Applications and Applied Mathematics: An International Journal (AAM)

Volume 16 | Issue 2 | Article 20

12-2021

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Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466

Applications and Applied Mathematics:
An International Journal
(AAM)

Vol. 16, Issue 2 (December 2021), pp. 1130 –1139

On some geometric properties of non-null curves via its position vectors in $\mathbb{R}_1^3$

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Received: August 11, 2021; Accepted: November 10, 2021

Abstract

In this work, the geometric properties of non-null curves lying completely on spacelike surface via its position vectors in the dimensional Minkowski 3-space $\mathbb{R}_1^3$ are studied. Also, we give a few portrayals for the spacelike curves which lie on certain subspaces of $\mathbb{R}_1^3$. Finally, we present an application to demonstrate our insights.

Keywords: Minkowski space-time; position vector; spacelike curve; osculating plane; subspaces

MSC 2010 No.: 53A04, 53B30, 53C50

1. Introduction

Curves are thought of as a geometric set of points, or locus, in local differential geometry. In the instance of a particle moving in Euclidean 3-space $\mathbb{R}^3$ or in Minkowski 3-space $\mathbb{R}_1^3$, we describe a curve to be the route followed by a particle moving in $\mathbb{R}^3$ or in $\mathbb{R}_1^3$. There are three types of curves in Euclidean space $\mathbb{R}^3$: rectifying, normal, and osculating curves that meet Cesaro's fixed point condition. That is, such curves' rectifying, normal, and osculating planes always provide a certain point. If all of a curve's normal or osculating planes pass through the same point in $\mathbb{R}^3$, the curve is spherical or planar, respectively. It is also known that if all rectifying planes of a non-planar curve in $\mathbb{R}^3$ pass through such a point, the curve's torsion and curvature ratio is a non-constant linear function.

In differential geometry, curves are a fundamental structure. They're a regular curve in Euclidean 3-space $\mathbb{R}^3$, which is a key component of the mathematical description of the world's 3-dimensional space see (Abazari et al. (2017); Akgun and Sivridag (2015); Ali (2012); Bukcu and Karacan (2016); Bektas and Kulahci (2017); Coken and Ciftci (2005); Fernandez, Gimenez and Lucas (2001); Ilarslan, Boyacioglu (2008); Ilarslan, Nesovic (2008); Nurkan, Gven and Karacan (2018); Saad et al. (2020); Yavari and Zarrati (2017)).

In the Euclidean space and the Minkowski space, the challenge of determining the syntactic structure of the position vector of an arbitrary space curve using intrinsic equations is still outstanding.

Throughout its lengthy history, curves theory remains one of the most fascinating problems in differential geometry, and it still being studied by many mathematicians today (see, for example Ali (2010)). The goal of these research is to derive curve position vectors in the Frenet frame. Furthermore, it is well-known in classical differential geometry that calculating the position vector of an arbitrary curve in standard frame is difficult. Position vectors of spacelike W-curves according to the standard frame of $\mathbb{R}^3_1$ were binned using vector differential equations in a recent work.

In Minkowski 3-space $\mathbb{R}^3_1$, there is a straightforward link between rectifying curves and Darboux vectors (centrodes), which are used to define curves of constant precession in mechanics, kinematics, and differential geometry. Obtaining precise parameter equations of rectifying and normal curves in Minkowski 3-space is a fascinating challenge. It is natural to apply certain additional conditions on the matching curve in order to generate such equations. In this work, the non-null curve $\alpha = \alpha(s)$ which lying completely on spacelike surface Ω via it position vectors in $\mathbb{R}^3_1$ are investigated and we give some properties for these curves which lie on some subspaces of $\mathbb{R}^3_1$.

2. Preliminaries

The 3-dimensional Minkowski space $\mathbb{R}^3_1$ is Euclidean 3-space $\mathbb{R}^3$ given with the standard flat metric

$$\Im = -(d\epsilon_1)^2 + (d\epsilon_2)^2 + (d\epsilon_3)^2,$$

where $(\epsilon_1, \epsilon_2, \epsilon_3) \in \mathbb{R}^3_1$.

Recall that a vector $\omega \in \mathbb{R}^3_1$ is called timelike if $\Im(\omega,\omega) < 0$, spacelike if $\Im(\omega,\omega) > 0$ or $\omega = 0$ and null, if $\Im(\omega,\omega) = 0$ and $\omega \neq 0$. So also, an subjective curve $\alpha(s)$ in $\mathbb{R}^3_1$ can be null, spacelike or timelike, if $\alpha'(s)$ are null, spacelike or timelike, respectively. A timelike or spacelike curve α has unit speed, if $\Im(\alpha'(s), \alpha'(s)) = \pm 1$ where $\alpha'(s) = d\alpha/ds$ (for more details see Lopez (2014); O'Neill (1983)).

Let $\{T, N, B, \kappa, \tau\}$ be the Frenet invariants of the non-null curve $\alpha = \alpha(s)$ with timelike principal normal vector in $\mathbb{R}^3_1$, then the Frenet formulae are given as (Lopez (2014)):

$$\begin{bmatrix} T(s) \\ N(s) \\ B(s) \end{bmatrix}_s = \begin{bmatrix} 0 & \kappa(s) & 0 \\ \kappa(s) & 0 & \tau(s) \\ 0 & \tau(s) & 0 \end{bmatrix} \begin{bmatrix} T(s) \\ N(s) \\ B(s) \end{bmatrix}, \tag{1}$$

where $\Im\,(T,T) = \Im\,(B,B) = 1,\ \Im\,(N,N) = -1$ and $\Im\,(T,N) = \Im\,(T,B) = \Im\,(N,B) = 0$.

Let $\chi : V \subset \mathbb{R}^3 \to \mathbb{R}^3_1$, $\chi(V) = \Omega$ and $\varrho : I \to V$ be respectively non-null embedding and regular curve. At that point we have the curve α on the surface Ω defined as $\alpha(s) = \chi(\varrho(s))$ and since χ is non-null embedding, we get unit timelike normal vector field η along Ω defined as

$$\eta = \frac{\chi_x \times \chi_y}{\| \chi_x \times \chi_y \|}.$$

Since Ω is a spacelike surface in this way, ready to choose another basis $\{T, \eta, \xi\}$ along α on Ω , where η is the unit normal and $\xi(s) = T(s) \times \eta(s)$. Note that η is a timelike vector and ξ is a spacelike vector along α satisfying the following relations:

$$T(s) \times \eta(s) = \xi(s), \eta(s) \times \xi(s) = -T(s), \xi(s) \times T(s) = \eta(s). \tag{2}$$

Comparing $\{T, \eta, \xi\}$ with $\{T, N, B\}$. Let φ indicate the angle between N and η. So, the relation matrix between $\{T, N, B\}$ and $\{T, \eta, \xi\}$ can be communicated as

$$\begin{bmatrix} T(s) \\ \xi(s) \\ \eta(s) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sinh\varphi & \cosh\varphi \\ 0 & \cosh\varphi & \sinh\varphi \end{bmatrix} \begin{bmatrix} T(s) \\ N(s) \\ B(s) \end{bmatrix}. \tag{3}$$

3. Some properties of a spacelike curve in $\mathbb{R}^3_1$

In this section, we'll grant a few properties of non-null curves to lie on subspaces of $\mathbb{R}^3_1$. Consider a spacelike surface Ω in $\mathbb{R}^3_1$ and let α be non-null curve lying completely on Ω via frame $\{T, \xi, \eta\}$. At that point there are three subspaces of $\mathbb{R}^3_1$ spanned by $\{T, \xi\}$, $\{T, \eta\}$ and $\{\xi, \eta\}$. Presently, we explore the position vector of non-null curve α to lie in these subspaces.

Theorem 3.1.

Let $\alpha = \alpha(s)$ be non-null curve with timelike principal normal vector with non-vanishing curvature lying completely on Ω in $\mathbb{R}^3_1$ via the frame $\{T, \xi, \eta\}$. Then, α lies on rectifying plane spanned by $\{T, \xi\}$ if and only if α written within the form

$$\alpha(s) = c\, T(s) + \left(\frac{1}{\kappa(s)\ \sinh\varphi}\right) \xi(s),$$

or

$$\alpha(s) = \left(-\frac{c\,\tau(s)\ \cosh\varphi}{\kappa(s)}\right) T(s) + c\,\xi(s),$$

where $\varphi(s) = constant$.

Proof:

Let α be non-null curve with timelike principal normal vector. Then, we can write

$$\alpha(s) = \beta(s)T(s) + \omega(s)\xi(s), \tag{4}$$

where β and ω are differentiable functions of s. From Equation (4) and using Equations (1) and (3) we have

$$\begin{aligned} \alpha'(s) = {} & [\,\beta'(s) + \omega(s)\kappa(s)\sinh\varphi\,]T(s) \\ & + \left[\beta(s)\kappa(s) + \omega'(s)\ \sinh\varphi + \omega(s)\cosh\varphi\left(\tau\,(s) + \frac{d\varphi}{ds}\right)\right]\xi(s) \\ & + \left[\omega'(s)\cosh\varphi + \omega(s)\sinh\varphi\left(\tau\,(s) + \frac{d\varphi}{ds}\right)\right]\eta(s), \end{aligned} \tag{5}$$

where $T(s) = \alpha'(s)$. So we can write

$$\begin{aligned} & \beta'(s) + \omega(s)\kappa(s)\sinh\varphi = 1, \\ & \beta(s)\kappa(s) + \omega'(s)\ \sinh\varphi + \omega(s)\cosh\varphi\left(\tau\,(s) + \frac{d\varphi}{ds}\right)=0, \\ & \omega'(s)\cosh\varphi + \omega(s)\sinh\varphi\left(\tau\,(s) + \frac{d\varphi}{ds}\right)=0. \end{aligned} \tag{6}$$

If we take $\beta(s)$ as a constant then

$$\omega(s) = \frac{1}{\kappa(s)\ \sinh\varphi}.$$

Hence, we have

$$\alpha(s) = c\,T(s) + \left(\frac{1}{\kappa(s)\ \sinh\varphi}\right)\xi(s). \tag{7}$$

If $\omega(s) = constant$ and $\varphi(s) = constant$ and using Equation (3) we find

$$\beta(s) = -\frac{c\,\tau\,(s)\ \cosh\varphi}{\kappa(s)}.$$

So, we have

$$\alpha(s) = \left(-\frac{c\,\tau\,(s)\ \cosh\varphi}{\kappa(s)}\right)T(s) + c\,\xi(s). \tag{8}$$

Which complete our proof. ■

Theorem 3.2.

Let $\alpha = \alpha(s)$ be non-null curve with timelike principal normal vector with non-vanishing curvature lying completely on Ω in $\mathbb{R}^3_1$ via the frame $\{T,\xi,\eta\}$. Then, α lies on osculating plane spanned by $\{T,\eta\}$ if and only if α written within the form

$$\begin{aligned} \alpha(s) = {} & \left[\frac{c(\kappa(s)\cosh\varphi\coth\varphi - \tau\,(s)\sinh\varphi\,)}{\kappa(s)}\ e^{-\int\kappa(s)\coth\varphi\,ds}\right]T(s) \\ & + \left[ce^{-\int\kappa(s)\coth\varphi\,ds}\right]\eta(s), \end{aligned}$$

where $\varphi(s) = constant$.

Proof:

Let α be non-null curve with timelike principal normal vector. Then, the curve α can expressed as

$$\alpha(s) = \lambda(s)T(s) + \mu(s)\eta(s), \tag{9}$$

where λ and μ are differentiable functions of s. Taking the derivative of Equation (9) and using Equations (1) and (3) we get

$$\begin{aligned}\alpha'(s) = {} & [\,\lambda'(s) + \mu(s)\kappa(s)\cosh\varphi]\,T(s), \\ & + \left[\lambda(s)\kappa(s) \ + \ \mu'(s)\cosh\varphi + \mu(s)\sinh\varphi\left(\tau\,(s) + \frac{d\varphi}{ds}\right)\right]\xi(s) \\ & + \left[\mu'(s)\ \sinh\varphi + \mu(s)\ \cosh\varphi\left(\tau\,(s) + \frac{d\varphi}{ds}\right)\right]\eta(s),\end{aligned} \tag{10}$$

where $T(s) = \alpha'(s)$. So we have

$$\begin{aligned}& \lambda'(s) + \mu(s)\kappa(s)\cosh\varphi = 1, \\ & \lambda(s)\kappa(s) \ + \ \mu'(s)\cosh\varphi + \mu(s)\sinh\varphi\left(\tau\,(s) + \frac{d\varphi}{ds}\right) = 0, \\ & \mu'(s)\sinh\varphi + \mu(s)\cosh\varphi\left(\tau\,(s) + \frac{d\varphi}{ds}\right) = 0.\end{aligned} \tag{11}$$

If we choose $\varphi(s) = constant$ in Equation (11) we find

$$\mu'(s)\sinh\varphi + \mu(s)\,\tau\,(s)\cosh\varphi = 0. \tag{12}$$

Solving the differential Equation (12) we find

$$\mu(s) = ce^{-\int \kappa(s)\coth\varphi\, ds}. \tag{13}$$

If we write Equation (13) in Equation (11) we find

$$\lambda(s) = \left(\frac{c}{\kappa(s)}\ e^{-\int \kappa(s)\coth\varphi\, ds}\right)(\kappa(s)\cosh\varphi\coth\varphi - \tau\,(s)\sinh\varphi\,). \tag{14}$$

Then,

$$\begin{aligned}\alpha(s) = {} & \left[\frac{c(\kappa(s)\cosh\varphi\coth\varphi - \tau\,(s)\sinh\varphi\,)}{\kappa(s)}\ e^{-\int \kappa(s)\coth\varphi\, ds}\right]T(s) \\ & + \left[ce^{-\int \kappa(s)\coth\varphi\, ds}\right]\eta(s).\end{aligned} \tag{15}$$

■

Theorem 3.3.

Let $\alpha = \alpha(s)$ be non-null curve with timelike principal normal vector with non-vanishing curvature lying completely on Ω in $\mathbb{R}_1^3$ via the frame $\{T, \xi, \eta\}$. Then, α lies on normal plane spanned by $\{\xi, \eta\}$ if and only if α written within the form

$$\alpha(s)=\left[\frac{1}{\kappa(s)\sinh\varphi}+e^{\int\tau(s)\coth\varphi\,ds}\coth\varphi\left[\int e^{-\int\tau(s)\coth\varphi\,ds}\left(\frac{\tau(s)}{\kappa(s)\sinh\varphi}\right)ds-c\right]\right]\xi(s)$$
$$+\left[e^{\int\tau(s)\coth\varphi\,ds}\left[-\int e^{-\int\tau(s)\coth\varphi\,ds}\left(\frac{\tau(s)}{\kappa(s)\sinh\varphi}\right)ds+c\right]\right]\eta(s),$$

where $\varphi(s)=constant$.

Proof:

Let α be non-null curve with timelike principal normal vector. Then, the curve α can expressed as

$$\alpha(s)=\varepsilon(s)\xi(s)+\nu(s)\eta(s), \tag{16}$$

where ε and ν are differentiable functions of s. From Equation (9) together Equations (1) and (3) we get

$$\begin{aligned}\alpha'(s)&=[\,\varepsilon(s)\kappa(s)\sinh\varphi+\nu(s)\kappa(s)\ \cosh\varphi]\,T(s),\\&+\left[\varepsilon'(s)\sinh\varphi+\nu'(s)\ \cosh\varphi+(\varepsilon(s)\cosh\varphi+\nu(s)\sinh\varphi)\left(\tau\ (s)+\tfrac{d\varphi}{ds}\right)\right]\xi(s),\\&+\left[\varepsilon'(s)\cosh\varphi+\nu'(s)\ \sinh\varphi+(\varepsilon(s)\sinh\varphi+\nu(s)\cosh\varphi)\left(\tau\ (s)+\tfrac{d\varphi}{ds}\right)\right]\eta(s),\end{aligned} \tag{17}$$

where $T(s)=\alpha'(s)$. So we can write

$$\begin{aligned}&\varepsilon(s)\kappa(s)\sinh\varphi+\nu(s)\kappa(s)\cosh\varphi=1,\\&\varepsilon'(s)\sinh\varphi+\nu'(s)\cosh\varphi+(\varepsilon(s)\cosh\varphi+\nu(s)\sinh\varphi)\left(\tau\ (s)+\tfrac{d\varphi}{ds}\right)=0,\\&\varepsilon'(s)\cosh\varphi+\nu'(s)\sinh\varphi+(\varepsilon(s)\sinh\varphi+\nu(s)\cosh\varphi)\left(\tau\ (s)+\tfrac{d\varphi}{ds}\right)=0.\end{aligned} \tag{18}$$

Now, if $\varphi(s)=constant$ in Equation (18) we have

$$\begin{aligned}&\varepsilon(s)\sinh\varphi+\nu(s)\cosh\varphi=\tfrac{1}{\kappa(s)},\\&\varepsilon'(s)\sinh\varphi+\nu'(s)\cosh\varphi+\tau\ (s)(\varepsilon(s)\cosh\varphi+\nu(s)\sinh\varphi)=0,\\&\varepsilon'(s)\cosh\varphi+\nu'(s)\sinh\varphi+\tau\ (s)(\varepsilon(s)\sinh\varphi+\nu(s)\cosh\varphi)=0.\end{aligned} \tag{19}$$

From the second two equations in Equation (19) we find

$$\nu'(s)=-\varepsilon(s)\tau\ (s). \tag{20}$$

If we write Equation (19) in Equation (20) we have

$$\nu'(s)-\nu(s)\tau(s)\coth\varphi=-\tfrac{\tau(s)}{\kappa(s)\sinh\varphi}. \tag{21}$$

So, the differential Equation (21) has the solution

$$\nu(s) = e^{\int \tau(s) \coth\varphi \, ds}\left[-\int e^{-\int \tau(s)\coth\varphi \, ds}\left(\frac{\tau(s)}{\kappa(s)\sinh\varphi}\right) ds + c\right]. \tag{22}$$

From Equation (19) in Equation (22) we get

$$\varepsilon(s) = \frac{1}{\kappa(s)\sinh\varphi} + e^{\int \tau(s)\coth\varphi \, ds}\coth\varphi\left[\int e^{-\int \tau(s)\coth\varphi \, ds}\left(\frac{\tau(s)}{\kappa(s)\sinh\varphi}\right) ds - c\right]. \tag{23}$$

Hence, we have

$$\begin{aligned}\alpha(s) = &\left[\frac{1}{\kappa(s)\sinh\varphi} + e^{\int \tau(s)\coth\varphi \, ds}\coth\varphi\left[\int e^{-\int \tau(s)\coth\varphi \, ds}\left(\frac{\tau(s)}{\kappa(s)\sinh\varphi}\right) ds - c\right]\right]\xi(s)\\ &+ \left[e^{\int \tau(s)\coth\varphi \, ds}\left[-\int e^{-\int \tau(s)\coth\varphi \, ds}\left(\frac{\tau(s)}{\kappa(s)\sinh\varphi}\right) ds + c\right]\right]\eta(s).\end{aligned} \tag{24}$$

■

Application 3.1.

Let $\alpha : I \subset \mathbb{R} \to \mathbb{R}^3_1$ be a regular spacelike curve with timelike principal normal lying on a spacelike surface Ω in $\mathbb{R}^3_1$ parametrized by $\alpha(s) = (2s, s^2 - 1, s^2 + 1)$ see Figure 1. Then, $\alpha(s)$'s Frenet apparatus can be written as:

$$\begin{aligned} T(s) &= \left(\frac{1}{\sqrt{2s^2-1}}, \frac{s}{\sqrt{2s^2-1}}, \frac{s}{\sqrt{2s^2-1}}\right),\\ N(s) &= \left(\frac{\sqrt{2}s}{\sqrt{2s^2-1}}, \frac{1}{\sqrt{4s^2-2}}, \frac{1}{\sqrt{4s^2-2}}\right),\\ B(s) &= \left(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right),\\ \kappa(s) &= \frac{1}{\sqrt{2}\sqrt{(2s^2-1)^3}}, \quad \tau(s) = 0. \end{aligned}$$

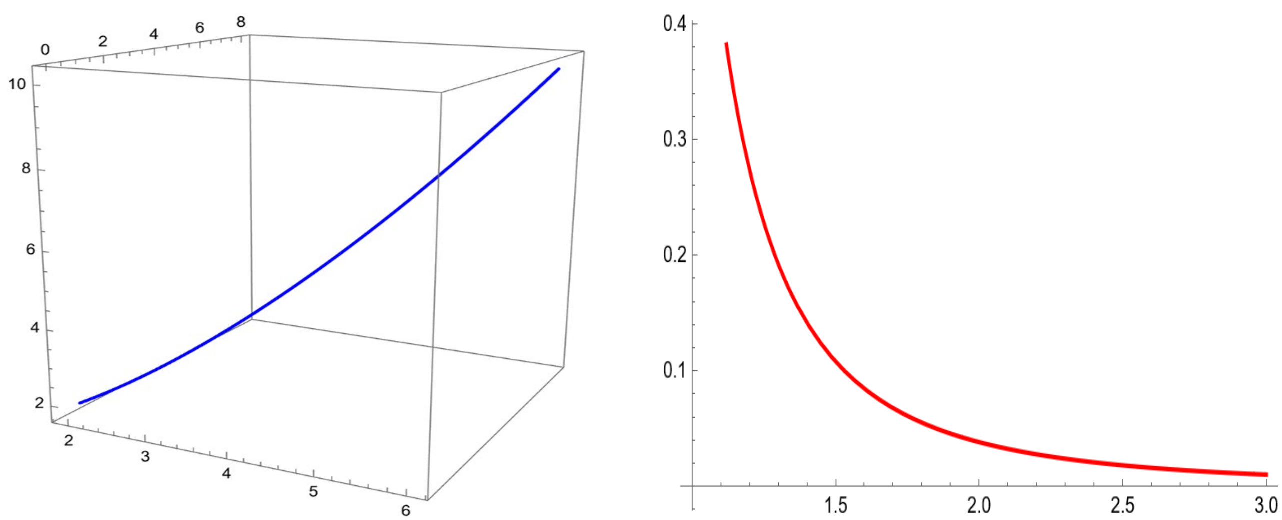

Figure 1. Spacelike curve $\alpha(s)$ via Frenet frame and behavior of $\kappa(s)$ for $s \in (1,3)$

Then, we get $\frac{d\varphi}{ds} = -\tau(s) = 0$, which implies that $\varphi(s) = constant$. So, the frame $\{T, \xi, \eta\}$ can be written as:

$$T(s) = \left(\frac{1}{\sqrt{2s^2-1}}, \frac{s}{\sqrt{2s^2-1}}, \frac{s}{\sqrt{2s^2-1}}\right),$$
$$\xi(s) = \left(\frac{\sqrt{2}s\sinh\varphi}{\sqrt{2s^2-1}}, \frac{\sinh\varphi}{\sqrt{4s^2-2}} - \frac{\cosh\varphi}{\sqrt{2}}, \frac{\sinh\varphi}{\sqrt{4s^2-2}} + \frac{\cosh\varphi}{\sqrt{2}}\right),$$
$$\eta(s) = \left(\frac{\sqrt{2}s\cosh\varphi}{\sqrt{2s^2-1}}, \frac{\cosh\varphi}{\sqrt{4s^2-2}} - \frac{\sinh\varphi}{\sqrt{2}}, \frac{\cosh\varphi}{\sqrt{4s^2-2}} + \frac{\sinh\varphi}{\sqrt{2}}\right).$$

The rectifying, osculating, and normal curves are obtained using theorems, as shown in Figures 2 - 4.

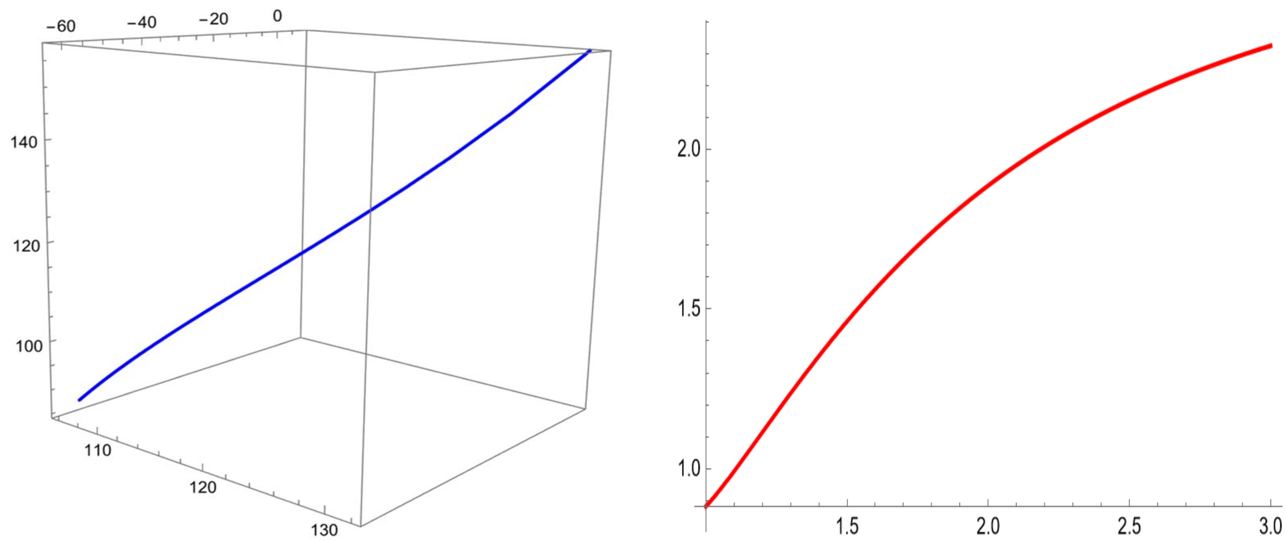

Figure 2. Rectifying curve $\alpha(s)$ and behavior of $\kappa(s)$ for $s \in (1,3)$

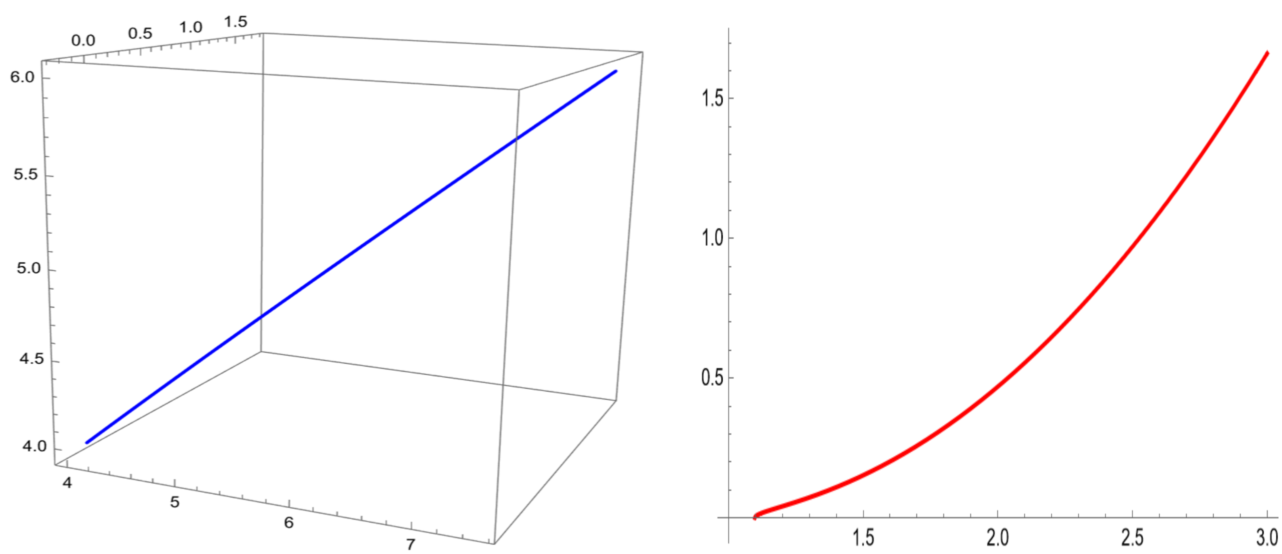

Figure 3. Osculating curve $\alpha(s)$ and behavior of $\kappa(s)$ for $s \in (1,3)$

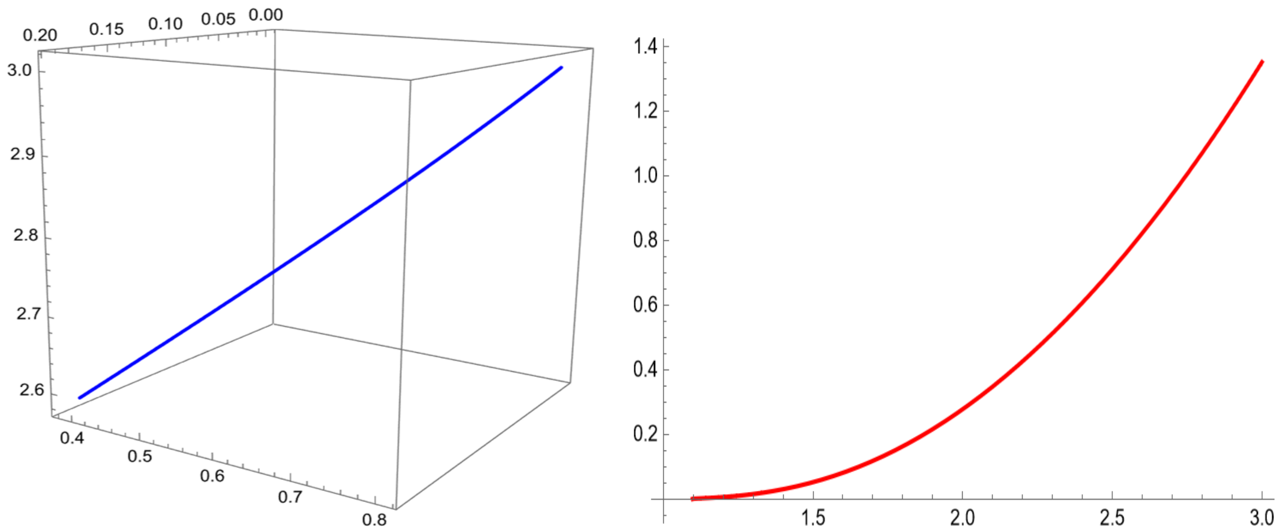

Figure 4. Normal curve $\alpha(s)$ and behavior of $\kappa(s)$ for $s \in (1,3)$

Conclusion

The geometric characteristics of non-null curves resting entirely on spacelike surface via their position vectors in the dimensions Minkowski 3-space $\mathbb{R}_1^3$ are investigated as a consequence of the results. We also offer a few representations for the spacelike curves that lie on the three subspaces of $\mathbb{R}_1^3$ spanned by subspaces $\{T, \xi\}$, $\{T, \eta\}$ and $\{\xi, \eta\}$.

Acknowledgment

The authors would like to thank Professor Aliakbar Haghighi (Editor in Chief) as well as the anonymous referees for their efforts and constructive criticism which have improved of the paper.

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