω is a sample space with elements denoted by ω; $\mathcal{F}$ is a $\sigma-$algebra on Ω; $\mathbb{P}$ is a measure with compact support on $[0, 1]$ and $\{\mathcal{F}_t\}_{t\in\mathbb{R}^+}$ is a filtration. A filtration is an increasing and right continuous class of sub $\sigma-$algebras of $\mathcal{F}$. The filtration is mathematically used to model the available information on the stochastic process under consideration at any particular point in time.

Let $\mathbf{R} = \{R_t\}_{t\in\mathbb{R}^+}$ denote the diffusion-perturbed risk model:

$$R(t) = x + ct + \sigma_R W_R(t) - \sum_{k=1}^{N_R(t)} Y_R(k), \quad t \in \mathbb{R}^+. \tag{1}$$

The number $\{N_R(t)\}_{t\in\mathbb{R}^+}$ is a homogeneous Poisson process with intensity λ that counts the number of claims/jumps received, $x \geq 0$ is initial capital, c is premium intensity, the distribution function of $Y_R(k)$ is F and we take it that $F(0) = 0$. Let $\mu = \mathbb{E}[Y_R(k)]$ be the mean value and $M_S(r) = \mathbb{E}\left[e^{rY_R(k)}\right]$ its moment-generating function. We define $c = (1 + \theta)\lambda\mu,\ \theta > 0$ is the premium loading factor.

We will take it that at least one of σ_R or λ is non-zero. The diffusion term $\sigma_R W_R$ represents random variations in the reserve process thus making it more realistic. The dual risk model to equation (1) regards c as an expense rate and $Y_R(k)$ as random gains. The gain arrival process is accordingly $N_R(t)$. The dual risk model is used in life annuity insurance. In Loke and Thomann (2018), such a model was studied and they were able to numerically compute the ruin probability when the gain distribution F was Exponential, Uniform and Pareto. They also found out the Laplace transform of the time of ruin.

In Kasozi and Paulsen (2005), the insurance company was allowed to enhance its financial base by investing in a safe asset. In this article, the company chooses to invest in a risky asset (which can be converted into a safe investment under some conditions). These considerations were not taken into account in Kizinevic and Šiaulys (2018), for example. The return on investments process is thus given by

$$B(t) = \delta t + \sigma_B W_B(t), \quad t \in \mathbb{R}^+, \; B(0) = 0, \tag{2}$$

where δ is the risk-free part of the investments so that $B(t) = \delta t$ the time t value of one dollar invested now is $e^{\delta t}$.; W_B is another diffusion motion so that the term $\sigma_B W_B(t)$ then takes account of random fluctuations in the investment returns; in fact, return on investments model corresponds to the famous Black and Scholes option pricing formula where the price of a stock is assumed to be governed by geometric Brownian motion (gBm):

$$S_t \;=\; S_0 \;+\; \int_0^t S_u \, dB_u, \tag{3}$$

where S_0 is the stock price at $t = 0$. The solution to the gBm is $S_t = S_0 \exp\{(\delta - \frac{1}{2}\sigma_B^2)t \;+\; \sigma_B W_{B,t}\}$.

The insurance model with stochastic investment returns process $X = \{X_t\}_{t \in \mathbb{R}^+}$ is a combination of Equations (1) and (2) with dynamics

$$dX_t = dR_t + X_{t^-} \, dB_t.$$

Accordingly, X is the solution of the linear stochastic differential equation

$$X_t = x + R_t + \int_0^t X_{u^-} \, dB_u, \tag{4}$$

where the non-negative constant $x = X_0 > 0$ is the initial capital and X_{u^-} denotes the company's surplus just prior to investing.

Formally, τ denotes the first entrance time of the risk process to $(-\infty, 0)$, that is, $\tau(x) = \inf\{t \in \mathbb{R}^+ : X_t < 0 | X_0 = x\}$. The associated probability is then $\psi(x) = \mathbb{P}(\tau_x < \infty)$. Equivalently, the survival probability is given by $\phi(x) = 1 - \psi(x)$. The so called net profit condition requires that $c > \lambda\mu$ otherwise ruins occurs a.s. In fact, if $c > \lambda\mu$ then $\mathbb{P}(\lim_{t\to\infty} X_t = \infty) = 1$.

The infinitesimal generator for X is given by the following integro-differential operator:

$$\begin{aligned} \mathfrak{A}g(x) = &\frac{1}{2}(\sigma_B^2 \, x^2 + \sigma_R^2)g''(x) + (\delta x + c)g'(x) \\ &+\lambda \int_0^\infty (g(x-u) - g(x))dF(u). \end{aligned} \tag{5}$$

Let $\psi(x)$ be bounded and twice continuously differentiable on $x \in \mathbb{R}^+$ with a bounded first derivative there and let $\psi(x)$ solve the integro-differential equation $\mathfrak{A}\psi(x) = 0$ on $x > 0$, together with the boundary conditions

$$\begin{aligned} \psi(x) &= 1 \quad \text{on } x < 0, \\ \psi(0) &= 1 \quad \text{if } \sigma_P^2 > 0, \\ \lim_{x\to\infty} \psi(x) &= 0. \end{aligned}$$

Then, $\psi(x) = \mathbb{P}(\tau_x < \infty)$. We thus can solve $\mathfrak{A}\phi(x) = \mathfrak{A}(1 - \psi(x)) = 0$. We have that $F(\infty) = \int_0^\infty f(u)du = 1$, $\mathfrak{A}\phi = 0$ takes the form

$$\frac{1}{2}(\sigma_B^2 x^2 + \sigma_R^2)\phi''(x) + (\delta x + c)\phi'(x) + \lambda\left(\int_0^x \phi(x-u)dF(u) - \phi(x)\right) = 0. \tag{6}$$

Equation (6) can be written as a Volterra integral equation of the second kind as in the following theorem.

Theorem 2.1.

The integro-differential equation (6) can be represented as the Volterra integral equation of the second kind,

$$\phi(x) + \int_0^x K(x,u)\phi(u)du = \beta(x), \tag{7}$$

where for the case with no diffusion, i.e., $\sigma_R^2 = \sigma_B^2 = 0$, $K(x,u)$ is given by

$$K(x,u) = \frac{\delta + \lambda\bar{F}(x-u)}{\delta x + c}, \tag{8}$$

with $\bar{F}(u) = 1 - F(u)$ and $\beta(x) = \frac{c\phi(0)}{\delta x + c}$. When there is diffusion, i.e., $\sigma_R^2 + \sigma_B^2 > 0$, $K(x,u)$ and $\beta(x)$ are given by

$$K(x,u) = \frac{2((2\delta - 3\sigma_B^2 + \lambda)u + c + \lambda G(x-u) - (\delta - \sigma_B^2 + \lambda)x)}{x^2\sigma_B^2 + \sigma_R^2}, \tag{9}$$

$$\beta(x) = \begin{cases} \dfrac{2c\phi(0)}{x\sigma_B^2}, & \text{if } \sigma_R^2 = 0, \\ \\ \dfrac{x\sigma_R^2\phi'(0)}{x^2\sigma_B^2 + \sigma_R^2}, & \text{if } \sigma_R^2 > 0. \end{cases} \tag{10}$$

Proof:

Integrating Equation (6) by parts on $[0,z]$ with respect to x gives

$$\begin{aligned}
0 &= \frac{1}{2}\int_0^z (\sigma_B^2 x^2 + \sigma_R^2)\phi''(x)dx + \int_0^z (\delta y + c)\phi'(x)dx - \lambda \int_0^z \phi(x)dx \\
&\quad + \lambda \int_0^z \int_0^x \phi(x-u)f(u)dudx \\
&= \frac{1}{2}\left(\left[(\sigma_B^2 x^2 + \sigma_R^2)\phi'(x)\right]_0^z - 2\sigma_B^2 \int_0^z x\phi'(x)dx \right) + \int_0^z (\delta x + c)\phi'(x)dx \\
&\quad + \lambda \int_0^z \int_0^x \phi(x-u)f(u)dudx - \lambda \int_0^z \phi(x)dx \\
&= \frac{1}{2}(\sigma_B^2 z^2 + \sigma_R^2)\phi'(z) - \frac{1}{2}\sigma_R^2\phi'(0) + \int_0^z ((\delta - \sigma_B^2)x + c)\phi'(x)dx \\
&\quad + \lambda \int_0^z \int_0^x \phi(v)f(x-v)dvdx - \lambda \int_0^z \phi(x)dx, \qquad (v = x-u) \\
&= \frac{1}{2}(\sigma_B^2 z^2 + \sigma_R^2)\phi'(z) - \frac{1}{2}\sigma_R^2\phi'(0) + \left[((\delta - \sigma_B^2)x + c)\phi(x)\right]_0^z \\
&\quad - (\delta - \sigma_B^2)\int_0^z \phi(x)dx + \lambda \int_0^z \int_v^z f(x-v)dx\phi(v)dv - \lambda \int_0^z \phi(x)dx \\
&= \frac{1}{2}(\sigma_B^2 z^2 + \sigma_R^2)\phi'(z) - \frac{1}{2}\sigma_R^2\phi'(0) + ((\delta - \sigma_B^2)z + c)\phi(z) - c\phi(0) \\
&\quad - (\delta - \sigma_B^2 + \lambda)\int_0^z \phi(v)dv + \lambda \int_0^z F(z-v)\phi(v)dv. \qquad (11)
\end{aligned}$$

Integrating Equation (11) by parts on $[0,x]$ with respect to z gives

$$\begin{aligned}
0 &= \frac{1}{2}\int_0^x (\sigma_B^2 z^2 + \sigma_R^2)\phi'(z)dz + \int_0^x ((\delta - \sigma_B^2)z + c)\phi(z)dz - \left(\frac{1}{2}\sigma_R^2\phi'(0) + c\phi(0)\right)x \\
&\quad - (\delta - \sigma_B^2 + \lambda)\int_0^x \int_0^z \phi(v)dvdz + \lambda \int_0^x \int_0^z F(z-v)\phi(v)dvdz \\
&= \frac{1}{2}\left[(\sigma_B^2 z^2 + \sigma_R^2)\phi(z)\right]_0^x - \sigma_B^2 \int_0^x z\phi(z)dz + \int_0^x ((\delta - \sigma_B^2)z + c)\phi(z)dz \\
&\quad - \left(\frac{1}{2}\sigma_R^2\phi'(0) + c\phi(0)\right)x - (\delta - \sigma_B^2 + \lambda)\int_0^x \int_v^y dz\phi(v)d(v) \\
&\quad + \lambda \int_0^x \int_v^x F(z-v)dz\phi(v)dv \\
&= \frac{1}{2}(\sigma_B^2 x^2 + \sigma_R^2)\phi(x) - \frac{1}{2}\sigma_R^2(\phi(0) + x\phi'(0)) - cx\phi(0) \qquad (12) \\
&\quad + \int_0^x ((2\delta - 3\sigma_B^2 + \lambda)z + c + \lambda G(x-z) - (\delta - \sigma_B^2 + \lambda)x)\phi(z)dz,
\end{aligned}$$

where $G(u) = \int_0^u F(z)dz$. Equation (12) can be written as

$$\phi(x) + 2\int_0^x \frac{(2\delta - 3\sigma_B^2 + \lambda)z + c + \lambda G(x-z) - (\delta - \sigma_B^2 + \lambda)x}{x^2\sigma_B^2 + \sigma_R^2}\phi(z)dz$$

$$= \frac{\sigma_R^2(\phi(0) + x\phi'(0)) + 2cx\phi(0)}{x^2\sigma_B^2 + \sigma_R^2}. \tag{13}$$

Finally, replace z with u. ■

Equation (7) can be written in a more general setting of a Volterra integral equation of the second kind, thus,

$$\phi(y) + c\int_a^y K(y, x, \phi(x))dx = h(y), \quad a \le y \le T. \tag{14}$$

Here, T is some finite time. The forcing function $h(y)$ and the kernel $K(y, x)$ are prescribed. The structure of the kernel dictates the numerical method appropriate for (14). The parameter c is often omitted but turns out to be of importance in certain theoretical investigations, say, stability. Set $c = 1$ and $a = 0$. We note from (7) that the forcing function is continuous in $0 \le y \le T$. Further, the kernel is continuous and satisfies a uniform Lipschitz condition in ϕ. Also $\phi \in [0, 1]$. The linear Volterra integral equation hence has a unique continuous solution, details are found in Linz (1985).

We now give highlights of the numerical solution of Equation (7) (for more details, see, e.g., Kasumo et al. (2020) and Kasozi and Paulsen (2005)). Other methods for solving such an equation, or versions of it, exist but are of less order and suffer from uniform convergence requirements (e.g., see Gatto and Baumgartner (2016), Baharum et al. (2018), and Assari and Dehghan (2018)). Using a fixed grid $x = 0, h, 2h, \ldots,$ the solution is of the form

$$\phi_n + h\sum_{i=1}^{n} \omega_i K_{n,i}\,\phi_i = \beta_n, \tag{15}$$

where ϕ_i is the numerical approximation to $\phi(ih)$, $K_{n,i} = K(nh, ih)$, $\phi_n = \phi(nh)$ and $\beta_n = \beta(nh)$. The ω_i are the integration weights. One of the block by block method in conjuction with Simpson's rule, best higher order methods for solving (7). Here, Simpson's rule gives

$$\phi_2 + \frac{h}{3}\left(K_{2,0}\,\phi_0 + 4K_{2,1}\,\phi_1 + K_{2,2}\,\phi_2\right) = \beta_2. \tag{16}$$

We don't know ϕ_1. The same rule with step size $\frac{h}{2}$, gives

$$\phi_1 + \frac{h}{6}\left(K_{1,0}\,\phi_0 + 4K_{1,\frac{1}{2}}\,\phi_{\frac{1}{2}} + K_{1,1}\,\phi_1\right) = \beta_1. \tag{17}$$

Interpolation approximates $\phi_{\frac{1}{2}} \approx \frac{3}{8}\phi_0 + \frac{3}{4}\phi_1 - \frac{1}{8}\phi_2$ with (17) gives

$$\phi_1 + \frac{h}{6}\left((K_{1,0} + \frac{3}{2}K_{1,\frac{1}{2}})\phi_0 + (K_{1,1} + 3K_{1,\frac{1}{2}})\phi_1 - \frac{1}{2}K_{1,\frac{1}{2}}\,\phi_2\right) = \beta_1. \tag{18}$$

Solving (16) and (18) for ϕ_1 and ϕ_2 yields the blocks of 2:

$$\phi_{2m+2} + h\sum_{i=0}^{2m} \omega_i K_{2m+2,i}\,\phi_i + \frac{h}{3}\,(K_{2m+2,2m}\,\phi_{2m} + 4K_{2m+2,2m+1}\,\phi_{2m+1}$$
$$+K_{2m+2,2m+2}\,\phi_{2m+2}) = \beta_{2m+2}, \tag{19}$$

and

$$\phi_{2m+1} + h\sum_{i=0}^{2m} \omega_i K_{2m+1,i}\,\phi_i + \frac{h}{6}\Big(K_{2m+1,2m}\,\phi_{2m} + 4K_{2m+1,2m+\frac{1}{2}}\,\phi_{2m+\frac{1}{2}}$$
$$+K_{2m+1,2m+1}\,\phi_{2m+1}) = \beta_{2m+1}. \tag{20}$$

Inserting $\phi_{2m+\frac{1}{2}} \approx \frac{3}{8}\phi_{2m} + \frac{3}{4}\phi_{2m+1} - \frac{1}{8}\phi_{2m+2}$ into (20) yields a set of two linear equations for ϕ_{2m+1} and ϕ_{2m+2}.

It follows from results in Linz (1985) that for fixed x so that $nh = x$, the solution satisfies

$$|\phi_n - \phi(x)| = O(h^4), \tag{21}$$

provided ϕ is four times continuously differentiable. On the other hand, $|\phi_{2m+2} - \phi_{2m+1}| = O(h^4)$ as well, so there may be some fairly large oscillations. To nullify the effect of the oscillations, we shall average the ϕ values at ∞ as seen in the sequel.

We solve (7) for $\phi(x)$ but we need the value of $\phi(0)$. We do not know the value of $\phi(0)$ except for the basic insurance process where $\phi(0) = 1 - \frac{\lambda\mu}{c}$ with adherence to the net profit conditions. The method we propose works well even when the net profit condition is violated and does not require advance knowledge of $\phi(0)$. It is already proved in Kasozi and Paulsen (2005) that if $\phi_a(x)$ is a solution to (7), then $\phi_a(\infty) = \lim_{x\to\infty} \phi_a(x)$ exists and for any $a_1 \neq a_2 \in \mathbb{R}$:

$$\phi_{\alpha,a_1,a_2}(x) = \alpha\phi_{a_1}(x) + (1-\alpha)\phi_{a_2}(x),$$

for some α. In particular, when $\alpha = \frac{1-\phi(0)-a_2}{a_1-a_2}$ we obtain $\phi_{\alpha,a_1,a_2}(0) = 1 - \phi(0)$ and $\phi_{\alpha,a_1,a_2}(x) = 1 - \phi(x)$. In the special case, $\alpha\phi_{a_1}(\infty) + (1-\alpha)\phi_{a_2}(\infty) = \phi(\infty) = 1$ and $\alpha = \frac{\phi_{a_2}(\infty)}{\phi_{a_2(\infty)} - \phi_{a_1}(\infty)}$. To counter the oscillations in the approximation of $\phi_{a_1}(\infty)$, we choose $\tilde{x}$ so that

$$\phi_{a_1}(\infty) = \frac{1}{1000}\sum_{j=n_0-999}^{n_0} \phi_j, \quad n_o = \frac{\tilde{x}}{h},$$

accordingly $\phi(x) = 1 - (\alpha - \alpha\phi_{a_1}(x)) = \alpha(1 - \phi_{a_1}(x))$.

3. Results and Discussion

In this article, all the equations like (7) have a unique and continuous solution in the range of integration because of the conditions on $\phi(0)$ and the kernels. Specifically, the solutions and the kernels are bounded and sufficiently smooth so that their derivatives exist. To make the article self-contained, we apply the method on Equation (22) which is in the same form as (7). Fortunately, the exact solution is known, and hence, magnitudes of the error have been computed. The equation is

$$\tilde{\phi}(x) = e^{-x} + \int_0^x e^{-(x-u)}(\tilde{\phi}(u) + e^{-\tilde{\phi}(u)})\, du; \quad 0 \le x \le 20, \tag{22}$$

whose exact solution is

$$\tilde{\phi}_{\text{exact}}(x) = \frac{\log(x+e)}{\log e}.$$

Let $\tilde{\phi}_h(x)$ be the solution from the Block-by-Block numerical method when step size h is used. Figure 1 is a comparison of the numerical method results when $h = 0.01$. All calculations were done on a DELL laptop Core (TM) i5-7200U, CPU at 2.50GHz 2.70GHz and 4GB RAM. The programming language preferred is FORTRAN with its Double Precision feature that improves the accuracy. The numerical results are reliable.

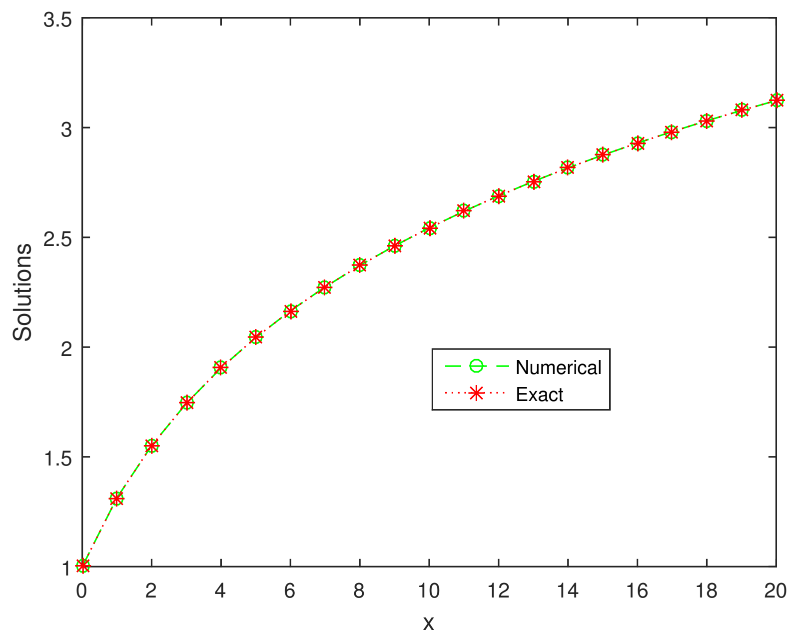

Figure 1. Both Numerical and Exact solutions to (22)

Table 1 gives the absolute percentage relative errors for some values of x computed using:

$$E_h(x) = \left| \frac{\tilde{\phi}_h(x) - \tilde{\phi}_{\text{exact}}(x)}{\tilde{\phi}_{\text{exact}}(x)} \cdot 100 \right|.$$

To 12 decimal places, the solution from the block by block numerical method compares well with the exact value of $\tilde{\phi}$ at the indicated values of x, in fact, it is very difficult to differentiate between the two. These reliable convergence results provide sufficient confidence to solve Equation (7), precise and accurate to, say, 6 decimal places.

Table 1. A comparison of Numerical and Analytical solutions when $h = 0.01$

x	$\tilde{\phi}_h(x)$	$\tilde{\phi}_{\text{exact}}(x)$	$E_h(x)$
0.00	1.000000000000	1.000000000000	0.000000000000
0.01	1.003672003746	1.003672003746	0.000000000000
0.02	1.007330656052	1.007330656052	0.000000000000
0.03	1.010975956917	1.010975956917	0.000000000000
0.04	1.014607906342	1.014607906342	0.000000000000
0.05	1.018226861954	1.018226861954	0.000000000000
0.06	1.021832704544	1.021832704544	0.000000000000
0.07	1.025425553322	1.025425553322	0.000000000000
0.50	1.168847680092	1.168847680092	0.000000000000
1.00	1.313261508942	1.313261628151	0.000009077345
5.00	2.043591976166	2.043591737747	0.000011666644
10.00	2.543040513992	2.543040513992	0.000000000000
15.00	2.874597787857	2.874597072601	0.000024881948
19.99	3.122729778290	3.122729778290	0.000000000000
20.00	3.123169183731	3.123169898987	0.000022901596

Table 2. Survival Probabilities for $c = 3,\ a_1 = 0.5,\ \alpha = 0.028818,\ \lambda = 2,\ \delta = 5\%$

x	$\phi(x)$
0	0.014409
10	0.193496
20	0.504712
30	0.784444
40	0.933185
50	0.984787
60	0.997362
70	0.999639
80	0.999960
90	0.999996
100	1.000000

Table 2 shows the results when the net profit condition is violated ($c < \lambda\mu$). The results are the same as those in Kasozi and Paulsen (2005). Without a diffusion in neither the insurance model nor the return on investments, the numerical solutions are accurate to 6 decimal places. For this case, $\phi_{a_1}(\infty) = -33.700504$. The jumps are exponentially distributed with mean $\mu = 2$. Next, we present survival probabilities of an insurance company with neither a diffusion in the insurance model nor in the return on investments model. Table 3 contains the results, accurate to 6 decimal places. The difference with Table 2 is that $c > \lambda\mu$. The jumps are exponentially distributed with mean, $\mu = 2$. When the premium rate is increased from 5 to 6, we see that the survival chances of the company increase substantially. This is in line with reality, when the premiums increase and higher than the expected claims, the insurance company is expected to survive to better levels. When the premium rate is 5, $a_1 = 0.3, \phi_{a_1}(\infty) = -1.802327$ and $\alpha = 0.356846$. Accordingly, when $c = 6$, $a_1 = 0.5, \phi_{a_1}(\infty) = -0.388109$ and $\alpha = 0.720405$. These probabilities can also be found by solving equation (7) using the numerical method for $\phi_{a_1}(x)$, then find $\phi_{a_1}(\infty)$, again stablised by averaging the last 1000 values then find $\phi(x) = \frac{\phi_{a_1}(x)}{\phi_{a_1}(\infty)}$.

Table 3. Survival Probabilities for the two premium rates, $\lambda = 2,\ \delta = 5\%$

x	$c = 5$	$c = 6$
0	0.249792	0.360202
10	0.828964	0.912387
20	0.970338	0.990276
30	0.995946	0.999101
40	0.999550	0.999929
50	0.999958	0.999995
60	0.999997	1.000000
70	1.000000	1.000000
80	1.000000	1.000000
90	1.000000	1.000000
100	1.000000	1.000000

Table 4 contains survival probabilities of an insurance company with a diffusion in the insurance model and in the return on investments model. In one case, $c < \lambda\mu$ and in the other, the net profit condition is strictly observed. The jumps are exponentially distributed with mean, $\mu = 2$. The probabilities are for premium rate $c = 3,\ \delta = 5\%, a_1 = 0.6,\ \phi_{a_1}(\infty) = 0.020241$ and premium rate $c = 5,\ \delta = 10\%,\ a_1 = 0.6,\ \phi_{a_1}(\infty) = 0.000544$.

Equation (1) in absence of a diffusion ($\sigma_R = 0$) is the famous Cramér-Lundberg model. The return on investments model is still the Black-Scholes type $(\sigma_B = 20\%)$. In this case, the Kernel, $K(x, u)$ and the forcing function $\beta(x)$ will be undefined at $x = 0$. This makes it impossible to start the block-by-block method. However, this singularity is removable by preforming a quadratic approximation at and near 0. The details are found in Kasozi (2019). The jumps are exponentially distributed with mean 2. The results are shown in the last column in Table 4. The other parameters are $c = 5,\ \delta = 10\%,\ a_1 = 0.6,\ \phi_{a_1}(\infty) = 2.301023$. The initial survival probability in this case is not 0 but $\phi(0) = 0.260754$.

Table 4. Survival Probabilities for different premium rates, diffusions and $\lambda = 2$

x	$c = 3,\ \sigma_R = 5\%,\ \sigma_B = 10\%$	$c = 5,\ \sigma_R = 5\%,\ \sigma_B = 10\%$	$c = 5,\ \sigma_R = 0,\ \sigma_B = 20\%$
5	0.070420	0.679884	0.645063
10	0.164028	0.871476	0.833828
20	0.419398	0.983004	0.961474
30	0.660892	0.998069	0.989300
40	0.822834	0.999792	0.996351
50	0.912509	0.999977	0.998513
60	0.957555	0.999997	0.999303
70	0.979280	1.000000	0.999636
80	0.989681	1.000000	0.999794
90	0.994720	1.000000	0.999875
100	0.997216	1.000000	0.999920
150	0.999824	1.000000	0.999986
200	0.999980	1.000000	0.999996
250	0.999997	1.000000	0.999998
300	0.999999	1.000000	0.999999
$\vdots$	$\vdots$	$\vdots$	$\vdots$
335	1.000000	1.000000	1.000000
340	1.000000	1.000000	1.000000
$\vdots$	1.000000	$\vdots$	$\vdots$

4. Conclusion

We have been able to compute ultimate survival probabilities in two cases: when there is a diffusion in each of the models, and the case when there is no diffusion, in the pure insurance model. This case translated into carefully handling the singularity in the associated kernel and forcing functions.

The study of the development in time of the insurer's capital is key because no company wants to suffer closure of business. It is important that this capital stays positive for decades; at least, this is what the shareholders anticipate. This work has demonstrated how the probability that the capital development stays positive is calculated. The dynamics of the capital development involve premiums received from the clients (insured) and the claim process. As earlier stated, this probability points to insurance risk, not mismanagement, and does not indicate that the insurer is broke. When the ultimate survival probability is low, then the managers of the company must take action as this indicates instability of the business. They then must undertake measures to raise this probability.

One such measure is raising premiums. There are several premium calculation principles that ensure that the premium charged is profitable and affordable. Whichever principle is employed, the premium rate is set with risk aversion in mind (also known as the net profit requirements). What is even more interesting is that the ultimate survival probability can be computed even when the profit requirements are violated. The other measure is by undertaking reinsurance arrangements.

In such an arrangement, the reinsurer takes on part or all the risk from the insurer in consideration of an agreed premium, usually dictated by the form of reinsurance at play: proportional or non-proportional. The assumption in both cases is that the reinsurer does not default on her obligations. Thirdly, the insurer can undertake investments with jumps in the investment process.

Future studies can take into account transaction costs associated with investments. The classical collective risk model is based on a compound Poisson process; one can describe it using a pure diffusion model. We have used a special case of exponential Levy model. One can go for a general Levy process.

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