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On Reducing the Linearization Coefficients of Some Classes of Jacobi Polynomials

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Abstract

This article is concerned with establishing some new linearization formulas of the modified Jacobi polynomials of certain parameters. We prove that the linearization coefficients involve hypergeometric functions of the type ${}_4F_3(1)$. Moreover, we show that the linearization coefficients can be reduced in several cases by either utilizing certain standard formulas, and in particular Pfaff-Saalschütz identity and Watson's theorem, or via employing the symbolic algebraic algorithms of Zeilberger, Petkovsek, and van Hoeij. New formulas for some definite integrals are obtained with the aid of the developed linearization formulas.

Keywords: Classical Jacobi polynomials; Chebyshev polynomials; linearization coefficients; generalized hypergeometric functions; symbolic algebraic computation; recurrenc relations; Watson's theorem

MSC 2010 No.: 42C10, 33A50, 33C25, 33D45

1. Introduction

The special functions are important in mathematical analysis and its applications (see, for example, Srivastava and Sing (2018) and Chaturvedi et al. (2020)). In particular, the Jacobi polynomials are crucial in theoretical and applied mathematical analysis. It is well-known that the class of Jacobi polynomials includes well-known six subclasses. The polynomials namely, ultraspherical, Legendre, and the first and second kinds of Chebyshev polynomials are symmetric Jacobi polynomials, while the third and fourth kinds of Chebyshev polynomials are nonsymmetric Jacobi polynomials (see, Rainville (1960), Andrews et al. (1999) and Abramowitz and Stegun (2012)). In the literature, there is a great concentration on the well-known four kinds of symmetric Jacobi polynomials from both theoretical and practical points of view. In this respect, there are several studies about ultraspherical polynomials and their various uses. For example, Abd-Elhameed and Napoli in (Abd-Elhameed and Napoli (2020)) have employed Legendre polynomials for handling some types of odd-order boundary value problems through innovative operational matrices of derivatives. Also, Elgindy and Smith-Miles in (Elgindy and Smith-Miles (2013)), treated boundary value problems, integral, and integro-differential equations using the ultraspherical integration matrices. Moreover, Doha and Abd-Elhameed utilized this kind of polynomials for the sake of obtaining numerical solutions of one and two-dimensional second-order differential equations (Doha and Abd-Elhameed (2002)). Also, Abd-Elhameed and Youssri in (Abd-Elhameed and Youssri (2019)) have developed new spectral solutions of solving fractional Riccati differential equations using the second-kind Chebyshev polynomials.

The Chebyshev polynomials of the first and second kinds $T_n(x)$ and $U_n(x)$ are the most commonly used polynomials among the four kinds of Chebyshev polynomials. In other words, the theoretical and practical studies concerning third and fourth kinds $V_n(x)$ and $W_n(x)$ are few if compared with the first and second kinds. However, all four kinds of polynomials have their roles. $U_n(x)$ has important parts in numerical integration. $V_n(x)$ and $W_n(x)$ are helpful in situations in which singularities occur at one endpoint (+1 or -1) but not at the other (Mason and Handscomb (2003)). Doha and Abd-Elhemeed (Doha and Abd-Elhameed (2014)) have developed new formulas for the coefficients of integrated expansions and integrals of Chebyshev polynomials of third and fourth kinds. For some other studies about these two classes of Jacobi polynomials, see, Doha et al. (2015) and Abd-Elhameed et al. (2016b).

The linearization of orthogonal polynomials and the connection coefficients problems between them are very important. In particular, the linearization and connection coefficients problems of ultraspherical and Jacobi polynomials have been investigated by many researchers. For some of these studies, one can be refereed to Askey and Gasper (1972), Gasper (1970 a), Gasper (1970 b), Hylleraas (1962), Rahman (1981) and Chaggara and Koepf (2010)). Other studies for these kinds of problems can be found in (Abd-Elhameed (2019), Doha and Abd-Elhameed (2016), Abd-Elhameed et al. (2016 a), Abd-Elhameed (2015a), Abd-Elhameed (2015 b), Maroni and da Rocha (2008), Doha and Ahmed (2004), Doha (2003), Sánchez-Ruiz (2001), Sánchez-Ruiz and Dehesa (2001)). Recently, some important nonlinear problems were solved by using some orthogonal polynomials. The main idea behind the treatment of the nonlinear terms in these equations was based on utilizing some linearization formulas of some orthogonal polynomials. For example, Abd-Elhameed in (Abd-Elhameed (2019)) solved a nonlinear Riccati differential equation using a specific linearization formula of Chebyshev polynomials of the third kind, while the same author in Abd-Elhameed (2021) handled the nonlinear one dimensional Burger's equation based on using the linearization formula of Chebyshev polynomials of the sixth kind.

The general linearization problem involves two important special problems. The first is the Clebsch-Gordan-type problem which is considered as the standard linearization problem, and the second is the well-known connection problem between two polynomial sets (see, Abd-Elhameed et al. (2016)).

The principal goals of the current paper are three-fold:

(1) Deriving a new linearization formula of Jacobi polynomials of certain parameters.
(2) Reducing the linearization coefficients of the derived linearization formula for some choices of the involved parameters, and hence deducing some new linearization formulas of third and fourth kinds of Chebyshev polynomials.
(3) Making use of the derived formulas to deduce closed formulas of some new definite integrals.

As far as we know, most of the derived formulas in this article are novel. To be more precise, the novelty of the paper can be summarized as follows:

(1) The main theorem which gives a linearization formula of certain Jacobi polynomials is new.
(2) The derived special linearization formulas of the main theorem are free of hypergeometric functions.

The rest of this paper is as follows. In the following section, we state and prove the basic theorem concerning a new linearization formula of certain Jacobi polynomials. In addition, some linearization formulas of third and fourth kinds of Chebyshev polynomials are given in simplified forms which do not involve any hypergeometric functions. These formulas are derived by using standard reduction formulas, or by using some symbolic algebraic algorithms, and in particular, the algorithms of Zeilberger, Petkovsek, and van Hoeij. Section 3 is devoted to introducing some new definite integrals involving products of Jacobi polynomials of certain parameters based on making use of the newly developed linearization formulas. We end the paper with some conclusions in Section 4.

2. Some Linearization Formulas of the Modified Jacobi Polynomials for Certain Parameters

This section is concerned with developing our main results. In the following, we will give some new linearization formulas of Jacobi polynomials of certain choices of their parameters.
From now on, we denote by $P_i^{(\mu,\nu)}(x)$ the normalized Jacobi polynomials that satisfy: $P_i^{(\mu,\nu)}(1) = 1,$ (see, Abd-Elhameed (2015 a)).

Theorem 2.1.

Let i be a nonnegative integer. For $\lambda, \mu, \nu > -1,$ the following linearization formula holds:

$$R_i^{\left(\lambda,\frac{1}{2}\right)}(x)R_i^{\left(\lambda+1,-\frac{1}{2}\right)}(x)=\sum_{k=0}^{2i}\frac{\binom{2i}{k}\left(\lambda+\frac{3}{2}\right)_k(\mu+1)_k(2i+2\lambda+3)_k}{(\lambda+2)_k(2\lambda+2)_k(k+\mu+\nu+1)_k}$$
$$\times\,{}_4F_3\left(\left.\begin{matrix}k-2i,k+\lambda+\frac{3}{2},k+2i+2\lambda+3,k+\mu+1\\ k+\lambda+2,k+2\lambda+2,2k+\mu+\nu+2\end{matrix}\right|1\right)R_k^{(\mu,\nu)}(x). \quad (1)$$

Proof:

At first, and with the aid of the hypergeometric form of the modified Jacobi polynomials $R_i^{(\mu,\nu)}(x)$, one can write (see, Rahman (1981)).

$$R_i^{\left(\lambda,\frac{1}{2}\right)}(x)R_i^{\left(\lambda+1,-\frac{1}{2}\right)}(x)=\,{}_2F_1\left(\left.\begin{matrix}-i,i+\lambda+\frac{3}{2}\\ \lambda+1\end{matrix}\right|\frac{1-x}{2}\right)\,{}_2F_1\left(\left.\begin{matrix}-i,i+\lambda+\frac{3}{2}\\ \lambda+2\end{matrix}\right|\frac{1-x}{2}\right), \quad (2)$$

and in virtue of a sauitable transformation formula (see, Bateman et al. (1953)), equation (2) can be turned into

$$R_i^{\left(\lambda,\frac{1}{2}\right)}(x)R_i^{\left(\lambda+1,-\frac{1}{2}\right)}(x)=\,{}_3F_2\left(\left.\begin{matrix}-2i,2i+2\lambda+3,\ \lambda+\frac{3}{2}\\ 2\lambda+2,\ \lambda+2\end{matrix}\right|\frac{1-x}{2}\right). \quad (3)$$

Making use of Lemma 1 that employed in Abd-Elhameed (2015 a) (see, p. 589 in Abd-Elhameed (2015 a)) that derived by Fields and Wimp (1961), and with suitable choices of the involved parameters, enables one to obtain the following relation

$${}_3F_2\left(\left.\begin{matrix}-2i,2i+2\lambda+3,\ \lambda+\frac{3}{2}\\ 2\lambda+2,\ \lambda+2\end{matrix}\right|\frac{1-x}{2}\right)$$
$$=\sum_{k=0}^{2i}\frac{\binom{2i}{k}\left(\lambda+\frac{3}{2}\right)_k(\mu+1)_k(2i+2\lambda+3)_k}{(\lambda+2)_k(2\lambda+2)_k(k+\mu+\nu+1)_k}$$
$$\times\,{}_4F_3\left(\left.\begin{matrix}k-2i,k+\lambda+\frac{3}{2},k+2i+2\lambda+3,k+\mu+1\\ k+\lambda+2,k+2\lambda+2,2k+\mu+\nu+2\end{matrix}\right|1\right) \quad (4)$$
$$\times\,{}_2F_1\left(\left.\begin{matrix}-k,k+\mu+\nu+1\\ \mu+1\end{matrix}\right|\frac{1-x}{2}\right),$$

which immediately yields

$$R_i^{(\lambda,\frac{1}{2})}(x)\, R_i^{(\lambda+1,-\frac{1}{2})}x) = \sum_{k=0}^{2i} \frac{\binom{2i}{k}\left(\lambda+\frac{3}{2}\right)_k (\mu+1)_k (2i+2\lambda+3)_k}{(\lambda+2)_k (2\lambda+2)_k (k+\mu+\nu+1)_k}$$
$$\times\, {}_4F_3\left(\left. \begin{matrix} k-2i, k+\lambda+\frac{3}{2}, k+2i+2\lambda+3, k+\mu+1 \\ k+\lambda+2, k+2\lambda+2, 2k+\mu+\nu+2 \end{matrix} \right| 1 \right) R_k^{(\mu,\nu)}(x).$$

Theorem 2.1 is now proved.

Corollary 2.1.

For every nonnegative integer i, the following linearization formula holds**:**

$$R_i^{(\frac{1}{2},\lambda)}(x)\, R_i^{(\frac{-1}{2},\lambda+1)}(x) = \frac{(\lambda+1)_i\, (\lambda+2)_i}{(\frac{3}{2})_i\, (\frac{1}{2})_i}$$
$$\times \sum_{k=0}^{2i} \frac{(-1)^k \binom{2i}{k}\left(\lambda+\frac{3}{2}\right)_k (\nu+1)_k (2i+2\lambda+3)_k}{(\lambda+2)_k (2\lambda+2)_k (k+\mu+\nu+1)_k}$$
$$\times\, {}_4F_3\left(\left. \begin{matrix} k-2i, k+\lambda+\frac{3}{2}, k+2i+2\lambda+3, k+\mu+1 \\ k+\lambda+2, k+2\lambda+2, 2k+\mu+\nu+2 \end{matrix} \right| 1 \right) R_k^{(\nu,\mu)}(x).$$

Proof:

If x in (1) is replaced by $-x$, then the above formula can be easily obtained.

2.1. Linearization Formulas of $V_i(x)$ and $W_i(x)$

Our goal in this section is to obtain some new linearization formulas of Chebyshev polynomials of third and fourth kinds $V_i(x)$ and $W_i(x)$. The key idea behind obtaining such formulas is to reduce the ${}_4F_3(1)$ that appears in (1). This reduction can be performed by using some standard formulas such as Pfaff-Saalschütz identity and Watson's theorem, or through utilizing some symbolic algebraic algorithms, and in particular the algorithms of Zeilberger, Petkovsek and van Hoeij. In this respect, we state and prove the following corollaries.

Corollary 2.2.

If we set $\lambda = -\frac{1}{2}$ and $\nu = \frac{1}{2}$, in relation (1), then the following linearization formula holds:

$$V_i(x)W_i(x) \tag{5}$$

$$= (2i+1)\sum_{k=0}^{2i} \frac{(-1)^k \binom{2i}{k} \, (2i+2)_k \, (\mu+1)_k \, \left(k-2i+\mu+\frac{1}{2}\right)_{2i-k}}{\left(\frac{3}{2}\right)_k \left(k+\mu+\frac{3}{2}\right)_k \left(2k+\mu+\frac{5}{2}\right)_{2i-k}} \, R_k^{\left(\mu,\frac{1}{2}\right)}(x).$$

Proof:

If we substitute by $\lambda = -\frac{1}{2}$ and $\nu = \frac{1}{2}$ in relation (1), and noting the two relations

$$R_i^{\left(-\frac{1}{2},\frac{1}{2}\right)}(x) = V_i(x), \qquad R_i^{\left(\frac{1}{2},-\frac{1}{2}\right)}(x) = \frac{W_i(x)}{2i+1},$$

then we get

$$\begin{aligned} V_i(x)W_i(x) = (2i+1) \sum_{k=0}^{2i} & \frac{\binom{2i}{k}(2i+2)_k(\mu+1)_k}{\left(\frac{3}{2}\right)_k \left(k+\mu+\frac{3}{2}\right)_k} \\ & \times {}_3F_2\left(\left.\begin{matrix} k-2i, k+2i+2, k+\mu+1 \\ k+\frac{3}{2}, 2k+\mu+\frac{5}{2} \end{matrix}\right| 1\right) R_k^{\left(\mu,\frac{1}{2}\right)}(x). \end{aligned} \tag{6}$$

Now, and based on the application of the well-known Pfaff-Saalschütz identity (see, Olver et al. (2010)), the ${}_3F_2(1)$ in (6) reduces to

$${}_3F_2\left(\left.\begin{matrix} k-2i, k+2i+2, k+\mu+1 \\ k+\frac{3}{2}, 2k+\mu+\frac{5}{2} \end{matrix}\right| 1\right) = \frac{(-1)^k \left(k-2i+\mu+\frac{1}{2}\right)_{2i-k}}{\left(2k+\mu+\frac{5}{2}\right)_{2i-k}},$$

and therefore, the following linearization formula holds:

$$\begin{aligned} & V_i(x)W_i(x) \\ & \quad = (2i+1)\sum_{k=0}^{2i} \frac{(-1)^k \binom{2i}{k} \, (2i+2)_k \, (\mu+1)_k \, \left(k-2i+\mu+\frac{1}{2}\right)_{2i-k}}{\left(\frac{3}{2}\right)_k \, \left(k+\mu+\frac{3}{2}\right)_k \, \left(2k+\mu+\frac{5}{2}\right)_{2i-k}} \, R_k^{\left(\mu,\frac{1}{2}\right)}(x). \end{aligned}$$

As special cases of (5), the following two linearization formulas of third and fourth kinds of Chebyshev polynomials can be obtained.

Corollary 2.3.

For every nonnegative integer i, the following two linearization formulas hold

$$V_i(x)W_i(x) = \sum_{k=0}^{2i} V_k(x), \tag{7}$$

$$V_i(x)W_i(x) = U_{2i}(x). \tag{8}$$

Proof:

If we set $\mu = \frac{-1}{2}, \frac{1}{2}$, respectively, in relation (5), then linerization formulas (7) and (8) can be obtained.

Remark 2.1.

If x in (7) is replaced by $-x$, then noting the identity: $V_i(-x) = (-1)^i\, W_i(x)$, the following linearization formula holds:

$$V_i(x)\, W_i(x) = \sum_{k=0}^{2i} (-1)^k\, W_k(x). \tag{9}$$

Corollary 2.4.

If we set $\lambda = -\frac{1}{2}, \nu = \mu$, and each is replaced by $\left(\mu - \frac{1}{2}\right)$ in relation (1), then we have

$$V_i(x)W_i(x) = \Gamma(\mu) \sum_{k=0}^{i} \frac{(-1)^{i+k} \binom{i}{k} (2k+1)\,(i+1)_k \left(\mu + \frac{1}{2}\right)_k}{\left(\frac{3}{2}\right)_k (k+\mu)_k\,(2k+\mu+1)_{i-k}\,\Gamma(k-i+\mu)}\, C_{2k}^{(\mu)}(x). \tag{10}$$

Proof:

Substitution of $\lambda = -\frac{1}{2}, \nu = \mu$ into relation (1) yields

$$\begin{aligned} V_i(x)W_i(x) = \quad & (2i+1) \sum_{k=0}^{2i} \frac{\binom{2i}{k} (2i+2)_k (\mu+1)_k}{\left(\frac{3}{2}\right)_k (k+2\mu+1)_k} \\ & \times {}_3F_2\left(\left.\begin{matrix} k-2i, k+2i+2, k+\mu+1 \\ k+\frac{3}{2}, 2k+2\mu+2 \end{matrix}\right| 1\right) R_k^{(\mu,\mu)}(x). \end{aligned} \tag{11}$$

Now, and by means of Watson's theorem (see, Olver et al. (2010)), the ${}_3F_2(1)$ in (11) can be reduced to the form

$${}_3F_2\left(\left.\begin{matrix} k-2i, k+2i+2, k+\mu+1 \\ k+\frac{3}{2}, 2k+2\mu+2 \end{matrix}\right| 1\right)$$

$$= \begin{cases} \dfrac{(-1)^{i-\frac{k}{2}}\ 2^{k-2i}(2i-k)!\left(\frac{1}{2}(k+1)-i+\mu\right)_{i-\frac{k}{2}}}{\left(i-\frac{k}{2}\right)!\left(k+\frac{3}{2}\right)_{i-\frac{k}{2}}\left(k+\mu+\frac{3}{2}\right)_{i-\frac{k}{2}}}, & k\ \ even, \\ 0, & k\ \ odd, \end{cases}$$

and therefore, the following linearization formula holds:

$$V_i(x)\,W_i(x) = \Gamma(\mu)\sum_{k=0}^{i} \frac{(-1)^{i+k}\binom{i}{k}\,(2k+1)\,(i+1)_k\left(\mu+\frac{1}{2}\right)_k}{\left(\frac{3}{2}\right)_k\,(k+\mu)_k\,(2k+\mu+1)_{i-k}\,\Gamma(k-i+\mu)}\,C_{2k}^{(\mu)}(x).$$

Corollary 2.5.

For every nonnegative integer *i*, the following two linerization formula hold

$$V_i(x)W_i(x) = 1 + 2\sum_{k=1}^{i} T_{2k}(x), \tag{12}$$

$$V_i(x)W_i(x) = \frac{\pi}{2}\sum_{k=0}^{i} \frac{(-1)^{i+k}\,(4k+1)\,(k+i)!}{(i-k)!\,\Gamma\left(k-i+\frac{1}{2}\right)\ \Gamma\left(k+i+\frac{3}{2}\right)} P_{2k}(x). \tag{13}$$

Proof:

Setting $\mu = 0$, and $\frac{1}{2}$, respectively in (10) yields the linerization formulas (12) and (13).

Remark 2.2.

The three linearization formulas (7), (9) and (12) lead to the following three well-known trigonometric identities:

$$\sum_{k=0}^{2i} \cos\left(\left(k+\frac{1}{2}\right)\theta\right) = \frac{\sin((2i+1)\theta)}{2\sin\left(\frac{\theta}{2}\right)},$$

$$\sum_{k=0}^{2i} (-1)^k \sin\left(\left(k+\frac{1}{2}\right)\theta\right) = \frac{\sin((2i+1)\theta)}{2\cos\left(\frac{\theta}{2}\right)},$$

$$1+2\sum_{k=1}^{i}\cos(2k\theta)=\frac{\sin((2i+1)\theta)}{\sin\theta}.$$

2.2. Some Other Linearization Formulas of $V_i(x)$ and $W_i(x)$

In this subsection, we give some other linearization formulas of products of Chebyshev polynomials of third and fourth kinds in reduced forms. The results are given in the following two corollaries.

Corollary 2.6.

For the case $\lambda=-\frac{1}{2}, \nu=\mu+1$, the following linearization formula is obtained:

$$V_i(x)W_i(x)=\frac{\sqrt{\pi}\ \Gamma\left(\mu+\frac{1}{2}\right)}{2^{2\mu+1}\,\Gamma(\mu+1)}$$

$$\times\left\{\sum_{m=0}^{i}\frac{(-1)^{m+i}\,(m+i)!\,\Gamma(2\,m+2\,\mu+2)}{(2m)!\,(i-m)!\,\Gamma\left(m-i+\mu+\frac{1}{2}\right)\Gamma\left(m+i+\mu+\frac{3}{2}\right)}R_{2\,m}^{(\mu,\mu+1)}(x)\right.$$
$$\left.+\sum_{m=0}^{i-1}\frac{(-1)^{m+i+1}(m+i+1)!\,\Gamma(2m+2\mu+3)}{(2m+1)!\,(i-m-1)!\,\Gamma\left(m-i+\mu+\frac{3}{2}\right)\Gamma\left(m+i+\mu+\frac{5}{2}\right)}R_{2\,m+1}^{(\mu,\mu+1)}(x)\right\}. \quad (14)$$

Proof:

If we substitute by $\lambda=\frac{-1}{2}, \nu=\mu+1$, then the linearization formula (1) is turned into

$$V_i(x)W_i(x)=\ (2i+1)\sum_{k=0}^{2i}\frac{\binom{2i}{k}(2i+2)_k(\mu+1)_k}{\left(\frac{3}{2}\right)_k(k+2\mu+2)_k}$$
$$\times\ {}_3F_2\left(\left.\begin{matrix}k-2i,k+2i+2,k+\mu+1\\ k+\frac{3}{2},2k+2\mu+3\end{matrix}\right|1\right)R_k^{(\mu,\mu+1)}(x). \quad (15)$$

Now, if we set

$$H_{j,i,\mu}=\ {}_3F_2\left(\left.\begin{matrix}k-j,-j+4i+2,\mu-j+2i+1\\ -j+2i+\frac{3}{2},2\mu-2j+4i+3\end{matrix}\right|1\right),$$

then with the aid of the celebrated algorithm of Zeilberger (Koepf (2014)), via the Maple software, and in particular, sumrecursion command, the following recurrence relation of order two is satisfied by $H_{j,i,\mu}$:

$$\begin{aligned}(1-j)(2\mu-j+2)(j-4i-3)(2\mu-2j+4i+3)\\ \times(2\mu-j+4i+4)\,H_{j-2,i,\mu}-(2j-4i-5)\\ \times(-6\mu-2j^2+4\mu j+8\,j\,i+8j-16i^2-8\,\mu\,i-24\,i-9)\\ \times(2\mu-2j+4i+5)H_{j-1,i,\mu}+(2j-4i-5)(2j-4i-3)\\ \times(2\mu-2j+4i+3)(2\mu-2j+4i+5)^2\,H_{j,i,\mu}\\ =0.\end{aligned} \tag{16}$$

with the following initial conditions

$$H_{0,i,\mu}=1,\quad H_{1,i,\mu}=\frac{1}{4i+2\mu+1},$$

which has the following exact solution:

$$H_{j,i,\mu}=\begin{cases}\dfrac{(-1)^{\frac{j}{2}}\;j!\left(\frac{1-j}{2}+\mu\right)_{\frac{j}{2}}}{2^j\;(\frac{j}{2})!\left(-j+2i+\frac{3}{2}\right)_{\frac{j}{2}}\left(-j+2i+\mu+\frac{3}{2}\right)_{\frac{j}{2}}}, & j\ \ even,\\[2ex] \dfrac{(-1)^{\frac{j-1}{2}}\,(j+1)!\left(-\frac{j}{2}+\mu+1\right)_{\frac{j-1}{2}}}{2^{j+1}\,(\frac{j+1}{2})!\left(-j+2i+\frac{3}{2}\right)_{\frac{j-1}{2}}\left(-j+2i+\mu+\frac{3}{2}\right)_{\frac{j+1}{2}}}, & j\ \ odd,\end{cases} \tag{17}$$

and therefore, the, ${}_3F_2(1)$ in (15) has the following reduction formula

$$\begin{aligned}&{}_3F_2\left(\left.\begin{matrix}k-2i,k+2i+2,k+\mu+1\\ k+\frac{3}{2},2k+2\mu+3\end{matrix}\,\right|1\right)\\ &=\begin{cases}\dfrac{(-1)^{i-\frac{k}{2}}\;2^{k-2i}(2i-k)!\left(\frac{1}{2}(k+1)-i+\mu\right)_{i-\frac{k}{2}}}{\left(i-\frac{k}{2}\right)!\left(k+\frac{3}{2}\right)_{i-\frac{k}{2}}\left(k+\mu+\frac{3}{2}\right)_{i-\frac{k}{2}}}, & k\ \ even,\\[2ex] \dfrac{(-1)^{i-(\frac{k+1}{2})}\;2^{k-2i-1}(-k+2i+1)!\;\left(\frac{k}{2}-i+\mu+1\right)_{i-(\frac{k+1}{2})}}{\left(i-(\frac{k-1}{2})\right)!\left(k+\frac{3}{2}\right)_{i-(\frac{k+1}{2})}\left(k+\mu+\frac{3}{2}\right)_{i-(\frac{k-1}{2})}}, & k\ \ odd.\end{cases}\end{aligned} \tag{18}$$

The last reduction formula enables one to write the linearization formula (15) in the form

$$V_i(x)W_i(x)=\frac{\sqrt{\pi}\,\Gamma\left(\mu+\frac{1}{2}\right)}{2^{2\mu+1}\Gamma(\mu+1)}$$

$$\times\left\{\sum_{m=0}^{i}\frac{(-1)^{m+i}\,(m+i)!\,\Gamma(2\,m+2\,\mu+2)}{(2m)!\,(i-m)!\,\Gamma\left(m-i+\mu+\frac{1}{2}\right)\Gamma\left(m+i+\mu+\frac{3}{2}\right)}\;R_{2m}^{(\mu,\mu+1)}(x)\right.$$

$$\left.+\sum_{m=0}^{i-1}\frac{(-1)^{m+i+1}\,(m+i+1)!\,\Gamma(2m+2\mu+3)}{(2m+1)!\,(i-m-1)!\,\Gamma\left(m-i+\mu+\frac{3}{2}\right)\Gamma\left(m+i+\mu+\frac{5}{2}\right)}\;R_{2m+1}^{(\mu,\mu+1)}(x)\right\},$$

and this completes the proof of Corollary 2.6.

Remark 2.3.

It is worthy to note that the recurrence relation (16) can be solved exactly through any suitable symbolic algorithms to obtain (17). For this purpose, the algorithms of Petkovsek (Koepf (2014)(Chapter 9)), or the improved version of van Hoeij (van Hoeij (1999)) may be used.

Corollary 2.7.

For the case $\lambda=-\frac{1}{2},\nu=\frac{3}{2}$, the following linearization formula holds:

$$V_i(x)\,W_i(x)=\frac{1}{2}\,(2i+1)\,\Gamma\left(\mu+\frac{1}{2}\right)$$

$$\times\sum_{k=0}^{2i}\frac{(-1)^k\binom{2i}{k}(2i+2)_k(\mu+1)_k(2k(2k+2\mu+5)-16i(i+1)+6\mu+3)}{(2k+3)\left(\frac{3}{2}\right)_k\left(k+\mu+\frac{5}{2}\right)_k\left(2k+\mu+\frac{7}{2}\right)_{2i-k}\Gamma\left(k-2i+\mu+\frac{3}{2}\right)}R_k^{\left(\mu,\frac{3}{2}\right)}(x). \tag{19}$$

Proof:

The substitution of $\lambda=-\frac{1}{2},\nu=\frac{3}{2}$ into relation (1), leads to the linearization formula

$$V_i(x)W_i(x)=(2i+1)\sum_{k=0}^{2i}\frac{\binom{2i}{k}(2i+2)_k(\mu+1)_k}{\left(\frac{3}{2}\right)_k\left(k+\mu+\frac{5}{2}\right)_k}\times{}_3F_2\left(\left.\begin{matrix}k-2i,k+2i+2,k+\mu+1\\ k+\frac{3}{2},2k+\mu+\frac{7}{2}\end{matrix}\right|1\right)R_k^{\left(\mu,\frac{3}{2}\right)}(x). \tag{20}$$

Now, if we set

$$G_{j,i,\mu} = {}_3F_2\left(\begin{matrix} -j, -j+4i+2, \mu-j+2i+1 \\ -j+2i+\frac{3}{2}, \mu-2j+4i+\frac{7}{2} \end{matrix}\middle| 1\right),$$

then the application of Zeilberger's algorithm enables one to obtain the following recurrence relation of order one which is satisfied by $G_{j,i,\mu}$:

$$\begin{aligned}&(2\mu-2j+3)(2j-4i-5)(2\,j(2\,\mu+8\,i+5)-4j^2-(2\,\mu+1)(4\,i+3))\\ &\times(2\mu-2\,j+8\,i+7)G_{j-1,i,\mu}+(2j-4i-3)\times(2\mu-4\,j+8i+7)(2\mu-4j+8i+9)\\ &\times(2\,j(2\,\mu+8\,i+9-10\,\mu-4\,j^2)-4\,(2\,\mu+5)i-17)G_{j,i,\mu}=0,\;\; G_{0,i,\mu}=1,\end{aligned}$$

which can be exactly solved to give

$$\begin{aligned}&{}_3F_2\left(\begin{matrix} -j, -j+4i+2, \mu-j+2i+1 \\ -j+2i+\frac{3}{2}, \mu-2j+4i+\frac{7}{2} \end{matrix}\middle| 1\right)\\ &= \frac{(-1)^j(-j+\mu+\frac{3}{2})_{j-1}}{2(2j-4i-3)(-2j+4i+\mu+\frac{7}{2})_j}\times\left(2j(2\mu+8i+5)-4j^2-(2\mu+1)(4i+3)\right).\end{aligned}$$

Accordingly, the ${}_3F_2(1)$ in (20) is equivalent to

$$\begin{aligned}&{}_3F_2\left(\begin{matrix} k-2i, k+2i+2, k+\mu+1 \\ k+\frac{3}{2}, 2k+\mu+\frac{7}{2} \end{matrix}\middle| 1\right)\\ &\qquad= \frac{(-1)^k\,\Gamma(\mu+\frac{1}{2})(2k(2k+2\mu+5)-16i(i+1)+6\mu+3)}{2(2k+3)\left(2k+\mu+\frac{7}{2}\right)_{2i-k}\Gamma(k-2i+\mu+\frac{3}{2})},\end{aligned} \tag{21}$$

and hence, the following linearization formula holds:

$$\begin{aligned}&V_i(x)\,W_i(x) = \frac{1}{2}\,(2i+1)\Gamma\left(\mu+\frac{1}{2}\right)\\ &\times\sum_{k=0}^{2i}\frac{(-1)^k\binom{2i}{k}(2i+2)_k(\mu+1)_k(2\,k\,(2k+2\mu+5)-16i(i+1)+6\mu+3)}{(2k+3)\left(\frac{3}{2}\right)_k\left(k+\mu+\frac{5}{2}\right)_k\left(2k+\mu+\frac{7}{2}\right)_{2i-k}\Gamma\left(k-2i+\mu+\frac{3}{2}\right)}R_k^{\left(\mu,\frac{3}{2}\right)}(x).\end{aligned}$$

3. New Formulas for Some Definite Integrals

In this section, we are interested in presenting some applications to the derived linearization formulas. We will introduce some new formulas for definite integrals involving three products of Jacobi polynomials of certain parameters by means of applying the developed linearization formulas in the previous section.

Theorem 3.1.

Let i, m be positive integers. The following identity holds:

$$\int_{-1}^{-1} (1-x)^{\mu}\,(1+x)^{\nu}\,R_i^{\left(\lambda,\frac{1}{2}\right)}(x) R_i^{\left(\lambda+1,\frac{-1}{2}\right)}(x) R_m^{(\mu,\nu)}(x)dx$$

$$= \frac{m!\,\binom{2i}{m}\,2^{2\lambda+\mu+\nu+2}\,\Gamma\left(m+\lambda+\frac{3}{2}\right)\,\Gamma(m+\nu+1)}{\sqrt{\pi}\,\Gamma(2i+2\lambda+3)}$$

$$\times\,\Gamma(m+2i+2\,\lambda+3)\,\Gamma(\lambda+1)\,\Gamma(\lambda+2)\,\Gamma(\mu+1) \tag{22}$$

$$\times\;{}_4\tilde{F}_3\left(\left.\begin{matrix} m-2i, m+\lambda+\frac{3}{2}, m+2i+2\lambda+3, m+\mu+1 \\ m+\lambda+2, m+2\lambda+2, 2m+\mu+\nu+2 \end{matrix}\right|1\right).$$

where the notation ${}_p\tilde{F}_q$ denotes the regularized generalized hypergeometric function (see Abd-Elhameed (2015 a)).

Proof:

The result in (22) can be followed if we multiply both sides of formula (1) by

$$(1-x)^{\mu}\,(1+x)^{\nu}\,R_m^{(\mu,\nu)}(x),$$

integrate over $(-1,1)$, and make use of the orthogonality *relation of* $R_m^{(\mu,\nu)}(x)$.

Now, since the ${}_4\tilde{F}_3$ in identity (22) can be written in several reduced forms for certain choices of the parameters involved as implemented in Section 2, so some integrals can be written in explicit forms free of any hypergeometric functions. The results are given in the following corollary.

Corollary 3.1.

For all nonnegative integers m and i, the following integrals formulas are valid

$$\int_{-1}^{1} \sqrt{x+1}\,\,(1-x)^{\mu} V_i(x)\,W_i(x)\,R_m^{\left(\mu,\frac{1}{2}\right)}(x)\,dx$$

$$= \frac{(-1)^m\,\pi\,2^{\frac{1}{2}-\mu}\Gamma(2\mu+1)\,\,(m+2i+1)!}{(2i-m)!\,\Gamma\left(m-2i+\mu+\frac{1}{2}\right)\Gamma\left(m+2i+\mu+\frac{5}{2}\right)}, \tag{23}$$

$$\int_{-1}^{1} (1-x^2)^{\mu-\frac{1}{2}} V_i(x) W_i(x) C_{2m}^{(\mu)}(x) dx = \frac{(-1)^{i+m}\, \pi\, 2^{1-2\mu}\, \Gamma(2\mu)\, (m+i)!}{(i-m)!\ \Gamma(m-i+\mu)\Gamma(m+i+\mu+1)}, \tag{24}$$

$$\int_{-1}^{1} (1-x)^{\mu}\ (1+x)^{\mu+1} V_i(x)\, W_i(x)\, R_m^{(\mu,\mu+1)}(x)\, dx = \frac{\pi\, \Gamma(2\,\mu+1)}{2^{2\,\mu}}$$

$$\times \begin{cases} \dfrac{(-1)^{\frac{m}{2}+i}(i+\frac{m}{2})!}{(i-\frac{m}{2})!\ \Gamma\left(\frac{1}{2}(m+1)-i+\mu\right)\Gamma\left(\frac{1}{2}(m+3)+i+\mu\right)}, & m\ even, \\ \dfrac{(-1)^{\frac{m+1}{2}+i}(i+\frac{m+1}{2})!}{(i-(\frac{m+1}{2}))!\,\Gamma\left(\frac{m}{2}-i+\mu+1\right)\Gamma\left(\frac{m}{2}+i+\mu+2\right)}, & m\ \ odd, \end{cases} \tag{25}$$

$$\int_{-1}^{1} (1-x)^{\mu}\ (1+x)^{\frac{3}{2}} V_i(x)\, W_i(x)\, R_m^{(\mu,\frac{3}{2})}(x)\, dx$$

$$= \frac{\pi(-1)^m \Gamma(2\mu+1)\,(m+2i+1)!\ (6\mu+2m(2\mu+2m+5)-16\,i(i+1)+3))}{2^{\mu+\frac{1}{2}}\,(2i-m)!\,\Gamma\left(m-2i+\mu+\frac{3}{2}\right)\ \Gamma\left(m+2i+\mu+\frac{7}{2}\right)}. \tag{26}$$

Proof:

The proof of Corollary 3.1 can be followed as an immediate consequence of formulas (5), (10), (14) and (19).

4. Conclusion

We have developed some new linearization formulas of Jacobi polynomials of special parameters. Some transformation formulas and some other standard formulas serve in the derivation of some linearization formulas. Furthermore, Some symbolic algebra such as Zeilberger, Petkovsek, and van Hoeij algorithms are also utilized.

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