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On the Invariant Subspaces of the Fractional Integral Operator

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Abstract

In operator theory, there is an important problem called the invariant subspace problem. This important problem of mathematics has been clear for more than half a century. However, the solution seems to be nowhere in sight. With this motivation, we investigate the invariant subspaces of the fractional integral operator in the Banach space with certain conditions in this paper. Also, by using the Duhamel product method, unicellularity of the fractional integral operator on some space is obtained and the description of the invariant subspaces is given.

Keywords: Riemann-Liouville fractional integration operator; Invariant subspaces; Lattice; Unicellularity; Duhamel product

MSC 2010 No.: 47A15, 34A08

1. Introduction

The famous invariant subspace problem is the question whether every bounded linear operator on an infinite dimensional separable Hilbert space has a nontrivial invariant subspace. On the history of the invariant subspace problem, the readers can consult in the book of Chalendar and Partington (2011). Also, questions concerning the existence of invariant subspaces for particular classes of operators have produced a wealth of interesting theorems and examples (see Nikolskii (1986)).

On the other hand, the Duhamel product was proposed by Wigley (1974). The Duhamel product has numerous applications in the boundary value problems of mathematical physics, the sloping beach problem, the theory of ordinary differential equations, the theory of operator equations of the some form and function Banach algebras (see, for example, Ivanova and Melikhov (2017), Wigley (1974), Karaev (2003), and Karaev (2018)).

The study of the lattice of invariant subspaces of the Volterra integration operator has a long history and extensive literature (see Nikolskii (1986)). The description of the invariant subspaces for some Volterra integration operator is essentially the problem posed in 1938 by Gelfand (1938) and first solved by Agmon (1949) who showed that all invariant subspaces of certain space have the form of a linearly ordered lattice. Note that an Volterra integration operator is an unicellular operator on the Banach spaces (see (Brodskii (1971); Nikolskii (1986))). In Ostapenko and Tarasov (1977), it was shown that a Volterra integration operator is unicellular on the certain space. It is well known (Gohberg and Krein (1970) and Nikolskii (1986)) the fractional integral operator is unicellular on the Lebesgue space. The invariant subspaces of the integration operator defined on the Sobolev space by Tsekanovskii (1965). Domanov and Malamud (2002) have extended these results for the fractional integral operator defined in the Sobolev spaces. Also, various applications of fractional integral operator can be found in (Agarwal et al. (2015); Jain and Agarwal (2019); Jena et al. (2020); Malamud (2019); Mohiuddine et al. (2021); Sarikaya et al. (2019); Tapdigoglu and Torebek (2020); Tapdigoglu and Torebek (2021); Torebek and Tapdigoglu (2017); Usta (2021); Usta et al. (2020); Yaying et al. (2020)). Note that some results related with non-trivial invariant subspaces and unicellularity problem for the integration operator in various spaces have been obtained with application of the Duhamel product in papers (Gürdal (2009a); Gürdal (2009b); Gürdal et al. (2015); Karaev (2005); Karaev (2018); Karaev and Gürdal (2011); Karaev et al. (2011); Saltan (2016); Saltan (2018); Tapdigoglu (2013); Tapdigoglu (2020)).

The aim of the present paper is to investigate the fractional integral operator defined on the certain space. The study is organized as follows. Section 2 presents the study and fundamental concepts. In Section 3, we prove the main results. That is, we describe the lattice of invariant subspaces and study the question of unicellularity of the some operator by using the Duhamel product.

2. Preliminaries

In this manuscript, we consider unicellularity problem for the fractional integral operator

$$J^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}f(t)\,dt,\ \ \mathrm{Re}\,\alpha>0,$$

which is the complex powers of the integration operator $J^1=\int_0^x f(t)\,dt$, where $f\in C_{\gamma}^{(n)}[0,1]$ and

$$C_{\gamma}^{(n)}[0,1]=\{f: f \text{ is continuously differentiable on } [0,1] \text{ up to order } n-1 \\ \text{and have the derivative } f^{(n)}(x)\in C_{\gamma}[0,1]\}.$$

Here $n=[\alpha]+1, C_{\gamma}[0,1]$ consists of the functions f such that $x^{\gamma}f(x)\in C[0,1]$ where γ is a complex number and $C[0,1]$ is the space of continuous functions on the segment $[0,1]$. If $\gamma=0$ then $C_{\gamma}^{(n)}[0,1]=C^{(n)}[0,1]$ and $C_{\gamma}[0,1]=C[0,1]$. A linear bounded operator defined on the space $C_{\gamma}^{(n)}[0,1]$ is said to be called unicellular if the lattice of invariant subspace of $C_{\gamma}^{(n)}[0,1]$ is totally ordered with respect to the inclusion operation, i.e., if $E_1,E_2\in \mathrm{Lat}C_{\gamma}^{(n)}[0,1]$ then $E_1\subset E_2$ or $E_2\subset E_1$. It is well known (Gohberg and Krein (1970); Nikolskii (1986)) that the fractional integral operator is unicellular on $L_p[0,1]$, $p\in[1,\infty)$. In other words, the lattices of invariant and hyperinvariant subspaces of the operator J^{α} are of the form

$$\mathrm{Lat}\ J^{\alpha}=\mathrm{HypLat}\ J^{\alpha}=\left\{E_a:=\chi_{[0,1]}L_p[0,1]:0\le a\le 1\right\}.$$

Consider the fractional integral operator

$$J^{\alpha}: f\to\int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)}f(t)dt, \tag{1}$$

where $\Gamma(.)$ is the Euler Gamma function and $\alpha\in\mathbb{C}$ with $\mathrm{Re}\,\alpha>0$. Here we suppose $f(x)\in C_{\gamma}[0,1]$ with $\gamma\in\mathbb{C}$, $0\le\mathrm{Re}\,\gamma<1$ and $\|f\|_{C_{\gamma}[0,1]}=\|x^{\gamma}f(x)\|_{C[0,1]}$ .The norm law on the space $C_{\gamma}^{(n)}[0,1]$ is defined as

$$\|f\|_{C_{\gamma}^{(n)}}=\sum_{k=0}^{n-1}\left|f^{(k)}(0)\right|+\left\|f^{(n)}\right\|_{C_{\gamma}},$$

If $n=0$, we set $C_{\gamma}^{(0)}[0,1]=C_{\gamma}[0,1]$.

Brickman and Fillmore (1967) showed that if $n\in\mathbb{N}_0$ and $\gamma\in\mathbb{C}$ $(0\le\mathrm{Re}\,\gamma<1)$, then the space $C_{\gamma}^{(n)}[0,1]$ consists of those and only those functions f which are represented in the form

$$f(x)=\frac{1}{(n-1)!}\int_0^x (x-t)^{n-1}\varphi(t)dt+\sum_{k=0}^{n-1}c_kx^k, \tag{2}$$

where $\varphi(t)\in C_{\gamma}[0,1]$ and c_k $(k=0,1,2,...,n-1)$ are constants such that

$$\varphi(t)=f^{(n)}(t)\,,\ c_k=\frac{f^{(k)}(0)}{k!}.$$

In particular, when $\gamma = 0$ the space $C_0^{(n)}[0,1]$ is the space of functions that continuously differentiable n times in the segment $[0,1]$.

Note that for $\operatorname{Re}\alpha \geq 0$ and $\gamma \in \mathbb{C}$ we have that the fractional integral operator J^α is a bounded operator from $C_\gamma[0,1]$ into $C_{\gamma-\alpha}[0,1]$ if $\operatorname{Re}\gamma > \operatorname{Re}\alpha$. In this case we have

$$\|J^\alpha\|_{C_{\gamma-\alpha}} \leq k\,\|f\|\,C_\gamma, \tag{3}$$

where $k_1 = \frac{\Gamma(\operatorname{Re}\alpha)|\Gamma(1-\operatorname{Re}_\gamma)|}{|\Gamma(\alpha)|\Gamma(1+\operatorname{Re}(\alpha-\gamma))}$. Additionally, if $\operatorname{Re}\gamma \leq \operatorname{Re}\alpha$, then J^α is bounded from $C_\gamma[0,1]$ into $C[0,1]$ with

$$\|J^\alpha f\|_C \leq k_2\,\|f\|\,C_\gamma, \tag{4}$$

where $k_2 = \Gamma(\operatorname{Re}\alpha)\left(\Gamma\left(1-\operatorname{Re}\gamma\right)/\left|\Gamma(\alpha)\right|\Gamma\left(1+\operatorname{Re}\left(\alpha-\gamma\right)\right)\right)$.

In this paper we investigate the fractional integral operator defined on the space $C_\gamma^{(n)}[0,1]$. We describe the lattice $\operatorname{Lat}J^\alpha$ of invariant subspaces and study the question of unicellularity of the operator J^α by using the Duhamel production. First, the invariant subspaces of the fractional integral operator J^α in $C_\gamma^{(n)}[0,1]$ are obtained and it is proved unicellularity of J^α defined on the special subspace of $C_\gamma^{(n)}[0,1]$. Later, by using the Duhamel product, unicellularity of J^α on $C_\gamma^{(n)}[0,1]$ is obtained and the description of the invariant subspaces is given.

3. Main Results

In this section we present some chain of invariant subspaces of the operator J^α. We define the following subspaces:

$$E_l^{(n)} = \left\{f \in C_\gamma^{(n)}[0,1] : f(0) = f'(0) = ... = f^{(n-l-1)}(0) = 0\right\},\ l = 0,1,...,n-1,$$

$$E_\lambda = \left\{f \in C_\gamma^{(n)}[0,1] : f(x) = 0,\ 0 \leq x \leq \lambda\right\},\ 0 < \lambda < 1.$$

It is clear that $E_\lambda \subset E_l^{(n)}$ for each $\lambda \in (0,1)$ and $l = 0,1,...,n-1$. Moreover, $E_\lambda \subset E_\mu$ if $\lambda > \mu$, $\lambda,\, \mu \in (0,1)$. Hence, we have

$$\{0\} \subset E_\lambda \subset E_\mu \subset E_0^{(n)} \subset ... \subset E_{n-1}^{(n)} \subset C_\gamma^{(n)}[0,1].$$

We can easily show that $E_l^{(n)}$ and E_λ are invariant subspaces of the operator J^α. Indeed, if $f(x) \in E_l^{(n)}$ then

$$g(x) = J^\alpha f(x) = \frac{1}{\Gamma(\alpha)}\int_0^x t^{\alpha-1} f(x-t)dt,$$

implies that

$$g^{(s)}(x) = \frac{1}{\Gamma(\alpha)}\frac{d^s}{dx^s}\int_0^x t^{\alpha-1}f(x-t)dt = \frac{1}{\Gamma(\alpha)}\int_0^x t^{\alpha-1}f^{(s)}(x-t)\,dt$$

$$= \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}f^{(s)}(t)\,dt,\ s = 0,1,...,n-l-1.$$

Therefore, $g\left(0\right)=g'(0)=...=g^{(s)}(0)=0$. Moreover,

$$g^{(n)}(x)=\frac{1}{\Gamma(\alpha)}\int\limits_0^x t^{\alpha-1}f^{(n)}\left(x-t\right)dt\in C_\gamma\left[0,1\right].$$

Consequently, $g(x)\in E_l^{(n)}$, i.e., $E_l^{(n)}$ is an invariant subspace of the operator J^α. Similarly, E_λ is also an invariant subspaces of J^α. Now consider the operator J^α defined on the space $E_0^{(n)}$. For convenience, we denote this operator by $J_{n,0}^\alpha$.

Theorem 3.1.

If $\operatorname{Re}\alpha>0$, the operator $J_{n,0}^\alpha$ is unicellular and

$$\text{Lat } J_{n,0}^\alpha=\left\{E_a^n:0\le a\le 1\right\},$$

where $E_a^n=\left\{f\in E_0^{(n)}:f(x)=0,\ x\in[0,a]\right\}$.

Proof:

Consider the operator $D_n=\frac{d^n}{dx^n}:E_0^{(n)}\to C_\gamma\left[0,1\right]$. If $f\in E_0^{(n)}$, then $g=D_nf=f^{(n)}\left(x\right)\in C_\gamma\left[0,1\right]$, and it implies that

$$\|g\|_{C_\gamma[0,1]}=\left\|f^{(n)}\left(x\right)\right\|_{C_\gamma[0,1]}=\|f\|_{C_\gamma^n[0,1]}=\|f\|_{E_0^{(n)}}.$$

Hence, the operator D_n isometrically maps $E_0^{(n)}$ on $C_\gamma\left[0,1\right]$. Moreover, the inverse operator is $D_n^{-1}=D_n^*=J^n:C_\gamma\left[0,1\right]\to E_0^{(n)}$. Since $J_{n,0}^\alpha=D_n^{-1}J_0^\alpha D_n$, where J_0^α is the operator J^α from $C_\gamma\left[0,1\right]$ to $C_{\gamma-\alpha}\left[0,1\right]$. Consequently, the operator $J_{n,0}^\alpha$ defined on $E_0^{(n)}$ is isometrically equivalent to the operator J_0^α defined on $C_\gamma\left[0,1\right]$. Since

$$\text{Lat } C_\gamma\left[0,1\right]=\left\{E_a:=X_{[0,1]}C_\gamma\left[0,1\right],\ 0\le a\le 1\right\},$$

(see Gohberg et al. (1986)), we reach the assertion of the theorem. ■

Theorem 3.2.

The operator $J_{n,l}^\alpha$ $(l=0,1,...,n)$ defined on the space $E_{n,l}^{(n)}$ is isometrically equivalent to the operator J_l^α defined on $C_\gamma^{(l)}\left[0,1\right]$.

Proof:

From (2), we have that if $f(x)\in C_\gamma^{(n)}\left[0,1\right]$, then

$$J^\alpha f(x)=\frac{1}{\Gamma\left(\alpha+n\right)}\int_0^x\left(x-s\right)^{\alpha+n-1}\varphi\left(s\right)ds+\sum_{k=0}^{n-1}\frac{f^{(k)}\left(0\right)}{(n-1)!}x^{\alpha+k},$$

where $\varphi(x)=f^{(n)}\left(x\right)\in C_\gamma\left[0,1\right]$. Therefore, if $f\in C_\gamma^{(n)}\left[0,1\right]$, then

$$J^\alpha f(x)=J^{\alpha+n}\varphi(x)+\sum_{k=0}^{n-1}\frac{f^{(n)}\left(0\right)}{(n-1)!}x^{\alpha+k}.$$

Since $0 \leq \operatorname{Re}\gamma < 1$ and $\operatorname{Re}\alpha > 0$, the operator $J^{\alpha+n}$ is bounded from $C_\gamma [0,1]$ into $C_{\gamma-\alpha}$ if $\operatorname{Re}(\gamma-\alpha) > n$. From here we obtain that the operator J^α is well defined on the space $C_\gamma^{(n)}[0,1]$ when $\operatorname{Re}(\gamma-\alpha) > n$. So we will assume that $\operatorname{Re}(\gamma-\alpha) > n$ is satisfied. Now consider the space $E_l^{(n)}$ which is an invariant subspace of the operator J^α. Denote the operator J^α acting in the space $E_l^{(n)}$ by $J_{n,l}^\alpha \ (k = 0,1,...,n-1)$. Additionally, let $J_{n,n}^\alpha$ be the fractional integral operator acting in $C_\gamma^{(n)}[0,1]$. We also denote $C_\gamma^{(n)}[0,1] = E_n^{(n)}$. Analogously we can prove that the operator $J_{n,l}^\alpha \ (l = 0,1,...,n-1)$ defined on $E_l^{(n)}$ is isometrically equivalent to the operator J_l^α defined on $C_\gamma^{(l)}[0,1]$. Indeed, it is clear that the operator $U_{n-l} = \frac{d^{n-l}}{dx^{n-l}} : E_l^{(n)} \to C_\gamma^{(l)}[0,1]$ isometrically maps $E_l^{(n)}$ on $C_\gamma^{(l)}[0,1]$. Moreover, $U_{n-l}^{-1} : U_{n-l}^{*} = J^{n-l} : C_\gamma^{(l)}[0,1] \to E_l^{(n)}$ is the inverse map. Now from the identity $J_{n,l}^\alpha = U_{n-l}^{-1} J_l^\alpha U_{n-l}$ we have our assertion. ■

Now we investigate the unicellularity problem for the operator J^α by using the Duhamel product.

Consider the Duhamel product of two functions f and g belonging to the space $C_\gamma^{(n)}[0,1]$:

$$(f \circledast g)(x) := \frac{d}{dx}\int_0^x f(x-t)g(t)dt = \int_0^x f'(x-t)g(t)dt + f(0)g(x). \tag{5}$$

The Duhamel product is commutative Banach algebra with a unity and has a wide applications in problems related to convolution operators. We apply the Duhamel product to reach results concerning with the unicellularity of the Rieman-Liouville fractional integral operator J^α. It is clear from (5) that kth $(0 \leq k \leq n)$ derivative of the Duhamel product is

$$(f \circledast g)^{(k)}(x) = \int_0^x f^{(k)}(x-t)g'(t)dt + \sum_{i=0}^{k-1} f^{(i)}(0)\, g^{(k-i)}(x) + f^{(k)}(x)\, g(0)\,,\ k = 0,1,...,n, \tag{6}$$

where $n = [\operatorname{Re}\alpha] + 1$. Note that all kth $(0 \leq k \leq n)$ derivatives are continuous in $[0,1]$ and nth the derivative belongs to $C_\gamma[0,1]$, so $x^\gamma f(x) x^\gamma g(x)$ are in $C[0,1]$.

Recall that

$$\begin{aligned} \|f\|_{C_\gamma^{(n)}} &= \sum_{k=0}^{n-1} \max_{x\in[0,1]} \left|f^{(a)}(x)\right| + \max_{x\in[0,1]} \left|x^\gamma f^{(n)}(x)\right| \\ &= \sum_{k=0}^{n-1} \left\|f^{(n)}\right\|_{C[0,1]} + \left\|x^\gamma f^{(n)}(x)\right\|_{C[0,1]}. \end{aligned} \tag{7}$$

Lemma 3.1.

If f and g are the functions belonging to the space $C_\gamma^{(n)}[0,1]$, then their Duhamel product (5) also belongs to the space $C_\gamma^{(n)}[0,1]$, and the inequality

$$\|f \circledast g\|_{C_\gamma^{(n)}} \leq \|f\|_{C_\gamma^{(n)}} \|g\|_{C_\gamma^{(n)}}\,, \tag{8}$$

is satisfied.

Proof:

$$\|f \circledast g\|_{C_\gamma^{(n)}} = \sum_{k=0}^{n-1} \max_{x\in[0,1]} \left|(f \circledast g)^{(n)}(x)\right| + \max_{x\in[0,1]} \left|x^\gamma (f \circledast g)^{(n)}(x)\right|.$$

Let $\max_{x\in[0,1]} \left|f^{(n)}(x)\right| = M_k(f)$ for $n = 0, 1, ..., n-1$. Then

$$\begin{aligned}
\|f \circledast g\|_{C_\gamma^{(n)}} &\leq \sum_{k=0}^{n-1} M_k(f) M_1(g) + \sum_{k=1}^{n-1}\sum_{i=0}^{n-1} M_i(f) M_{k-i}(g) \\
&+ \sum_{k=0}^{n-1} M_k(f) M_0(g) + \max_{x\in[0,1]} \left|x^\gamma f^{(n)}(x)\right| M_1(g) \\
&+ \sum_{i=0}^{n-1} M_{(n-i)}(g) \max_{x\in[0,1]} \left|x^\gamma f^{(i)}(x)\right| + M_0(g) \max_{x\in[0,1]} \left|x^\gamma f^{(n)}(x)\right| \\
&\leq \sum_{k=0}^{n-1} M_k(f) \sum_{i=0}^{n-1} M_i(g) + \left(M_n^{(\gamma)}(f) M_1(g) + \sum_{i=0}^{n-1} M_{(n-i)}(g) M_i(f)\right) \\
&+ M_0(g) M_n^{(\gamma)}(f),
\end{aligned}$$

where $M_n^{(\gamma)}(f) = \max_{x\in[0,1]} \left|x^\gamma f^{(n)}(x)\right|$.

Consequently,

$$\begin{aligned}
\|f \circledast g\|_{C_\gamma^{(n)}} &\leq \sum_{k=0}^{n-1} \left(M_k(f) + M_n^{(\gamma)}(f)\right) \sum_{i=0}^{n-1} M_i(g) + M_n^{(\gamma)}(f) \sum_{i=0}^{n-1} M_i(g) \\
&+ \sum_{i=0}^{n-1} M_i(g) \sum_{i=0}^{n-1} M_i(f) \\
&\leq \|f\|_{C_\gamma^{(n)}} \|g\|_{C_\gamma^{(n)}}.
\end{aligned}$$

Then the desired result has been obtained. ∎

The inequality (8) shows that the operator

$$D_f(g) := f \circledast g,\ g \in C_\gamma^{(n)},$$

is continuous in the space $C_\gamma^{(n)}$. We also obtain from the definition of the Duhamel product

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha+1)} \frac{d}{dx} \int_0^x (x-t)^\alpha f(t)\, dt = \frac{x^\alpha}{\Gamma(\alpha+1)} \circledast f(x). \tag{9}$$

Lemma 3.2.

Let $f \in C_\gamma^{(n)}[0,1]$, where $0 < \gamma < 1$ and $n = [\operatorname{Re}\alpha] + 1$. Then, $f \in \operatorname{Cyc}(J^\alpha)$ if and only if $f(0) \neq 0$.

Proof:

By the formula (9), we have

$$\begin{aligned}
&\operatorname{span}\left\{f, J^{\alpha} f, \left(J^{\alpha}\right)^{2} f, \ldots, \left(J^{\alpha}\right)^{m} f, \ldots\right\} \\
&= \operatorname{span}\left\{1 \circledast f, \frac{x^{\alpha}}{\Gamma(\alpha+1)} \circledast f, \frac{x^{2\alpha}}{\Gamma(2\alpha+1)} \circledast f, \ldots, \frac{x^{m\alpha}}{\Gamma(m\alpha+1)} \circledast f\right\} \\
&= \operatorname{span}\left\{D_f 1, D_f \frac{x^{\alpha}}{\Gamma(\alpha+1)}, \ldots, D_f \frac{x^{m\alpha}}{\Gamma(m\alpha+1)}, \ldots\right\} \\
&= \overline{D_f \operatorname{span}\left\{\frac{x^{m\alpha}}{\Gamma(m\alpha+1)} : m \geq 0\right\}},
\end{aligned}$$

i.e.,

$$\operatorname{span}\left\{\left(J^{\alpha}\right)^{m} f : m \geq 0\right\} = \overline{D_f \operatorname{span}\left\{\frac{x^{m\alpha}}{\Gamma(m\alpha+1)} : m \geq 0\right\}}. \tag{10}$$

From (10) we obtain that $f \in \operatorname{Cyc}(J^{\alpha})$ if and only if $\overline{D_f M} = C_{\gamma}^{(n)}[0,1]$ where $M = \operatorname{span}\left\{\frac{x^{m\alpha}}{\Gamma(m\alpha+1)} : m \geq 0\right\}$. Consider the function sequence $e_m(x) = \frac{x^{m\alpha}}{\Gamma(m\alpha+1)}$, $m = 0, 1, 2, \ldots$. Since $n = [\operatorname{Re}\alpha] + 1$ we have that the functions

$$\frac{d^k}{dx^k} e_m(x) = \frac{m^k \alpha(\alpha-1)\ldots(\alpha-n+1)}{\Gamma(m\alpha+1)} x^{m\alpha-k}, \ k = 0, 1, \ldots, n-1,$$

and

$$x^{\gamma} e_m^{(n)}(x) = \frac{m^k \alpha(\alpha-1)\ldots(\alpha-n+1)\, x^{m\alpha-n+\gamma}}{\Gamma(m\alpha+1)}, \ 0 < \gamma < 1,$$

are continuous on $[0,1]$. So we obtain $e_m(x) \in C_{\gamma}^{(n)}[0,1]$. Consequently

$$\begin{aligned}
M &= \operatorname{span}\left\{\frac{x^{m\alpha}}{\Gamma(m\alpha+1)} : m \geq 0\right\} = \operatorname{span}\left\{\frac{x^m}{m!} : m \geq 0\right\} \\
&= C_{\gamma}^{(n)}[0,1],
\end{aligned}$$

and we set $f \in \operatorname{Cyc}(J^{\alpha})$ if and only if $\overline{D_f C_{\gamma}^{(n)}[0,1]} = C_{\gamma}^{(n)}[0,1]$.

According to (5), we have that if g is the $\circledast-$inverse of f we obtain $(f \circledast g)(0) = f(0)\, g(0) = 1$ which implies $f \neq 0$. Hence, if D_f is invertible operator, then $f(0) \neq 0$. Now let $f(0) \neq 0$. Then we can written the operator $D_f = D_{f-f(0)}$, i.e. $h(x) = f(x) - f(0)$, and I is the identity operator. Consequently we have $h(0) = 0$ and by this

$$(D_h g)(x) = \frac{d}{dx} \int_0^x h(x-t) g(t) dt = \int_0^x h'(x-t) g(t) dt. \tag{11}$$

We denote

$$K_{h'} g(x) = \int_0^x h'(x-t) g(t) dt. \tag{12}$$

Then

$$K_{h'}^2 g(x) = \int_0^x h'(x-t)\int_0^t h'(x-s)ds\int_0^s h'(s-\xi)g(\xi)d\xi, \tag{13}$$

and generally

$$K_{h'}^m g(x) = \int_0^x h'(x-t_1)dt_1\int_0^{t_1} h'(t_1-t_2)dt_2 ... \int_0^{t_{m-1}} h'(t_{m-1}-t_m)g(t_m)dt_m, \tag{14}$$
$$m = 1,2,3,... .$$

It is easy to obtain from (14)

$$\left|\left(K_{h'}^m g\right)(x)\right| \leq \|h\|^m \|g\| \frac{x^m}{m!},\ m = 0,1,2,... . \tag{15}$$

Here $\|.\|$ denotes the norm in $C_\gamma^{(n)}[0,1]$. For convenience let us set

$$F_{m-1}(t_1) = \int_0^{t_1} h'(t_1-t_2)dt_2\int_0^{t_2} ... \int_0^{t_{m-1}} h'(t_{m-1}-t_m)g(t_m)dt_m. \tag{16}$$

Then (14) is written as

$$K_{h'}^m g(x) = \int_0^x h'(x-t_1)F_{m-1}(t_1)dt_1. \tag{17}$$

Now using (17) we have

$$\left(K_{h'}^m g\right)'(x) = h'(0)F_{m-1}(x) + \int_0^x h''(x-t_1)F_{m-1}(t_1)dt,$$

$$\left(K_{h'}^m g\right)''(x) = h'(0)F_{m-1}^{'}(x) + h''(0)F_{m-1}(x) + \int_0^x h''(x-t_1)F_{m-1}(t_1)dt,$$

and generally

$$\left(K_{h'}^m g\right)^{(j)}(x) = h^{'}(0)F_{m-1}^{(j-1)}(x) + h^{''}(0)F_{m-1}^{(j-2)}(x) + ... + h^{(j-1)}(0)F_{m-1}(x)$$
$$+\int_0^x h^{(j)}(x-t)F_{m-1}(t_1)dt, j = 1,2,... . \tag{18}$$

First of all we can obtain from (7)

$$\left|F_{m-1}(t_1)\right| \leq \|h\|^{m-1} \|g\| \frac{t_1^{m-1}}{(m-1)!},\ m = 1,2,... . \tag{19}$$

By this, we have

$$\left|\left(K_{h'}^{m}g\right)'(x)\right| \leq \|h\|^{m}\|g\|\left(\frac{x^{m-1}}{(m-1)!}+\frac{x^{m}}{m!}\right) \leq \|h\|^{m}\|g\|\frac{(1+x)^{m}}{m!}, \tag{20}$$

$$\begin{aligned}\left|\left(K_{h'}^{m}g\right)''(x)\right| &\leq \|h\|^{m}\|g\|\left(\frac{x^{m-2}}{(m-2)!}+\frac{x^{m-1}}{(m-1)!}+\frac{x^{m-1}}{(m-1)!}+\frac{x^{m}}{m!}\right) \\ &\leq \|h\|^{m}\|g\|\frac{(2+x)^{m}}{m!},\end{aligned} \tag{21}$$

and in general it is proved by induction that

$$\left|\left(K_{h'}^{m}g\right)^{(j)}(x)\right| \leq \|h\|^{m}\|g\|\frac{(j+x)^{m}}{m!}, j=1,2,...,n. \tag{22}$$

Now from estimation (22) we find that

$$\begin{aligned}\|K_{h'}^{m}g\| &= \sum_{\gamma=0}^{n-1}\left\|\left(K_{h'}^{m}g\right)^{(j)}\right\|_{C[0,1]}+\|K_{h'}^{m}g\|_{C[0,1]}^{(n)} \\ &\leq \|h\|^{m}\|g\|\frac{(1+n)^{m}}{m!}.\end{aligned} \tag{23}$$

From the estimation (23) we have that

$$\|K_{h'}^{m}\| \leq \|h\|^{m}\frac{(1+n)^{m}}{m!}, \tag{24}$$

which implies

$$\|K_{h'}^{m}\|^{\frac{1}{m}} \leq \|h\|\frac{1+n}{(m!)^{\frac{1}{m}}}. \tag{25}$$

Since $\frac{1}{(n!)^{\frac{1}{m}}} \to 0$ as $m \to \infty$ we have $K_{h'}$ is quasi-nilpotent operator and the operator D_f is therefore invertible. This completes the proof. ■

Lemma 3.3.

If $g \in E_k^{(n)}$ $(k=0,1,...,n-1)$ and $g(x) \neq 0$ for every $x \in (0,\varepsilon)$ where $\varepsilon > 0$ is arbitrary then

$$\operatorname{span}\left\{(J^{\alpha})^{m}g : m \geq 0\right\} = E_k^{(n)}.$$

Proof:

Let $g \in E_n^{(n)}$. Then

$$g(x) = \frac{1}{(n-1)!}\int_{0}^{t}(x-t)^{n-1}\varphi(t)dt+\sum_{j=k+1}^{n-1}\frac{f^{j}(0)}{j!}x^{j}, \tag{26}$$

where $\varphi(t) = C_\gamma[0,1]$, $\varphi(t) = g^{(n)}(t)$. Then,

$$\begin{aligned}
(J^\alpha)^m g(x) &= \frac{1}{\Gamma(\alpha m)} \int_0^x (x-t)^{\alpha m - 1} g(t)dt \\
&= \frac{1}{\Gamma(\alpha m)} \frac{1}{(n-1)!} \int_0^x (x-t)^{\alpha m-1} dt \int_0^t (t-s)^{n-1} \varphi(s)ds + \\
&+ \frac{1}{\Gamma(\alpha m)} \int_0^x (x-t)^{\alpha m-1} \left(\sum_{j=k+1}^{n-1} \frac{g^{(j)}(0)}{j!} .....dt \right) \\
&= \frac{1}{\Gamma(n+\alpha m)} \int_0^x (x-t)^{n+\alpha m-1} \varphi(s)ds + \sum_{j=k+1}^{n-1} \frac{g^{(j)}(0)}{\Gamma(\alpha m+j+1)} x^{\alpha m+j} \\
&= \frac{d}{dx} \left(\frac{1}{\Gamma(n+\alpha m+1)} \int_0^x (x-t)^{n+\alpha m} \varphi(s)ds \right) \\
&+ \sum_{j=k+1}^{n-1} \frac{g^{(j)}(0)}{\Gamma(\alpha m+j+1)} x^{\alpha m+j},
\end{aligned}$$

i.e.,

$$(J^\alpha)^m g(x) = \frac{1}{\Gamma(n+\alpha m+1)} \left(x^{n+\alpha m} \circledast g^{(n)} \right) + \sum_{j=k+1}^{n-1} \frac{g^{(j)}(0)}{\Gamma(\alpha m+j+1)}. \tag{27}$$

First consider the case for $E^{(n-1)}$. If $g \in E^{(n-1)}$, then (27) takes the form of

$$(J^\alpha)^m g(x) = \left(\frac{x^{n+\alpha m}}{\Gamma(n+\alpha m+1)} \circledast g^{(n)} \right) = D_{g^{(n)}} \frac{x^{n+\alpha m}}{\Gamma(n+\alpha m+1)}. \tag{28}$$

Therefore,

$$\begin{aligned}
\operatorname{span}\{(J^\alpha)^m g : m \geq 0\} &= \overline{D_{g^{(n)}} \operatorname{span}\left\{ \frac{x^{n+\alpha m}}{\Gamma(n+\alpha m+1)} \right\}} \\
&= \overline{D_{g^{(n)}} \operatorname{span}\left\{ \frac{x^{n+k}}{k!} : k > 0 \right\} = E^{(n-1)}}.
\end{aligned}$$

Hence, $\operatorname{span}\left\{ (J^\alpha)^m g : m \geq 0, g \in E_0^{(n)} \right\} = E_0^{(n)}$. Using the similar arguments (see Tapdigoglu (2013)) it can be obtained from (27) that

$$\operatorname{span}\left\{ (J^\alpha)^m g : m \geq 0, g \in E_k^{(n)} \right\} = E_k^{(n)},\ k = 0, 1, ..., n-1.$$

Then the desired result has been obtained. ■

By the standard techniques (see Nikolskii (1986)), it is easy prove that $f \in \operatorname{Cyc}(J^\alpha / E_\lambda)$ if and only if $f \in E_\lambda / E_\mu$ for every $\mu > \lambda$.

Now from (2) and Lemma 2.1, 2.2, 2.3 we have the following statement.

Theorem 3.3.

The fractional integration operator $J^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int\limits_0^x (x-t)^{\alpha} f(t)dt$ is unicellular in the space $C_{\gamma}^{(n)}[0,1]$, where $n = [\operatorname{Re}\alpha] + 1$, $0 < \gamma < 1$, i.e.,

$$\operatorname{Lat}(J^{\alpha}) = \left\{ E_{\lambda}, E_k^{(n)}, 0 < \lambda < 1, k = 0, 1, ..., n-1 \right\}.$$

4. Conclusion

The study is devoted to the invariant subspace problem for the fractional integral operator. The invariant subspace problem for the bounded linear operators is well known open problem in mathematics and the question on the existence of the invariant subspaces of every bounded linear operator in a Banach space has not solved yet. So it is interesting to answer this question for special operators as Volterra integral operator, fractional integral operator, etc. From the presented study, we conclude that the invariant subspaces of the fractional integral operator in certain space are obtained and it is proved unicellularity of the fractional integral operator defined on the special subspace of this space. Also, by using the Duhamel product unicellularity of the fractional integral operator is obtained and the description of the invariant subspaces is given. For future work, it will be interesting to examine the same problems for the fractional integral operator in the Sobolev space.

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