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Analysis of Resonant Curve in the Earth-Moon System under the Effect of Resistive Force and Earth's Equatorial Ellipticity

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Abstract

In the present paper, we have determined the equations of motion of the Moon in spherical coordinate system using the gravitational potential of Earth. Using perturbation, equations of motion are reduced to a second order differential equation. From the solution, two types of resonance are observed: (i) due to the frequencies–rate of change of Earth's equatorial ellipticity parameter and Earth's rotation rate, and (ii) due to the frequencies–angular velocity of the bary-center around the sun and Earth's rotation rate. Resonant curves are drawn where oscillatory amplitude becomes infinitely large at the resonant points. The effect of Earth's equatorial ellipticity parameter and resistive force on the resonant curve is analyzed. From the graphs it is observed that the effect of Earth's equatorial ellipticity on the resonant curve is very small while the effect of resistive force is significant. It is also observed that oscillatory amplitude decreases when the magnitude of resistive force increases. Finally, the phase portrait is analyzed when the system is free from forces. Orbits in the phase space are also studied by applying the method of Poincare section. A necessary condition for the bifurcation is derived at the end.

Keywords: Resonance; Earth's equatorial ellipticity; Oblateness; Resistive force; Earth-Moon system; Obliquity; Perturbation; Phase portrait; Poincare section; Bifurcation

MSC 2010 No.: 37N05, 70F07, 70F15, 34F15

1. Introduction

In our solar system, motion of celestial bodies is influenced by many forces such as solar radiation pressure, solar gravitational field, Earth's gravitational force, etc. The phenomenon of resonance, phase portrait, Poincare section and bifurcation has been studied by many authors by including many forces while in our paper we have investigated the resonant curves, phase portrait, Poincare section and bifurcation theory in the Earth-Moon system including Earth's equatorial ellipticity parameter and resistive force. Using Lagrange's planetary equation, Crook (1962) studied the gravitational attraction effect of Sun-Moon on the orbital elements of satellite around the Earth.

Frick and Garber (1962) studied the motion of the synchronous satellite around the Earth by considering the gravitational attraction of the Moon and the Sun. Also Earth's oblateness has been considered. Hagihara (1972) investigated the problems of resonance of geo-centric satellite around the Earth due to oblateness. Garfinkel (1982) surveyed the problems of resonance in celestial mechanics and formulated the method for the resonance problems and found the solutions. Shaw and Holmes (1983) investigated bifurcation, the phase portrait for unforced, undamped system and phase space using method of Poincare section for one degree of freedom. Zaborszky et al. (1988) investigated the phase portrait, stability boundary and stability region for nonlinear dynamics. In the planar elliptic restricted three-body problem, Henrard (1988) reviewed the study of resonance and showed the chaotic behavior in the motion of the celestial body.

Gidea et al. (2007) investigated the motion of the infinitesimal mass and reconstruct the phase space in planar circular restricted three-body problem. Farmiga and Moraes (2011) studied the motion of the artificial satellite between the frequencies of the orbital mean motion and the rotational motion of the Earth. They also studied the phase space by taking one fixed resonant angle of the resulting system. Tsumoto et al. (2012) studied the bifurcation for the nonlinear differential equation and formulated this problem as the boundary value problem and solved it by using newton's method. In the planar restricted three-body problem, Narayan and Singh (2014) studied the linear stability of triangular equilibrium points and the resonance problems by considering the photo gravitational effects in the elliptic and circular case.

Yadav et al. (2016) studied the resonance of synchronous satellite between the rotation rate of the Earth and angular rate of the Earth-Moon system using perturbation technique. They also found resonance between spin rate of the Earth and angular velocity of a synchronous satellite of the Earth. Licea et al. (2019) studied the phase portrait and provided a graphical tool to represent it in n-dimensional space. Yadav et al. (2021) studied the resonant curve between the frequencies $\dot{\gamma}$ and $\dot{\theta}_E$ in a geo-synchronous satellite and shown the effect of Earth's equatorial ellipticity parameter and resistive force on the resonant curve.

In the present study, Luni-Solar resonance is investigated with the help of resonant curves. Effect of Earth's equatorial ellipticity parameter and resistive force on the resonant curves are analyzed. From the graphs it is shown that the oscillatory amplitude decreases on increasing the magnitude of resistive force while effect of Earth's equatorial ellipticity parameter on oscillatory amplitude is very small.

The present study on Luni-Solar resonance have many important applications to manned lunar mission design, stationkeeping on a spacecraft in the vicinity of Libration points orbit as part of trajectory design in the Earth-Moon system and communication system in the vicinity of the Moon.

We have organized this paper as follows. The configuration of the problem has been described in Section 2. In Section 3, we have derived the equations of motion of the Moon in spherical coordinate system. By defining the perturbations in r_m and θ_m relative to the Moon, we obtained a second order linear differential equation in Section 4. In Section 5, we have shown the effect of Earth's equatorial ellipticity and resistive force in the resonant curves due to the frequencies $\dot{\gamma}$, $\dot{\theta}_E$ and $\dot{\alpha}$. In Section 6, we analysed the phase portrait for unforced and undamped and studied the phase space using the method of Poincare section. We also derived a condition for bifurcation to occur. Finally, in Section 7, we analysed and discussed the results.

2. Configuration of the Problem

We define the following coordinate systems for the problem.

(1) (X, Y, Z) as an inertial coordinate system with center of the Sun as the origin and XY plane as the ecliptic plane and the vectors $\vec{x}_1$, $\vec{y}_1$ and $\vec{z}_1$ are the unit vectors. (Figure 1)
(2) (X', Y', Z') be the coordinate system of Earth-Moon system with bary-center as the origin (Figure 1).
(3) (X'', Y'', Z'') be the coordinate system with origin at the center of Earth and all the axes parallel to XYZ system (Figure 1).
(4) (x, y, z) coordinate system with origin at the center of Earth, xy-plane is in the Earth's equatorial plane. X'' axes is along the x axes which is in the direction of vernal equinox (Figure 1).

We consider (r_m, θ_m, ϕ_m) as the spherical system of the Moon and center of Earth as origin, where r_m is the radial distance of the Moon from the center of the Earth, θ_m is the Moon longitude and ϕ_m is the latitude of the Moon in the direction of vernal equinox with unit vectors $\vec{r}_1$, $\vec{n}_1$ and $\vec{n}_2$.

In Figure 1, the relations between unit vectors are as follows,

$$
\begin{aligned}
\vec{r}_1 &= a_x\vec{x}_1 + a_y\vec{y}_1 + a_z\vec{z}_1,\\
\vec{n}_1 &= b_x\vec{x}_1 + b_y\vec{y}_1 + b_z\vec{z}_1,\\
\vec{n}_2 &= c_x\vec{x}_1 + c_y\vec{y}_1 + c_z\vec{z}_1,
\end{aligned}
$$

where

$$
\begin{aligned}
a_x &= \cos\theta_m \ \cos\phi_m, & a_y &= \sin\theta_m \ \cos\phi_m \ \cos\epsilon - \sin\epsilon \ \sin\phi_m,\\
a_z &= \sin\theta_m \ \cos\phi_m \ \sin\epsilon + \cos\epsilon \ \sin\phi_m, & b_x &= -\sin\theta_m,\\
b_y &= \cos\theta_m \ \cos\epsilon, & b_z &= \cos\theta_m \ \sin\epsilon,\\
c_x &= -\cos\theta_m \ \sin\phi_m, & c_y &= -\sin\theta_m \ \sin\phi_m \ \cos\epsilon - \sin\epsilon \ \cos\phi_m,\\
c_z &= -\sin\theta_m \ \sin\phi_m \ \sin\epsilon + \cos\epsilon \ \cos\phi_m.
\end{aligned}
$$

The acceleration and velocity components along the direction $\vec{r}_1$, $\vec{n}_1$ and $\vec{n}_2$ are given by

$$a_{r_m} = \ddot{\vec{r}}_m.\vec{r}_1 = \ddot{r}_m - r_m\dot{\theta}_m^2 \cos^2\phi_m - r_m\dot{\phi}_m^2, \tag{1}$$

$$a_{\theta_m} = \ddot{\vec{r}}_m.\vec{n}_1 = \frac{1}{r_m \cos\phi_m}\frac{d}{dt}(r_m^2\dot{\theta}_m \cos^2\phi_m), \tag{2}$$

$$a_{\phi_m} = \ddot{\vec{r}}_m.\vec{n}_2 = \frac{1}{r_m}\frac{d}{dt}(r_m^2\dot{\phi}_m), \tag{3}$$

$$v_{r_m} = \dot{r}_m, \tag{4}$$

$$v_{\theta_m} = r_m\,\dot{\theta}_m \cos\phi_m, \tag{5}$$

$$v_{\phi_m} = r_m\,\dot{\phi}_m. \tag{6}$$

3. Equations of Motion of the Moon M in Spherical Coordinate System

$\ddot{\vec{r}}_m$ is a acceleration of the Moon relative to Earth and it can be expressed as

$$\ddot{\vec{r}}_m = \frac{1}{M_m}(\vec{F}_{ME} + \vec{F}_{MS}) - \frac{1}{M_E}(\vec{F}_{ES} + \vec{F}_{EM}) - \frac{b}{M_m}\vec{v}, \tag{7}$$

where

- M_m is the mass of the Moon,
- M_E is the mass of the Earth,
- $\vec{F}_{MS}$ and $\vec{F}_{ME}$ are the gravitation force of the Sun and the Earth on Moon, respectively,
- $\vec{F}_{ES}$ and $\vec{F}_{EM}$ are gravitational force of the Sun and the Moon on the Earth, respectively,
- $\vec{v}$ is the velocity of M,
- $b = \frac{F}{c}$ (F is the measure of the radiation pressure and c is velocity of light).

By following the procedure of Yadav et al. (2021), the components of force of $\vec{F}_{ME}$ along the direction of $\vec{r}_1$, $\vec{n}_1$ and $\vec{n}_2$ can be expressed as

$$\frac{\vec{F}_{ME}.\vec{r}_1}{M_m} = -\frac{g_0R_0^2}{r_m^2} + \frac{3J_2g_0R_0^4}{r_m^4}\left(\frac{3\sin^2\phi_m - 1}{2}\right) - \frac{9J_2^{(2)}g_0R_0^4\cos^2\phi_m \cos 2\gamma}{r_m^4}, \tag{8}$$

$$\frac{\vec{F}_{ME}.\vec{n}_1}{M_m} = -\frac{6J_2^{(2)}g_0R_0^4 \cos\phi_m \sin 2\gamma}{r_m^4}, \tag{9}$$

$$\frac{\vec{F}_{ME}.\vec{n}_2}{M_m} = -\frac{3J_2g_0R_0^4 \cos\phi_m \sin\phi_m}{r_m^4} - \frac{6J_2^{(2)}g_0R_0^4 \cos\phi_m \sin\phi_m \cos 2\gamma}{r_m^4}, \tag{10}$$

where

$r_m =$ radial distance of the moon from the centre of the Earth,
$R_0 = 63781\ Km =$ radius of the Earth,
$g_0 = 7.3156608 \times 10^7\ km/day^2 =$ gravitational acceleration at the Earth's surface,
$J_2 = 1.08219 \times 10^{-3} =$ Earth oblateness coefficient,
$J_2^{(2)} = 2.32 \times 10^{-6} =$ equatorial ellipticity coefficient,
$\gamma = \theta - \theta_E$,
$\phi_m =$ latitude of the Moon.

Since $\vec{F}_{MS}$ is the gravitational force of Sun to Moon,

$$\frac{\vec{F}_{MS}}{M_m} = -\frac{g_1 R_1^2}{\rho_1^3}\vec{\rho}_1, \tag{11}$$

where $\vec{\rho}_1$ is vector from the Sun to the Moon, g_1 is gravitational acceleration at the surface of Sun and R_1 is the radius of the Sun.

We can write $\vec{\rho}_1$ as

$$\vec{\rho}_1 = \vec{R} + \vec{r}_m - \frac{\vec{r}_m}{\mu}, \tag{12}$$

where $\mu = (M_m + M_E)/M_m$, $\vec{R}$ is a vector from the Sun to the center of mass of Earth-Moon system and $\frac{\vec{r}_m}{\mu}$ is a vector from the center of mass of Earth-Moon system to the Earth's center.

Substituting the value of $\vec{\rho}_1$ from Equation (12) in (11), we get

$$\frac{\vec{F}_{MS}}{M_m} = -\frac{g_1 R_1^2}{R^3}\left[1 - \frac{3(\vec{R}.\vec{r}_m)}{R^2} + \frac{3(\vec{R}.\vec{r}_m)}{\mu R^2}\right]\left(\vec{R} + \vec{r}_m - \frac{\vec{r}_m}{\mu}\right). \tag{13}$$

Now, the components of force $\vec{F}_{MS}$ along the direction of $\vec{r}_1$, $\vec{n}_1$ and $\vec{n}_2$ are given by

$$\frac{\vec{F}_{MS}.\vec{r}_1}{M_m} = -\frac{g_1 R_1^2}{R^3}\left[1 - \frac{3(\vec{R}.\vec{r}_m)}{R^2} + \frac{3(\vec{R}.\vec{r}_m)}{\mu R^2}\right]\left(\vec{R}.\vec{r}_1 + r_m - \frac{r_m}{\mu}\right), \tag{14}$$

$$\frac{\vec{F}_{MS}.\vec{n}_1}{M_m} = -\frac{g_1 R_1^2}{R^3}\left[1 - \frac{3(\vec{R}.\vec{r}_m)}{R^2} + \frac{3(\vec{R}.\vec{r}_m)}{\mu R^2}\right](\vec{R}.\vec{n}_1), \tag{15}$$

$$\frac{\vec{F}_{MS}.\vec{n}_2}{M_m} = -\frac{g_1 R_1^2}{R^3}\left[1 - \frac{3(\vec{R}.\vec{r}_m)}{R^2} + \frac{3(\vec{R}.\vec{r}_m)}{\mu R^2}\right](\vec{R}.\vec{n}_2). \tag{16}$$

Now, the gravitational force of the Sun on the Earth $\vec{F}_{ES}$ is given as,

$$\frac{\vec{F}_{ES}}{M_E} = -\frac{g_1 R_1^2}{r_{10}^3}\vec{r}_{10}, \tag{17}$$

where $\vec{r}_{10}$ = a vector from the center of the Sun to the Earth; it can be be represented as

$$\vec{r}_{10} = \vec{R} - \frac{\vec{r}_m}{\mu}. \tag{18}$$

Substituting the value of $\vec{r}_{10}$ from Equation (18) in (17), we get

$$\frac{\vec{F}_{ES}}{M_E} = -\frac{g_1 R_1^2}{R^3}\left[1 + \frac{3(\vec{R}.\vec{r}_m)}{\mu R^2}\right]\left(\vec{R} - \frac{\vec{r}_m}{\mu}\right). \tag{19}$$

Now, the components of force $\vec{F}_{ES}$ along the direction of $\vec{r}_1$, $\vec{n}_1$ and $\vec{n}_2$ are given by

$$\frac{\vec{F}_{ES}.\vec{r}_1}{M_E} = -\frac{g_1 R_1^2}{R^3}\left[1 + \frac{3(\vec{R}.\vec{r}_m)}{\mu R^2}\right]\left(\vec{R}.\vec{r}_1 - \frac{r_m}{\mu}\right), \tag{20}$$

$$\frac{\vec{F}_{ES}.\vec{n}_1}{M_E} = -\frac{g_1 R_1^2}{R^3}\left[1 + \frac{3(\vec{R}.\vec{r}_m)}{\mu R^2}\right](\vec{R}.\vec{n}_1), \tag{21}$$

$$\frac{\vec{F}_{ES}.\vec{n}_2}{M_E} = -\frac{g_1 R_1^2}{R^3}\left[1 + \frac{3(\vec{R}.\vec{r}_m)}{\mu R^2}\right](\vec{R}.\vec{n}_2). \tag{22}$$

Now, the gravitational force of the Moon on the Earth $\vec{F}_{EM}$ is given by

$$\frac{\vec{F}_{EM}}{M_E} = -\frac{g_2 R_2^2}{r_m^3}\vec{r}_m, \tag{23}$$

where g_2 is gravitational attraction at the surface of the Moon and R_2 is the radius of the Moon.

The components of force $\vec{F}_{EM}$ along the direction $\vec{r}_1$, $\vec{n}_1$ and $\vec{n}_2$ are given by

$$\frac{\vec{F}_{EM}.\vec{r}_1}{M_E} = -\frac{g_2 R_2^2}{r_m^2}, \tag{24}$$

$$\frac{\vec{F}_{EM}.\vec{n}_1}{M_E} = 0, \tag{25}$$

$$\frac{\vec{F}_{EM}.\vec{n}_2}{M_E} = 0. \tag{26}$$

Also, we have

$$\vec{R} = R\left(\vec{x}_1 \cos\alpha + \vec{y}_1 \sin\alpha\right), \tag{27}$$

$$\vec{r}_m = r_m \vec{r}_1, \tag{28}$$

$$\dot{\alpha}^2 = \frac{g_1 R_1^2}{R^3}, \tag{29}$$

$$\dot{\nu}^2 = \frac{g_2 R_2^2}{r_m^3}\mu. \tag{30}$$

By using Equations (1), (4), (8), (14), (20), (24), (27), (28), (29) and (30) in Equation (7), we get

$$\ddot{r}_m - r_m\dot{\theta}_m^2\cos^2\phi_m - r_m\dot{\phi}_m^2 = -\frac{g_0R_0^2}{r_m^2} + \frac{3J_2g_0R_0^4}{r_m^4}\left(\frac{3\sin^2\phi_m - 1}{2}\right)$$
$$-\frac{9J_2^{(2)}g_0R_0^4\cos^2\phi_m\ \cos 2\gamma}{r_m^4}$$
$$-\frac{g_1R_1^2}{R^3}\left[1-\frac{3(\vec{R}.\vec{r}_m)}{R^2}+\frac{3(\vec{R}.\vec{r}_m)}{\mu R^2}\right](\vec{R}.\vec{r}_1 + r_m - \frac{r_m}{\mu})$$
$$+\frac{g_1R_1^2}{R^3}\left[1+\frac{3(\vec{R}.\vec{r}_m)}{\mu R^2}\right](\vec{R}.\vec{r}_1 - \frac{r_m}{\mu}) + \frac{g_2R_2^2}{r_m^2} - \frac{b}{M_m}\dot{r}_m \tag{31}$$

$$= -\frac{g_0R_0^2}{r_m^2} + \frac{3J_2g_0R_0^4}{r_m^4}\left(\frac{3\sin^2\phi_m - 1}{2}\right) - \frac{9J_2^{(2)}g_0R_0^4\cos^2\phi_m\ \cos 2\gamma}{r_m^4}$$
$$+\frac{3r_m^2\dot{\alpha}^2}{2R}\left(\frac{\mu-2}{\mu}\right)[\cos(\theta_m+\alpha)\ K_1 + \cos(\theta_m-\alpha)\ K_2 - \sin\alpha\ K_3]$$
$$+\frac{3r_m\dot{\alpha}^2}{4}\Big[\cos(2\theta_m+2\alpha)\ K_4 + \cos(2\theta_m-2\alpha)\ K_5 + \cos 2\theta_m\ K_6 + \cos 2\alpha\ K_7$$
$$+\sin(\theta_m+2\alpha)\ K_8 + \sin(\theta_m-2\alpha)\ K_9 + \sin\theta_m\ K_{10} + K_{11}\Big]$$
$$+\frac{\dot{\nu}^2r_m}{\mu} - r_m\dot{\alpha}^2 - B\dot{r}_m, \tag{32}$$

where values of $K_i's$ are given in Appendix (A).

Now, by using Equations (2), (5), (9), (15), (21), (25), (27), (28), (29) and (30) in Equation (7), we get

$$\frac{1}{r_m\ \cos\phi_m}\frac{d}{dt}(r_m^2\dot{\theta}_m\ \cos^2\phi_m) = -\frac{6J_2^{(2)}g_0R_0^4\ \cos\phi_m\ \sin 2\gamma}{r_m^4}$$
$$-\frac{g_1R_1^2}{R^3}\left[1-\frac{3(\vec{R}.\vec{r}_m)}{R^2}+\frac{3(\vec{R}.\vec{r}_m)}{\mu R^2}\right](\vec{R}.\vec{n}_1) + \frac{g_1R_1^2}{R^3}\left[1+\frac{3(\vec{R}.\vec{r}_m)}{\mu R^2}\right](\vec{R}.\vec{n}_1)$$
$$-\frac{b}{M_m}\ r_m\ \dot{\theta}_m\ \cos\phi_m \tag{33}$$

$$= -\frac{6J_2^{(2)}g_0R_0^4\ \cos\phi_m\ \sin 2\gamma}{r_m^4} + 3r_m\dot{\alpha}^2\Big[\sin(2\theta_m+2\alpha)\ M_1 + \sin 2\theta_m\ M_2$$
$$+\sin(2\theta_m-2\alpha)\ M_3 + \cos(\theta_m-2\alpha)\ M_4 + \cos(\theta_m+2\alpha)\ M_5 + \cos 2\theta_m\ M_6$$
$$+\cos(2\theta_m+2\alpha)\ M_7 + \cos(2\theta_m-2\alpha)\ M_8 + \cos\theta_m\ M_9\Big] - B\ r_m\ \dot{\theta}_m\ \cos\phi_m, \tag{34}$$

where values of $M_i's$ are given in Appendix (B).

Finally, by using Equations (3), (6), (10), (16), (22), (26), (27), (28), (29) and (30) in Equation (7), we get

$$\frac{1}{r_m}\frac{d}{dt}(r_m^2\dot{\phi}_m) = -\frac{3J_2 g_0 R_0^4\ \cos\phi_m\ sin\phi_m}{r_m^4} - \frac{6J_2^{(2)} g_0 R_0^4\ \cos\phi_m\ \sin\phi_m\ \cos 2\gamma}{r_m^4}$$
$$-\frac{g_1 R_1^2}{R^3}\left[1 - \frac{3(\vec{R}.\vec{r}_m)}{R^2} + \frac{3(\vec{R}.\vec{r}_m)}{\mu R^2}\right](\vec{R}.\vec{n}_2) + \frac{g_1 R_1^2}{R^3}\left[1 + \frac{3(\vec{R}.\vec{r}_m)}{\mu R^2}\right](\vec{R}.\vec{n}_2) - B\ r_m\ \dot{\phi}_m, \quad (35)$$

where $B = \frac{b}{M_m}$ (M_m is mass of the Moon and b is damping constant).

4. Perturbations Relative to the Moon Including Resistive Force and Latitude ϕ_m

We define the following perturbations relative to the Moon as follows,

$$r_m = r_{em} + \Delta r_m,$$
$$\dot{\theta}_m = \dot{\theta}_E + \Delta\dot{\theta}_m,$$

where r_{em} is the steady-state value of orbital radius of the Moon.

Substituting the values of r_m and $\dot{\theta}_m$ in Equation (32) and (34) (neglecting second and higher order perturbations), we get

$$\Delta\ddot{r}_m + B\ \Delta\dot{r}_m - \left(\dot{\theta}_E^2\Delta r_m + r_{em}\dot{\theta}_E^2 + 2r_{em}\dot{\theta}_E\Delta\dot{\theta}_m\right)\cos^2\phi_m = -\frac{g_0 R_0^2}{r_{em}^2} + \frac{3J_2 g_0 R_0^4(3\sin^2\phi_m - 1)}{2r_{em}^4}$$
$$-\frac{9J_2^{(2)} g_0 R_0^4}{r_{em}^4}\ \cos^2\phi_m\ \cos 2\gamma + \frac{3r_{em}^2\dot{\alpha}^2}{2R}\left(\frac{\mu-2}{\mu}\right)\Bigg[\cos(\theta_m+\alpha)\ K_1 + \cos(\theta_m-\alpha)\ K_2$$
$$-\sin\alpha\ K_3\Bigg] + \frac{3r_{em}\dot{\alpha}^2}{4}\Bigg[\cos(2\theta_m+2\alpha)\ K_4 + \cos(2\theta_m-2\alpha)\ K_5 + \cos 2\theta_m\ K_6 + \cos 2\alpha\ K_7$$
$$+\sin(\theta_m+2\alpha)\ K_8 + \sin(\theta_m-2\alpha)\ K_9 + \sin\theta_m\ K_{10} + K_{11}\Bigg] + \frac{\dot{\nu}^2 r_{em}}{\mu} - r_{em}\dot{\alpha}^2, \quad (36)$$

$$\frac{d}{dt}\left(r_{em}^2\dot{\theta}_E + r_{em}^2\Delta\dot{\theta}_m + 2r_{em}\dot{\theta}_E\Delta r_m\right)\cos^2\phi_m + B\ \Bigg(r_{em}^2\dot{\theta}_E + r_{em}^2\Delta\dot{\theta}_m$$
$$+2r_{em}\dot{\theta}_E\Delta r_m\Bigg)\cos^2\phi_m = -\frac{6J_2^{(2)} g_0 R_0^4\ \cos^2\phi_m\ \sin 2\gamma}{r_{em}^3}$$
$$+3\ r_{em}^2\dot{\alpha}^2\ \cos\phi_m\ \Bigg[\sin(2\theta_m+2\alpha)\ M_1 + \sin 2\theta_m\ M_2 + \sin(2\theta_m-2\alpha)\ M_3$$
$$+\cos(\theta_m-2\alpha)\ M_4 + \cos(\theta_m+2\alpha)\ M_5 + \cos 2\theta_m\ M_6 + \cos(2\theta_m+2\alpha)\ M_7$$
$$+\cos(2\theta_m-2\alpha)\ M_8 + \cos\theta_m\ M_9\Bigg]. \quad (37)$$

On solving the linear differential equation (37), we get

$$
\begin{aligned}
&(r_{em}^2\dot{\theta}_E + r_{em}^2\Delta\dot{\theta}_m + 2r_{em}\dot{\theta}_E\Delta r_m)\ \cos^2\phi_m = -\frac{6J_2^{(2)}g_0R_0^4}{r_{em}^3}\frac{B}{B^2+4\dot{\gamma}^2}\left(\sin 2\gamma - \frac{2\ \dot{\gamma}\ \cos 2\gamma}{B}\right)\\
&+3r_{em}^2\dot{\alpha}^2\ \cos\phi_m\left[\frac{BM_1}{B^2+4(\dot{\theta}_E+\Delta\dot{\theta}_m+\dot{\alpha})^2}\left(\sin(2(\theta_m+\alpha))\right.\right.\\
&\left.-\frac{2(\dot{\theta}_E+\Delta\dot{\theta}_m+\dot{\alpha})}{B}\cos(2(\theta_m+\alpha))\right)\\
&+\frac{BM_2}{B^2+4(\dot{\theta}_E+\Delta\dot{\theta}_m)^2}\left(\sin(2(\theta_m)) - \frac{2\dot{\theta}_E}{B}\cos(2(\theta_m))\right)\\
&+\frac{BM_3}{B^2+4(\dot{\theta}_E+\Delta\dot{\theta}_m-\dot{\alpha})^2}\left(\sin(2(\theta_m-\alpha)) - \frac{2(\dot{\theta}_E+\Delta\dot{\theta}_m-\dot{\alpha})}{B}\cos(2(\theta_m-\alpha))\right)\\
&+\frac{BM_4}{B^2+(\dot{\theta}_E+\Delta\dot{\theta}_m-2\ \dot{\alpha})^2}\left(\cos(\theta_m-2\ \alpha) + \frac{(\dot{\theta}_E+\Delta\dot{\theta}_m-2\ \dot{\alpha})}{B}\sin(\theta_m-2\ \alpha)\right)\\
&+\frac{BM_5}{B^2+(\dot{\theta}_E+\Delta\dot{\theta}_m+2\ \dot{\alpha})^2}\left(\cos(\theta_m+2\ \alpha) + \frac{(\dot{\theta}_E+\Delta\dot{\theta}_m+2\ \dot{\alpha})}{B}\sin(\theta_m+2\ \alpha)\right)\\
&+\frac{BM_6}{B^2+4\ \dot{\theta}_E^2}\left(\cos(2\theta_m) + \frac{2(\dot{\theta}_E+\Delta\dot{\theta}_m)}{B}\sin(2\theta_m)\right)\\
&+\frac{BM_7}{B^2+4(\dot{\theta}_E+\Delta\dot{\theta}_m+\dot{\alpha})^2}\left(\cos(2(\theta_m+\alpha)) - \frac{2(\dot{\theta}_E+\Delta\dot{\theta}_m+\dot{\alpha})}{B}\sin(2(\theta_m+\alpha))\right)\\
&+\frac{BM_8}{B^2+4(\dot{\theta}_E+\Delta\dot{\theta}_m-\dot{\alpha})^2}\left(\cos(2(\theta_m-\alpha)) - \frac{2(\dot{\theta}_E+\Delta\dot{\theta}_m-\dot{\alpha})}{B}\sin(2(\theta_m-\alpha))\right)\\
&\left.+\frac{BM_9}{B^2+\ \dot{\theta}_E^2}\left(\cos(\theta_m) + \frac{2(\dot{\theta}_E+\Delta\dot{\theta}_m)}{B}\sin(\theta_m)\right)\right] + C\ e^{-Bt},
\end{aligned}
\tag{38}
$$

where C is a integrating constant to determined by initial condition at $t = t_0$, $\Delta r_m = \Delta r_{m0}$, $\Delta\dot{\theta}_m = 0$, $\theta_m = \theta_{m0}$ and $\alpha = \alpha_0$; Then, we get the value of C as

$$
\begin{aligned}
C =&(r_{em}^2\dot{\theta}_E + 2r_{em}\dot{\theta}_E\Delta r_{m0})\ \cos^2\phi_m\ e^{Bt_0} + \frac{6J_2^{(2)}g_0R_0^4\ e^{Bt_0}}{r_{em}^3}\frac{B}{B^2+4\dot{\gamma}^2}\ (\sin 2\dot{\gamma}t_0 - \frac{2\ \dot{\gamma}\ \cos 2\dot{\gamma}t_0}{B})\\
&-3\ e^{Bt_0}r_{em}^2\dot{\alpha}^2\ \cos\phi_m\left[\frac{BM_1}{B^2+4(\dot{\theta}_E+\dot{\alpha})^2}\left(\sin(2(\theta_{m0}+\alpha_0))\right.\right.\\
&\left.-\frac{2(\dot{\theta}_E+\dot{\alpha})}{B}\cos(2(\theta_{m0}+\alpha_0))\right) + \frac{BM_2}{B^2+4\dot{\theta}_E^2}\left(\sin(2(\theta_{mo})) - \frac{2\dot{\theta}_E}{B}\cos(2(\theta_{m0}))\right)\\
&+\frac{BM_3}{B^2+4(\dot{\theta}_E-\dot{\alpha})^2}\left(\sin(2(\theta_{m0}-\alpha_0)) - \frac{2(\dot{\theta}_E-\dot{\alpha})}{B}\cos(2(\theta_{m0}-\alpha_0))\right)
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{BM_4}{B^2+(\dot\theta_E-2\,\dot\alpha)^2}\left(\cos(\theta_{m0}-2\,\alpha_0)+\frac{(\dot\theta_E-2\,\dot\alpha)}{B}\sin(\theta_{m0}-2\,\alpha_0)\right)\\
&+ \frac{BM_5}{B^2+(\dot\theta_E+2\,\dot\alpha)^2}\left(\cos(\theta_{m0}+2\,\alpha_0)+\frac{(\dot\theta_E+2\,\dot\alpha)}{B}\sin(\theta_{m0}+2\,\alpha_0)\right)\\
&+ \frac{BM_6}{B^2+4\,\dot\theta_E^2}\left(\cos(2\theta_{m0})+\frac{2\dot\theta_E}{B}\sin(2\theta_{m0})\right)\\
&+ \frac{BM_7}{B^2+4(\dot\theta_E+\dot\alpha)^2}\left(\cos(2(\theta_{m0}+\alpha_0))-\frac{2(\dot\theta_E+\dot\alpha)}{B}\sin(2(\theta_{m0}+\alpha_0))\right)\\
&+ \frac{BM_8}{B^2+4(\dot\theta_E-\dot\alpha)^2}\left(\cos(2(\theta_{m0}-\alpha_0))-\frac{2(\dot\theta_E-\dot\alpha)}{B}\sin(2(\theta_{m0}-\alpha_0))\right)\\
&+ \frac{BM_9}{B^2+\,\dot\theta_E^2}\left(\cos(\theta_{m0})+\frac{2\dot\theta_E}{B}\sin(\theta_{m0})\right)\Bigg].
\end{aligned}
$$

Substituting the value of C in Equation (38), we get

$$
\begin{aligned}
&r_{em}\Delta\dot\theta_m\ \cos^2\phi_m = \ r_{em}\ \dot\theta_E\ \cos^2\phi_m(\ e^{-B(t_0-t)}-1)+2\ \dot\theta_E\ cos^2\phi_m\ (\Delta r_{m0}\ e^{-B(t_0-t)}-\Delta r_m)\\
&-\frac{6J_2^{(2)}g_0R_0^4}{r_{em}^4}\frac{B}{B^2+4\dot\gamma^2}\ (\sin 2\gamma-\frac{2\ \dot\gamma\ \cos 2\gamma}{B})+3r_{em}\dot\alpha^2\ \cos\phi_m\ \Bigg[\ \frac{BM_1}{B^2+4(\dot\theta_E+\Delta\dot\theta_m+\dot\alpha)^2}\times\\
&\left(\sin(2(\theta_m+\alpha))-\frac{2(\dot\theta_E+\Delta\dot\theta_m+\dot\alpha)}{B}\cos(2(\theta_m+\alpha))\right)\\
&+ \frac{BM_2}{B^2+4(\dot\theta_E+\Delta\dot\theta_m)^2}\left(\sin(2(\theta_m))-\frac{2\dot\theta_E}{B}\cos(2(\theta_m))\right)\\
&+ \frac{BM_3}{B^2+4(\dot\theta_E+\Delta\dot\theta_m-\dot\alpha)^2}\left(\sin(2(\theta_m-\alpha))-\frac{2(\dot\theta_E+\Delta\dot\theta_m-\dot\alpha)}{B}\cos(2(\theta_m-\alpha))\right)\\
&+ \frac{BM_4}{B^2+(\dot\theta_E+\Delta\dot\theta_m-2\,\dot\alpha)^2}\left(\cos(\theta_m-2\,\alpha)+\frac{(\dot\theta_E+\Delta\dot\theta_m-2\,\dot\alpha)}{B}\sin(\theta_m-2\,\alpha)\right)\\
&+ \frac{BM_5}{B^2+(\dot\theta_E+\Delta\dot\theta_m+2\,\dot\alpha)^2}\left(\cos(\theta_m+2\,\alpha)+\frac{(\dot\theta_E+\Delta\dot\theta_m+2\,\dot\alpha)}{B}\sin(\theta_m+2\,\alpha)\right)\\
&+ \frac{BM_6}{B^2+4\,\dot\theta_E^2}\left(\cos(2\theta_m)+\frac{2(\dot\theta_E+\Delta\dot\theta_m)}{B}\sin(2\theta_m)\right)\\
&+ \frac{BM_7}{B^2+4(\dot\theta_E+\Delta\dot\theta_m+\dot\alpha)^2}\left(\cos(2(\theta_m+\alpha))-\frac{2(\dot\theta_E+\Delta\dot\theta_m+\dot\alpha)}{B}\sin(2(\theta_m+\alpha))\right)\\
&+ \frac{BM_8}{B^2+4(\dot\theta_E+\Delta\dot\theta_m-\dot\alpha)^2}\left(\cos(2(\theta_m-\alpha))-\frac{2(\dot\theta_E+\Delta\dot\theta_m-\dot\alpha)}{B}\sin(2(\theta_m-\alpha))\right)
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{BM_9}{B^2 + \dot{\theta}_E^2}\left(\cos(\theta_m) + \frac{2(\dot{\theta}_E + \Delta\dot{\theta}_m)}{B}\sin(\theta_m)\right)\Bigg]\\
&+ \frac{6J_2^{(2)} g_0 R_0^4 \, e^{B(t_0-t)}}{r_{em}^4}\frac{B}{B^2+4\dot{\gamma}^2}\,(\sin 2\dot{\gamma}t_0 - \frac{2\,\dot{\gamma}\; cos 2\dot{\gamma}t_0}{B}) - 3\, e^{B(t_0-t)} r_{em}\dot{\alpha}^2 \, \cos\phi_m \times\\
&\Bigg[\frac{BM_1}{B^2+4(\dot{\theta}_E+\dot{\alpha})^2}\left(\sin(2(\theta_{m0}+\alpha_0)) - \frac{2(\dot{\theta}_E+\dot{\alpha})}{B}\cos(2(\theta_{m0}+\alpha_0))\right)\\
&+ \frac{BM_2}{B^2+4\dot{\theta}_E^2}\left(\sin(2(\theta_{mo})) - \frac{2\dot{\theta}_E}{B}\cos(2(\theta_{m0}))\right)\\
&+ \frac{BM_3}{B^2+4(\dot{\theta}_E-\dot{\alpha})^2}\times\left(\sin(2(\theta_{m0}-\alpha_0)) - \frac{2(\dot{\theta}_E-\dot{\alpha})}{B}\cos(2(\theta_{m0}-\alpha_0))\right)\\
&+ \frac{BM_4}{B^2+(\dot{\theta}_E-2\,\dot{\alpha})^2}\left(\cos(\theta_{m0}-2\,\alpha_0) + \frac{(\dot{\theta}_E-2\,\dot{\alpha})}{B}\sin(\theta_{m0}-2\,\alpha_0)\right)\\
&+ \frac{BM_5}{B^2+(\dot{\theta}_E+2\,\dot{\alpha})^2}\left(\cos(\theta_{m0}+2\,\alpha_0) + \frac{(\dot{\theta}_E+2\,\dot{\alpha})}{B}\sin(\theta_{m0}+2\,\alpha_0)\right)\\
&+ \frac{BM_6}{B^2+4\,\dot{\theta}_E^2}\left(\cos(2\theta_{m0}) + \frac{2\dot{\theta}_E}{B}\sin(2\theta_{m0})\right)\\
&+ \frac{BM_7}{B^2+4(\dot{\theta}_E+\dot{\alpha})^2}\left(\cos(2(\theta_{m0}+\alpha_0)) - \frac{2(\dot{\theta}_E+\dot{\alpha})}{B}\sin(2(\theta_{m0}+\alpha_0))\right)\\
&+ \frac{BM_8}{B^2+4(\dot{\theta}_E-\dot{\alpha})^2}\left(\cos(2(\theta_{m0}-\alpha_0)) - \frac{2(\dot{\theta}_E-\dot{\alpha})}{B}\sin(2(\theta_{m0}-\alpha_0))\right)\\
&+ \frac{BM_9}{B^2+\dot{\theta}_E^2}\left(\cos(\theta_{m0}) + \frac{2\dot{\theta}_E}{B}\sin(\theta_{m0})\right)\Bigg].
\end{aligned}
$$

Putting the value of $r_{em}\Delta\dot{\theta}_m\cos^2\phi_m$ in Equation (36) and ignoring the non-periodic or secular terms, we get

$$
\begin{aligned}
&\Delta\ddot{r}_m + B\Delta\dot{r}_m + 3\dot{\theta}_E^2\cos^2\phi_m\Delta r_m = P_1\,\sin 2\gamma + P_2\,\cos 2\gamma + P_3\,\sin(2\theta_m+2\alpha)\\
&+ P_4\,\sin 2\theta_m + P_5\,\sin(\theta_m-2\alpha) + P_6\,\sin(\theta_m+2\alpha) + P_7\,\sin(2\theta_m-2\alpha)\\
&+ P_8\,\sin\theta_m + P_9\,\sin\alpha + P_{10}\,\cos(2\theta_m+2\alpha) + P_{11}\,\cos(2\theta_m) + P_{12}\,\cos(2\theta_m-2\alpha)\\
&+ P_{13}\,\cos(\theta_m-2\alpha) + P_{14}\,\cos(\theta_m+2\alpha) + P_{15}\,\cos\theta_m + P_{16}\,\cos(\theta_m+\alpha)\\
&+ P_{17}\,\cos(\theta_m-\alpha) + P_{18}\,\cos(2\,\alpha), \qquad (39)
\end{aligned}
$$

where values of $P_i's$ are given in Appendix (C). This is a second order non-homogeneous differential equation.

Hence, the particular solution of Equation (39) is given by

$$\begin{aligned}\Delta r_m = &Q_1 \ \sin 2\gamma + Q_2 \ \sin 2(\theta_m + \alpha) + Q_3 \ \sin 2\theta_m + Q_4 \ \sin(\theta_m - 2\alpha) \\ &+ Q_5 \ \sin(\theta_m + 2\alpha) + Q_6 \ \sin 2(\theta_m - \alpha) + Q_7 \ \sin \theta_m + Q_8 \ \sin(\theta_m + \alpha) \\ &+ Q_9 \ \sin(\theta_m - \alpha) + Q_{10} \ \sin 2\alpha + Q_{11} \ \sin \alpha + Q_{12} \ \cos 2\gamma + Q_{13} \ \cos 2(\theta_m + \alpha) \\ &+ Q_{14} \ \cos 2\theta_m + Q_{15} \ \cos(\theta_m - 2\alpha) + Q_{16} \ \cos(\theta_m + 2\alpha) + Q_{17} \ \cos 2(\theta_m - \alpha) \\ &+ Q_{18} \ \cos 2\theta_m + Q_{19} \ \cos \alpha + Q_{20} \ \cos(\theta_m + \alpha) + Q_{21} \ \cos(\theta_m - \alpha) + Q_{22} \ \cos 2\alpha, \qquad (40)\end{aligned}$$

where values of $Q_i^{\prime}s$ are given in Appendix (D).

5. Effect of Resistive Force on the Resonant Curves

5.1. Case 1 : When $\phi_m = 0$ (Moon M lies in the equatorial plane)

In Figure 2, when $\phi_m = 0$ and $B = 0$, oscillatory amplitude $A = \left[\frac{(4\dot{\gamma}^2 - 3\dot{\theta}_E^2 \cos^2 \phi_m) \ P_2 + 2 \ P_1 \dot{\gamma} \ B}{-4 \ \dot{\gamma}^2 B^2 - (4\dot{\gamma}^2 - 3\dot{\theta}_E^2 \cos^2 \phi_m)^2}\right]$ (corresponding to the term $Q_{12} \ \cos 2\gamma$) becomes indeterminate. So, resonance occurs at the point where $\frac{\dot{\gamma}^2}{\dot{\theta}_E^2} = \frac{3}{4}$.

In Figure 3, we observed that the oscillatory amplitude decreases when the resistive force increases.

In Figure 4, when $\phi_m = 0$ and $B = 1$, oscillatory amplitude $A = \left[\frac{-2 \ P_{18} \dot{\alpha} \ B}{-4 \ \alpha^2 B^2 - (4\dot{\alpha}^2 - 3\dot{\theta}_E^2 \cos^2 \phi_m)^2}\right]$ (corresponding to the term $Q_{10} \ \sin 2\alpha$) becomes indeterminate.

In Figure 5, we observed that the oscillatory amplitude decreases when the resistive force increases.

In Figure 6, when $\phi_m = 0$ and $B = 0$, oscillatory amplitude $A = \left[\frac{(4\dot{\alpha}^2 - 3\dot{\theta}_E^2 \cos^2 \phi_m) \ P_{18}}{-4 \ \dot{\alpha}^2 B^2 - (4\dot{\alpha}^2 - 3\dot{\theta}_E^2 \cos^2 \phi_m)^2}\right]$ (corresponding to the term $Q_{22} \ \cos 2\alpha$) becomes indeterminate. So, resonance occurs at the point where $\frac{\dot{\alpha}^2}{\dot{\theta}_E^2} = \frac{3}{4}$.

In Figure 7, we observed that the oscillatory amplitude decreases when the resistive force increases.

5.2. Case 2 : When ϕ (latitude of the Moon) = $\frac{\pi}{3}$

In Figure 8 (a, b), resonant occurs corresponding to the terms $Q_{12} \ \cos 2\gamma$ and $Q_{11} \sin \alpha$ when $B = 0$ while in Figure (8 (c, d)), resonant occurs corresponding to the terms $Q_{19} \ \cos \alpha$ and $Q_{10} \ \sin 2\alpha$ when $B = 1$.

In Figure 9, we observed that the oscillatory amplitude decreases when the resistive force increases.

6. Phase Portrait, Poincare Section and Bifurcation Theory

6.1. Phase Portrait

The continuous family of closed curves are called Phase portrait. Phase plane is the parametric curves for undamped and unforced of the differential Equation (39)

$$\Delta \dot{s}_m = -B\ \Delta s_m - 3\dot{\theta}_E^2 \cos^2 \phi_m\ \Delta r_m,$$

where $\Delta s_m = \Delta \dot{r}_m$ which trace the path of the solution.

In Figure 10, the phase portrait has been analyzed when the system is free from forces.

6.2. Phase Space $(t, \Delta r_m, \Delta s_m)$ using the method of Poincare Section

With the help of Equation (39), we describe the phase space $(t, \Delta r_m, \Delta s_m)$ by using the method of Poincare section $\sum$ with periodicity $\frac{2\pi}{\dot{\omega}_i}$ in t.

We draw the graphs (Figure 11 (a, b, c, d)) corresponding to the following differential equations,

$$\begin{aligned} \Delta \dot{r}_m = \Delta s_m, \qquad & \Delta \dot{s}_m = P_2\ \cos(\dot{\omega_1})t, && \text{Figure } 11(a), \\ & \Delta \dot{s}_m = P_{16}\ \cos(\dot{\omega_2})t, && \text{Figure } 11(b), \\ & \Delta \dot{s}_m = P_{17}\ \cos(\dot{\omega_3})t, && \text{Figure } 11(c), \\ & \Delta \dot{s}_m = P_{18}\ \cos(\dot{\omega_4})t, && \text{Figure } 11(d), \end{aligned} \tag{41}$$

for $\Delta s_m > 0$, $\Delta \dot{r}_m \neq 0$ is everywhere transverse to $\sum$, where

$$\begin{aligned} \omega_1 &= 2\gamma, \\ \omega_2 &= \theta_E + \alpha, \\ \omega_3 &= \theta_E - \alpha, \\ \omega_4 &= 2\alpha. \end{aligned}$$

6.3. Bifurcation Theory

Consider the differential equation

$$\Delta \ddot{r}_m + B\Delta \dot{r}_m + 3\dot{\theta}_E^2 \cos^2 \phi_m \Delta r_m = P_2 \cos 2\dot{\gamma}t, \tag{42}$$

with initial condition $\Delta r_m(t_0) = 0$ and $\Delta \dot{r}_m(t_0) = \Delta s_0$.

Hence, the general solution of Equation (42) is

$$\begin{aligned} \Delta r_m(t, t_0, \Delta s_0) =& e^{-\frac{B}{2}(t-t_0)} \left[A\ \cos(\Omega(t-t_0)) + B\ \sin(\Omega(t-t_0))\right] + C_1\ \cos 2\dot{\gamma}t \\ &+ C_2\ \sin 2\dot{\gamma}t, \end{aligned} \tag{43}$$

$$\Delta s_1 = \Delta \dot{r}_m(t_1, t_0, \Delta s_0), \tag{44}$$

where

$$\begin{aligned}
A =&\frac{-(3\dot\theta_E^2\cos^2\phi_m-4\dot\gamma^2)P_2\cos 2\dot\gamma t_0-2B\dot\gamma P_2\sin 2\gamma t_0}{(3\theta_m^2\cos^2\phi_m \;\dot-4\gamma)^2+4B^2\dot\gamma^2},\\
B =&\frac{1}{\Omega}\Bigg[\Delta s_0+\sin 2\dot\gamma t_0\frac{2\dot\gamma(3\dot\theta_E^2\cos^2\phi_m-4\dot\gamma^2)P_2-2B^2\dot\gamma P_2}{(3\theta_m^2\cos^2\phi_m \;\dot-4\gamma)^2+4B^2\dot\gamma^2}\\
&-\cos 2\dot\gamma t_0\frac{\frac{B}{2}(3\dot\theta_E^2\cos^2\phi_m-4\dot\gamma^2)P_2+4B\dot\gamma^2P_2}{(3\theta_m^2\cos^2\phi_m \;\dot-4\gamma)^2+4B^2\dot\gamma^2}\Bigg],\\
\Omega =&\sqrt{3\dot\theta_E^2\cos^2\phi_m-\frac{B^2}{4}},\; C_1=\frac{((3\dot\theta_E^2\cos^2\phi_m-4\dot\gamma^2)P_2}{(3\theta_m^2\cos^2\phi_m \;\dot-4\gamma)^2+4B^2\dot\gamma^2},\\
C_2 =&\frac{2\;B\;\dot\gamma P_2}{(3\theta_m^2\cos^2\phi_m \;\dot-4\gamma)^2+4B^2\dot\gamma^2}.
\end{aligned}$$

Differentiate Equation (43) with respect to t_0 and Δs_0, so we get

$$\begin{aligned}
\frac{\partial t_1}{\partial t_0}&=\frac{e^{-\frac{B}{2}(t-t_0)}}{\Omega\;\Delta s_1}\left[\Omega\Delta s_0\;\cos(\Omega(t_1-t_0))+(P_2\;\cos 2\dot\gamma t_0-\frac{B}{2}\Delta s_0)\sin(\Omega(t_1-t_0))\right],\\
\frac{\partial t_1}{\partial(\Delta s_0)}&=\frac{e^{-\frac{B}{2}(t-t_0)}}{\Omega\;\Delta s_1}\sin(\Omega(t_1-t_0)).
\end{aligned}$$

Now, differentiate Equation (44) with respect to t_0 and Δs_0, and we get

$$\begin{aligned}
\frac{\partial(\Delta s_1)}{\partial t_0}=&\frac{e^{-\frac{B}{2}(t-t_0)}}{\Omega\;\Delta s_1}\Bigg[\Bigg((P_2\cos 2\dot\gamma t_1-\frac{B}{2}\Delta s_1)(P_2\cos 2\dot\gamma t_0-\frac{B}{2}\Delta s_0)\Bigg)\sin(\Omega(t_1-t_0))\\
&+\Omega\;\cos(\Omega(t_1-t_0)(\Delta s_0(P_2\cos 2\dot\gamma t_1-\frac{B}{2}\Delta s_1)-\Delta s_1(P_2\cos 2\dot\gamma t_0-\frac{B}{2}\Delta s_0))\Bigg],
\end{aligned}$$

and

$$\frac{\partial(\Delta s_1)}{\partial\Delta s_0}=\left[\Omega\Delta s_1\;\cos(\Omega(t_1-t_0)-(P_2\cos 2\dot\gamma t_1-\frac{B}{2}\Delta s_1)\sin(\Omega(t_1-t_0))\right].$$

Now, we determine the determinant (D) and the trace (T) of a matrix

$$\begin{bmatrix}\dfrac{\partial t_1}{\partial t_0} & \dfrac{\partial t_1}{\partial(\Delta s_0)}\\[2ex] \dfrac{\partial(\Delta s_1)}{\partial t_0} & \dfrac{\partial(\Delta s_1)}{\partial(\Delta s_0)}\end{bmatrix} \tag{45}$$

$$\Longrightarrow\; D=\frac{\Delta s_0}{\Delta s_1}e^{-B(t-t_0)},$$

and

$$\begin{aligned}
T=&\frac{e^{-\frac{B}{2}(t-t_0)}}{\Omega\;\Delta s_1}\Big[\Omega\Delta s_0\cos(\Omega(t_1-t_0))+(P_2\;\cos 2\dot\gamma t_0-\frac{B}{2}\Delta s_0)\sin(\Omega(t_1-t_0))\\
&+\Omega\Delta s_1\cos(\Omega(t_1-t_0))-(P_2\;\cos 2\dot\gamma t_1-\frac{B}{2}\Delta s_0)\sin(\Omega(t_1-t_0))\Big].
\end{aligned}$$

Now, the values of D and T on a period one orbit, that is, $\Delta s_0 = \Delta s_1$ and $(t_1 - t_0) = \frac{\pi}{\dot\gamma}$. Therefore, we get

$$\hat{D} = e^{-B\frac{\pi}{\dot\gamma}},$$

and

$$\hat{T} = \frac{e^{-B\frac{\pi}{2\dot\gamma}}}{\Omega \Delta s_1}\left[2\ \Omega\Delta s_1\ \cos(\Omega\frac{\pi}{\dot\gamma}) + P_2\cos 2\dot\gamma t_0\ \sin(\Omega\frac{\pi}{\dot\gamma}) - P_2\ \cos 2\dot\gamma t_1\ \sin(\Omega\frac{\pi}{\dot\gamma})\right].$$

The eigenvalues are given by

$$\lambda_{1,2} = \frac{1}{2}\left(\hat{T} \pm \sqrt{\hat{T}^2 - 4\hat{D}}\right). \tag{46}$$

Hence, the condition for $\lambda = \pm 1$ is given by

$$\hat{D} \mp \hat{T} + 1 = 0. \tag{47}$$

Using the values of $\hat{D}$ and $\hat{T}$ in Equation (47), we get

$$e^{-B\frac{\pi}{\dot\gamma}} \mp 2\frac{e^{-B\frac{\pi}{2\dot\gamma}}}{\Omega\Delta s_1}\left[\ \Omega\Delta s_1\ \cos(\Omega\frac{\pi}{\dot\gamma}) + \frac{1}{2}\left(P_2\cos 2\dot\gamma t_0\ \sin(\Omega\frac{\pi}{\dot\gamma}) - P_2\ \cos 2\dot\gamma t_1\ \sin(\Omega\frac{\pi}{\dot\gamma})\right)\right] + 1 = 0; \tag{48}$$

This is the quadratic equation in $e^{-B\frac{\pi}{2\dot\gamma}}$. The necessary condition for real roots of Equation (47) lies between 0 and 1 is

$$0 < e^{-B\frac{\pi}{2\dot\gamma}} < 1,$$

and this gives,

$$\pm\ \left[\Omega\Delta s_1\ \cos(\Omega\frac{\pi}{\dot\gamma}) + \frac{1}{2}\left(P_2\cos 2\dot\gamma t_0\ sin(\Omega\frac{\pi}{\dot\gamma}) - P_2\ \cos 2\dot\gamma t_1\ \sin(\Omega\frac{\pi}{\dot\gamma})\right)\right] > \Omega\Delta s_1. \tag{49}$$

We observed Equation (48) depends on the system parameter B (resistive force) and $\dot\gamma$ and Equation (49) give the neccessary condition for the bifurcation to occur.

7. Conclusion

By taking the gravitational potential of Earth and Moon as a satellite of Earth, we determined the equations of motion of the Moon in spherical coordinate system. By using the perturbations technique, equations of motion are reduced to a second order differential equation. From the solution, two types of resonance are observed – (a) Luni-Solar resonance due to the frequencies angular velocity of the bary-center around the Sun and Earth's rotation rate and (b) resonance due to rate of change of Earth's equatorial ellipticity parameter and Earth's rotation rate. At the resonance points, resonant curves are drawn where oscillatory amplitude becomes infinitely large. Effect of resistive force and Earth's equatorial ellipticity parameter on the resonant curves is analyzed. It is observed that the effect of Earth's equatorial ellipticity parameter on the resonant curves is very small while effect of resistive force on the resonant curves is significant. From different graphs, we conclude that oscillatory amplitude decreases when the magnitude of resistive force increases. Phase portrait are analyzed when the system is free from forces. Using the method of Poincare secton, phase space are drawn. Finally, a neccessary condition for bifurcation to occur is derived.

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Appendix A

$$K_1=\cos\phi_m-\cos\phi_m\ \cos\epsilon,\ K_2=\cos\phi_m+\cos\phi_m\ \cos\epsilon,\ K_3=2\ \sin\phi_m,$$
$$K_4=\frac{\cos^2\phi_m}{2}-\cos^2\phi_m\ \cos\epsilon+\frac{\cos^2\phi_m\ \cos^2\epsilon}{2},$$
$$K_5=\frac{\cos^2\phi_m}{2}+\cos^2\phi_m\ \cos\epsilon+\frac{\cos^2\phi_m\ \cos^2\epsilon}{2},\ K_6=\cos^2\phi_m-\cos^2\phi_m\ \cos^2\epsilon,$$
$$K_7=\cos^2\phi_m-\cos^2\phi_m\ \cos^2\epsilon-2\ \sin^2\epsilon\sin^2\phi_m,$$
$$K_8=-2\ \sin\epsilon\ \sin\phi_m\ \cos\phi_m(1-\cos\epsilon),\ K_9=2\ \sin\epsilon\ \sin\phi_m\ \cos\phi_m(1+\cos\epsilon),$$
$$K_{10}=-4\ \sin\epsilon\ \sin\phi_m\ \cos\phi_m\ \cos\epsilon,\ K_{11}=\cos^2\phi_m+\cos^2\phi_m\ \cos^2\epsilon+\ \sin^2\epsilon\ \sin^2\phi_m.$$

Appendix B

$$M_1=\frac{-\ \cos\phi_m}{8}+\frac{\cos\phi_m\ \cos\epsilon}{4},\ M_2=\frac{-\ \cos\phi_m}{4},\ M_3=\frac{-\ \cos\phi_m}{8}-\frac{\cos\phi_m\ \cos\epsilon}{4},$$
$$M_4=\frac{\sin\epsilon\ \cos\epsilon}{4}(1+\sin\phi_m),\ M_5=\frac{\sin\epsilon\ \cos\epsilon}{4}(\sin\phi_m-1),\ M_6=\frac{-3\ \cos\phi_m\ \cos^2\epsilon}{8},$$
$$M_7=\frac{\cos\phi_m\ \cos^2\epsilon}{8},\ M_8=\frac{\cos\phi_m\ \cos^2\epsilon}{8},\ M_9=\frac{-\ \sin\epsilon\ \cos\epsilon\sin\phi_m}{2}.$$

Appendix C

$$P_1=\frac{-12J_2^{(2)}\dot\theta_E g_0R_0^4\cos^2\phi_m}{r_{em}^4}\frac{B}{B^2+4\dot\gamma^2},\ P_2=\frac{12J_2^{(2)}g_0R_0^4\cos^2\phi_m}{r_{em}^4}\left(\frac{2\dot\theta_E\ \dot\gamma}{B^2+4\dot\gamma^2}-\frac{3}{4}\right),$$
$$P_3=\frac{6\ r_{em}\ \dot\theta_E\ \dot\alpha^2\ \cos\phi_m B\ M_1}{(B^2+4(\dot\theta_E+\dot\alpha)^2)}+\frac{12\ r_{em}\ \dot\theta_E\ \dot\alpha^2\ \cos\phi_m\ (\dot\theta_E+\dot\alpha)M_7}{(B^2+4(\dot\theta_E+\dot\alpha)^2)},$$
$$P_4=\frac{6\ r_{em}\ \dot\theta_E\ \dot\alpha^2\ \cos\phi_m B\ M_2}{(B^2+4\dot\theta_E^2)}+\frac{12\ r_{em}\ \dot\theta_E^2\ \dot\alpha^2\ \cos\phi_m\ M_6}{(B^2+4\dot\theta_E^2)},$$
$$P_5=\frac{6\ r_{em}\ \dot\theta_E\ \dot\alpha^2\ \cos\phi_m\ (\dot\theta_E-2\ \dot\alpha)M_4}{(B^2+(\dot\theta_E-2\ \dot\alpha)^2)}+\frac{3\ r_{em}\ \dot\alpha^2\ K_9}{4},$$
$$P_6=\frac{6\ r_{em}\ \dot\theta_E\ \dot\alpha^2\ \cos\phi_m\ (\dot\theta_E+2\ \dot\alpha)M_5}{B^2+(\dot\theta_E+2\ \dot\alpha)^2)}+\frac{3\ r_{em}\ \dot\alpha^2\ K_8}{4},$$
$$P_7=\frac{6\ r_{em}\ \dot\theta_E\ \dot\alpha^2\ \cos\phi_m\ BM_3}{(B^2+4(\dot\theta_E-\dot\alpha)^2)}+\frac{12\ r_{em}\ \dot\theta_E\ \dot\alpha^2\ \cos\phi_m\ (\dot\theta_E-\dot\alpha)M_8}{(B^2+4(\dot\theta_E-\dot\alpha)^2)},$$
$$P_8=\frac{6\ r_{em}\ \dot\theta_E^2\ \dot\alpha^2\ \cos\phi_m\ M_9}{(B^2+\dot\theta_E^2)}+\frac{3\ r_{em}\ \dot\alpha^2\ K_{10}}{4},\ P_9=\frac{-3\ r_{em}^2\ \dot\alpha^2\ K_1}{2\ R}\left(\frac{\mu-2}{\mu}\right),$$

$$P_{10} = \frac{-12\ r_{em}\ \dot\theta_E\ \dot\alpha^2\ \cos\phi_m B\ (\dot\theta_E+\dot\alpha)M_1}{(B^2+4(\dot\theta_E+\dot\alpha)^2)} + \frac{6\ r_{em}\ \dot\theta_E\ \dot\alpha^2\ \cos\phi_m B\ M_7}{(B^2+4(\dot\theta_E+\dot\alpha)^2)} + \frac{3\ r_{em}\ \dot\alpha^2 K_4}{4},$$

$$P_{11} = \frac{-12\ r_{em}\ \dot\theta_E^2\ \dot\alpha^2\ \cos\phi_m\ M_2}{(B^2+4\dot\theta_E^2)} + \frac{3\ r_{em}\ \dot\alpha^2\ K_6}{4} + \frac{6\ r_{em}\ \dot\theta_E\ \dot\alpha^2\ \cos\phi_m B\ M_6}{(B^2+4\dot\theta_E^2)},$$

$$P_{12} = \frac{-12\ r_{em}\ \dot\theta_E\ \dot\alpha^2\ \cos\phi_m\ (\dot\theta_E-\dot\alpha)M_3}{(B^2+4(\dot\theta_E-\dot\alpha)^2)} + \frac{6\ r_{em}\ \dot\theta_E\ \dot\alpha^2\ \cos\phi_m\ BM_8}{(B^2+4(\dot\theta_E-\dot\alpha)^2)} + \frac{3\ r_{em}\ \dot\alpha^2 K_5}{4},$$

$$P_{13} = \frac{6\ r_{em}\ \dot\theta_E\ \dot\alpha^2\ \cos\phi_m\ B\ M_4}{(B^2+(\dot\theta_E-2\ \dot\alpha)^2)},\ P_{14} = \frac{6\ r_{em}\ \dot\theta_E\ \dot\alpha^2\ \cos\phi_m\ B\ M_5}{B^2+(\dot\theta_E+2\ \dot\alpha)^2)},$$

$$P_{15} = \frac{6\ r_{em}\ \dot\theta_E\ \dot\alpha^2\ \cos\phi_m\ B\ M_9}{(B^2+\dot\theta_E^2)},\ P_{16} = \frac{3\ r_{em}^2\ \dot\alpha^2\ K_1}{2\ R}\left(\frac{\mu-2}{\mu}\right),$$

$$P_{17} = \frac{3\ r_{em}^2\ \dot\alpha^2\ K_2}{2\ R}\left(\frac{\mu-2}{\mu}\right),\ P_{18} = \frac{3\ r_{em}\ \dot\alpha^2 K_7}{4}.$$

Appendix D

$$Q_1 = \left[\frac{(4\dot\gamma^2-3\dot\theta_E^2\cos^2\phi_m)\ P_1 - 2\ P_2\dot\gamma\ B}{-4\ \dot\gamma^2B^2-(4\dot\gamma^2-3\dot\theta_E^2\cos^2\phi_m)^2}\right],$$

$$Q_2 = \left[\frac{(4(\dot\theta_E+\dot\alpha)^2-3\dot\theta_E^2\cos^2\phi_m)\ P_3 - 2\ P_{10}(\dot\theta_E+\dot\alpha)\ B}{-4\ (\dot\theta_E+\dot\alpha)^2B^2-(4(\dot\theta_E+\dot\alpha)^2-3\dot\theta_E^2\cos^2\phi_m)^2}\right],$$

$$Q_3 = \left[\frac{(4\dot\theta_E^2-3\dot\theta_E^2\cos^2\phi_m)\ P_4 - 2\ P_{11}\dot\theta_E\ B}{-4\ \dot\theta_E^2B^2-(4\dot\theta_E^2-3\dot\theta_E^2\cos^2\phi_m)^2}\right],$$

$$Q_4 = \left[\frac{((\dot\theta_E-2\dot\alpha)^2-3\dot\theta_E^2\cos^2\phi_m)\ P_5 -\ P_{13}(\dot\theta_E-2\dot\alpha)\ B}{-(\dot\theta_E-2\dot\alpha)^2B^2-((\dot\theta_E-2\dot\alpha)^2-3\dot\theta_E^2\cos^2\phi_m)^2}\right],$$

$$Q_5 = \left[\frac{((\dot\theta_E+2\dot\alpha)^2-3\dot\theta_E^2\cos^2\phi_m)\ P_6 -\ P_{14}(\dot\theta_E+2\dot\alpha)\ B}{-(\dot\theta_E+2\dot\alpha)^2B^2-((\dot\theta_E+2\dot\alpha)^2-3\dot\theta_E^2\cos^2\phi_m)^2}\right],$$

$$Q_6 = \left[\frac{(4(\dot\theta_E-\dot\alpha)^2-3\dot\theta_E^2\cos^2\phi_m)\ P_7 - 2\ P_{12}(\dot\theta_E-\dot\alpha)\ B}{-4\ (\dot\theta_E-\dot\alpha)^2B^2-(4(\dot\theta_E+\dot\alpha)^2-3\dot\theta_E^2\cos^2\phi_m)^2}\right],$$

$$Q_7 = \left[\frac{(\dot\theta_E^2-3\dot\theta_E^2\cos^2\phi_m)\ P_8 -\ P_{15}\dot\theta_E\ B}{-\ \dot\theta_E^2B^2-(\dot\theta_E^2-3\dot\theta_E^2\cos^2\phi_m)^2}\right],$$

$$Q_8 = \left[\frac{-\ P_{16}(\dot\theta_E+\dot\alpha)\ B}{-\ (\dot\theta_E+\dot\alpha)^2B^2-((\dot\theta_E+\dot\alpha)^2-3\dot\theta_E^2\cos^2\phi_m)^2}\right],$$

$$Q_9 = \left[\frac{-\ P_{17}(\dot\theta_E-\dot\alpha)\ B}{-\ (\dot\theta_E-\dot\alpha)^2B^2-((\dot\theta_E-\dot\alpha)^2-3\dot\theta_E^2\cos^2\phi_m)^2}\right],$$

$$Q_{10} = \left[\frac{-2\ P_{18}\dot{\alpha}\ B}{-4\ \alpha^2 B^2 - (4\dot{\alpha}^2 - 3\dot{\theta}_E^2 \cos^2\phi_m)^2}\right],$$

$$Q_{11} = \left[\frac{(\dot{\alpha}^2 - 3\dot{\theta}_E^2 \cos^2\phi_m)\ P_9}{-\dot{\alpha}^2 B^2 - (\dot{\alpha}^2 - 3\dot{\theta}_E^2 \cos^2\phi_m)^2}\right],$$

$$Q_{12} = \left[\frac{(4\dot{\gamma}^2 - 3\dot{\theta}_E^2 cos^2\phi_m)\ P_2 + 2\ P_1\dot{\gamma}\ B}{-4\ \dot{\gamma}^2 B^2 - (4\dot{\gamma}^2 - 3\dot{\theta}_E^2 \cos^2\phi_m)^2}\right],$$

$$Q_{13} = \left[\frac{(4(\dot{\theta}_E + \dot{\alpha})^2 - 3\dot{\theta}_E^2 \cos^2\phi_m)\ P_{10} + 2\ P_3(\dot{\theta}_E + \dot{\alpha})\ B}{-4\ (\dot{\theta}_E + \dot{\alpha})^2 B^2 - (4(\dot{\theta}_E + \dot{\alpha})^2 - 3\dot{\theta}_E^2 \cos^2\phi_m)^2}\right],$$

$$Q_{14} = \left[\frac{(4\dot{\theta}_E^2 - 3\dot{\theta}_E^2 \cos^2\phi_m)\ P_{11} + 2\ P_4\dot{\theta}_E\ B}{-4\ \dot{\theta}_E^2 B^2 - (4\dot{\theta}_E^2 - 3\dot{\theta}_E^2 \cos^2\phi_m)^2}\right],$$

$$Q_{15} = \left[\frac{((\dot{\theta}_E - 2\dot{\alpha})^2 - 3\dot{\theta}_E^2 \cos^2\phi_m)\ P_{13} +\ P_5(\dot{\theta}_E - 2\dot{\alpha})\ B}{-(\dot{\theta}_E - 2\dot{\alpha})^2 B^2 - ((\dot{\theta}_E - 2\dot{\alpha})^2 - 3\dot{\theta}_E^2 \cos^2\phi_m)^2}\right],$$

$$Q_{16} = \left[\frac{((\dot{\theta}_E + 2\dot{\alpha})^2 - 3\dot{\theta}_E^2 \cos^2\phi_m)\ P_{14} +\ P_6(\dot{\theta}_E + 2\dot{\alpha})\ B}{-(\dot{\theta}_E + 2\dot{\alpha})^2 B^2 - ((\dot{\theta}_E + 2\dot{\alpha})^2 - 3\dot{\theta}_E^2 \cos^2\phi_m)^2}\right],$$

$$Q_{17} = \left[\frac{(4(\dot{\theta}_E - \dot{\alpha})^2 - 3\dot{\theta}_E^2 \cos^2\phi_m)\ P_{12} + 2\ P_7(\dot{\theta}_E - \dot{\alpha})\ B}{-4\ (\dot{\theta}_E + \dot{\alpha})^2 B^2 - (4(\dot{\theta}_E - \dot{\alpha})^2 - 3\dot{\theta}_E^2 \cos^2\phi_m)^2}\right],$$

$$Q_{18} = \left[\frac{(\dot{\theta}_E^2 - 3\dot{\theta}_E^2 \cos^2\phi_m)\ P_{15} +\ P_8\dot{\theta}_E\ B}{-\ \dot{\theta}_E^2 B^2 - (\dot{\theta}_E^2 - 3\dot{\theta}_E^2 \cos^2\phi_m)^2}\right],$$

$$Q_{19} = \left[\frac{P_9\dot{\alpha}\ B}{-\alpha^2 B^2 - (\dot{\alpha}^2 - 3\dot{\theta}_E^2 \cos^2\phi_m)^2}\right],$$

$$Q_{20} = \left[\frac{((\dot{\theta}_E + \dot{\alpha})^2 - 3\dot{\theta}_E^2 \cos^2\phi_m)\ P_{16}}{-\ (\dot{\theta}_E + \dot{\alpha})^2 B^2 - ((\dot{\theta}_E + \dot{\alpha})^2 - 3\dot{\theta}_E^2 \cos^2\phi_m)^2}\right],$$

$$Q_{21} = \left[\frac{((\dot{\theta}_E - \dot{\alpha})^2 - 3\dot{\theta}_E^2 \cos^2\phi_m)\ P_{17}}{-\ (\dot{\theta}_E - \dot{\alpha})^2 B^2 - ((\dot{\theta}_E - \dot{\alpha})^2 - 3\dot{\theta}_E^2 \cos^2\phi_m)^2}\right],$$

$$Q_{22} = \left[\frac{(4\dot{\alpha}^2 - 3\dot{\theta}_E^2 \cos^2\phi_m)\ P_{18}}{-4\ \dot{\alpha}^2 B^2 - (4\dot{\alpha}^2 - 3\dot{\theta}_E^2 \cos^2\phi_m)^2}\right].$$

Appendix E

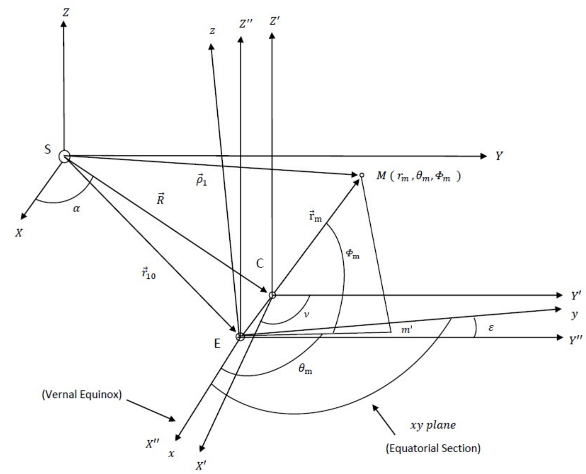


Figure 1. Configuration of the problem

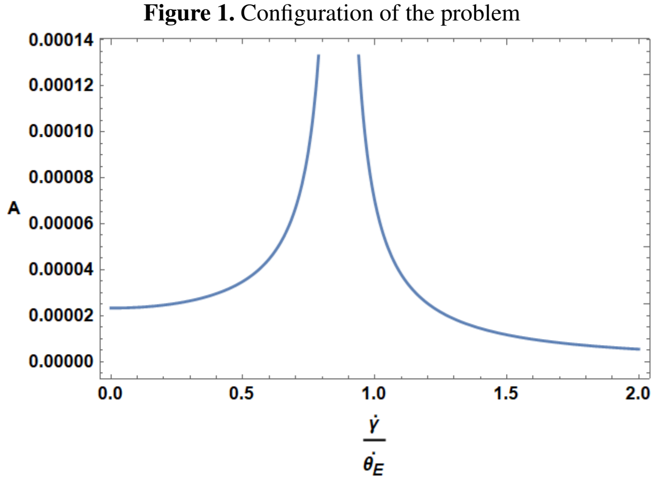


Figure 2. Resonant curve when B = 0 and $\phi_m = 0$

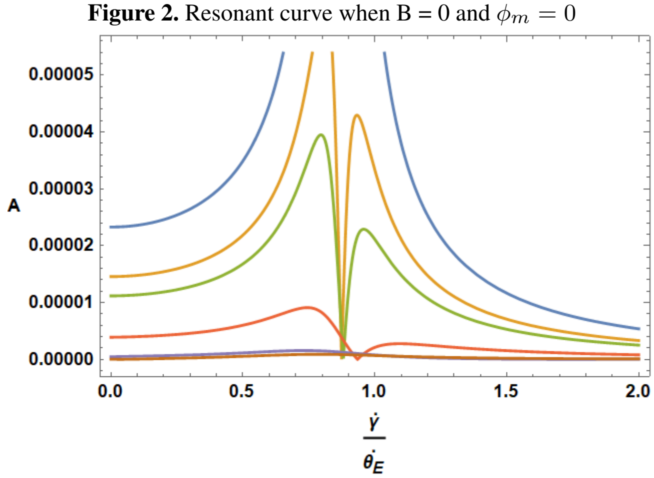


Figure 3. Resonant curves for different values of B

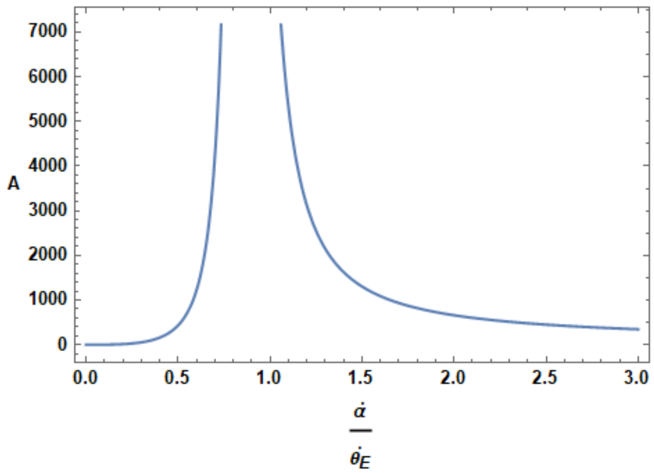


Figure 4. Resonant curve when B = 0 and $\phi_m = 0$

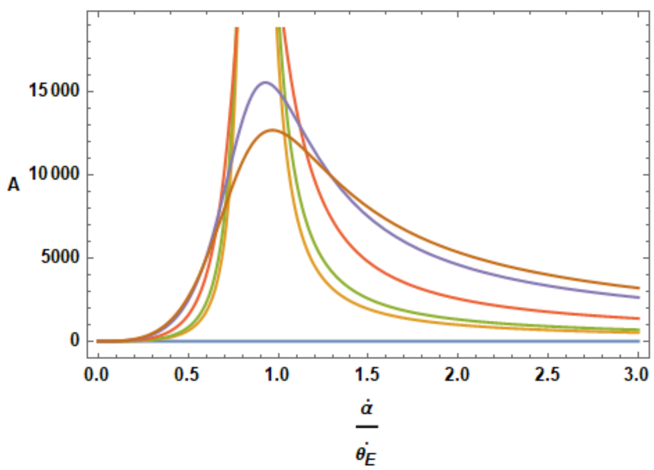


Figure 5. Resonant curves for different values of B

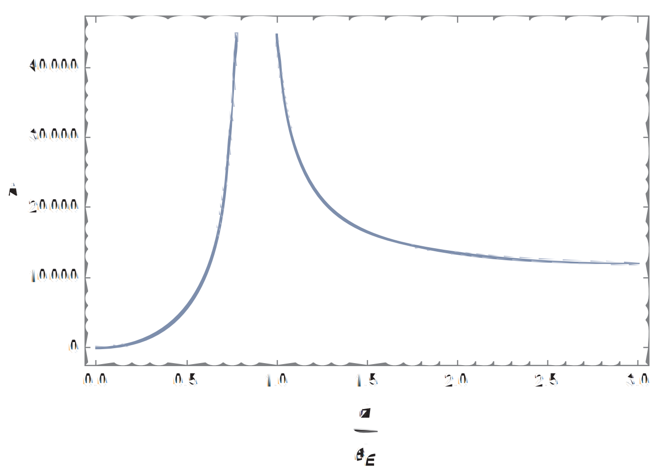


Figure 6. Resonant curve when B = 0 and $\phi_m = 0$

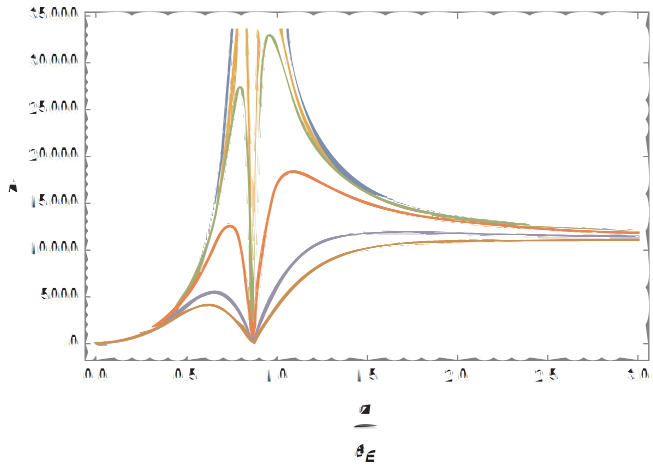


Figure 7. Resonant curves for different values of B

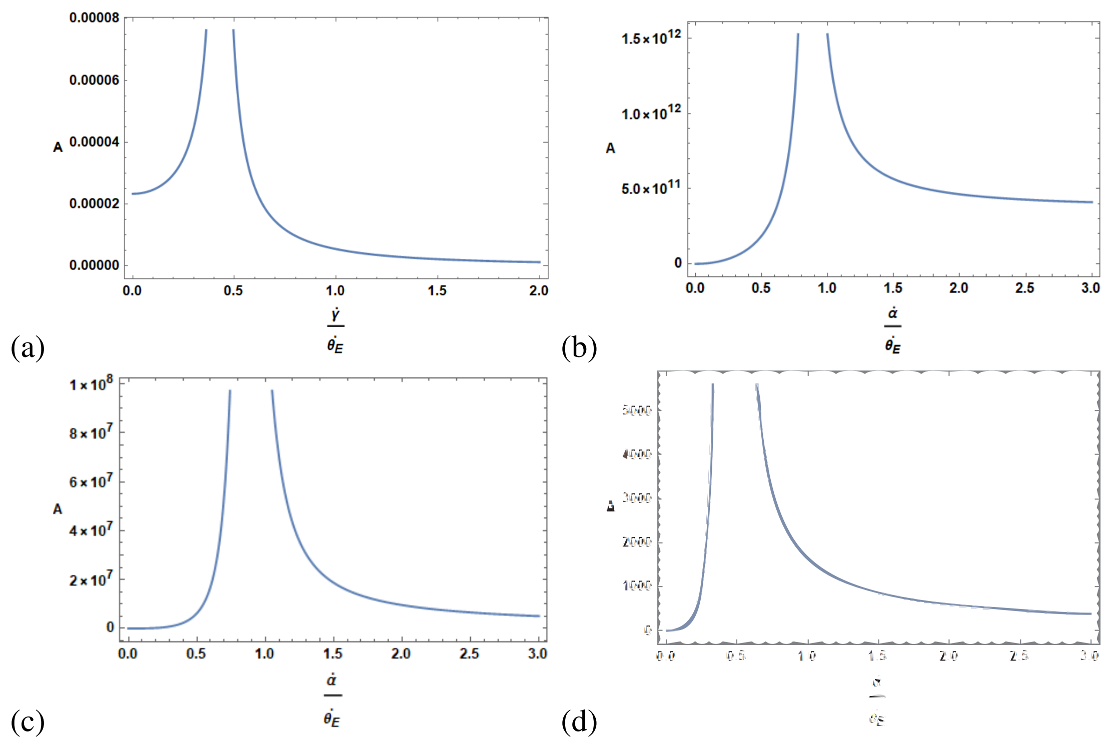


Figure 8. Resonant curve when $\phi_m = \frac{\pi}{3}$

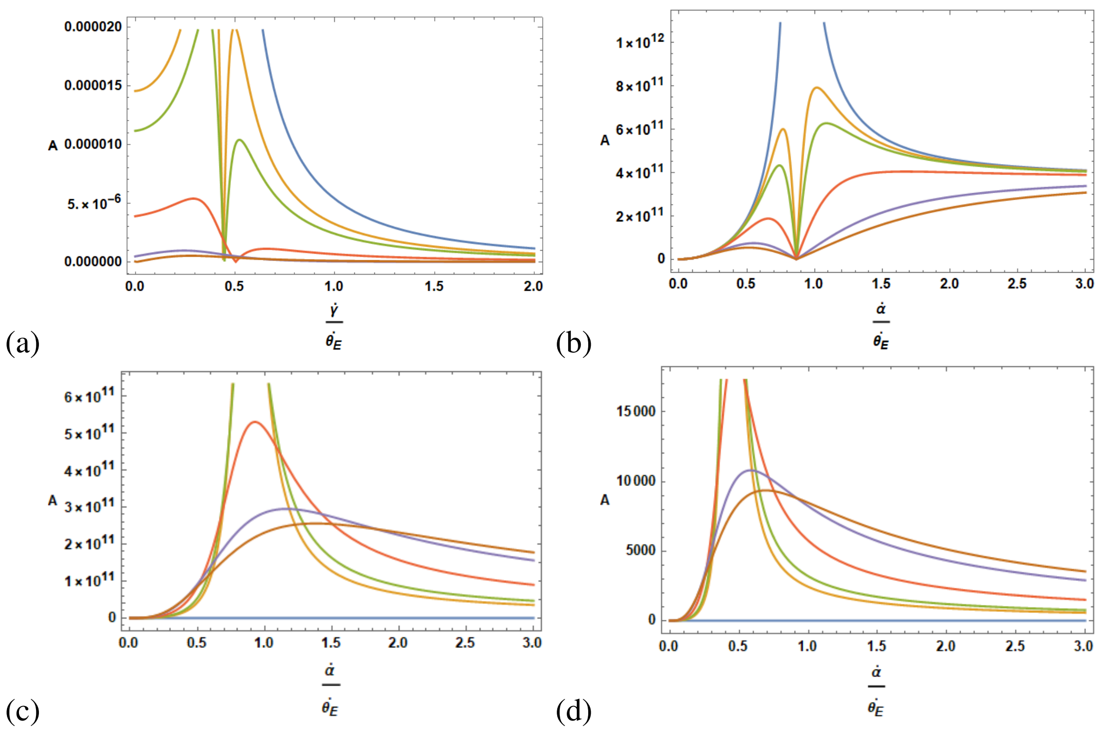
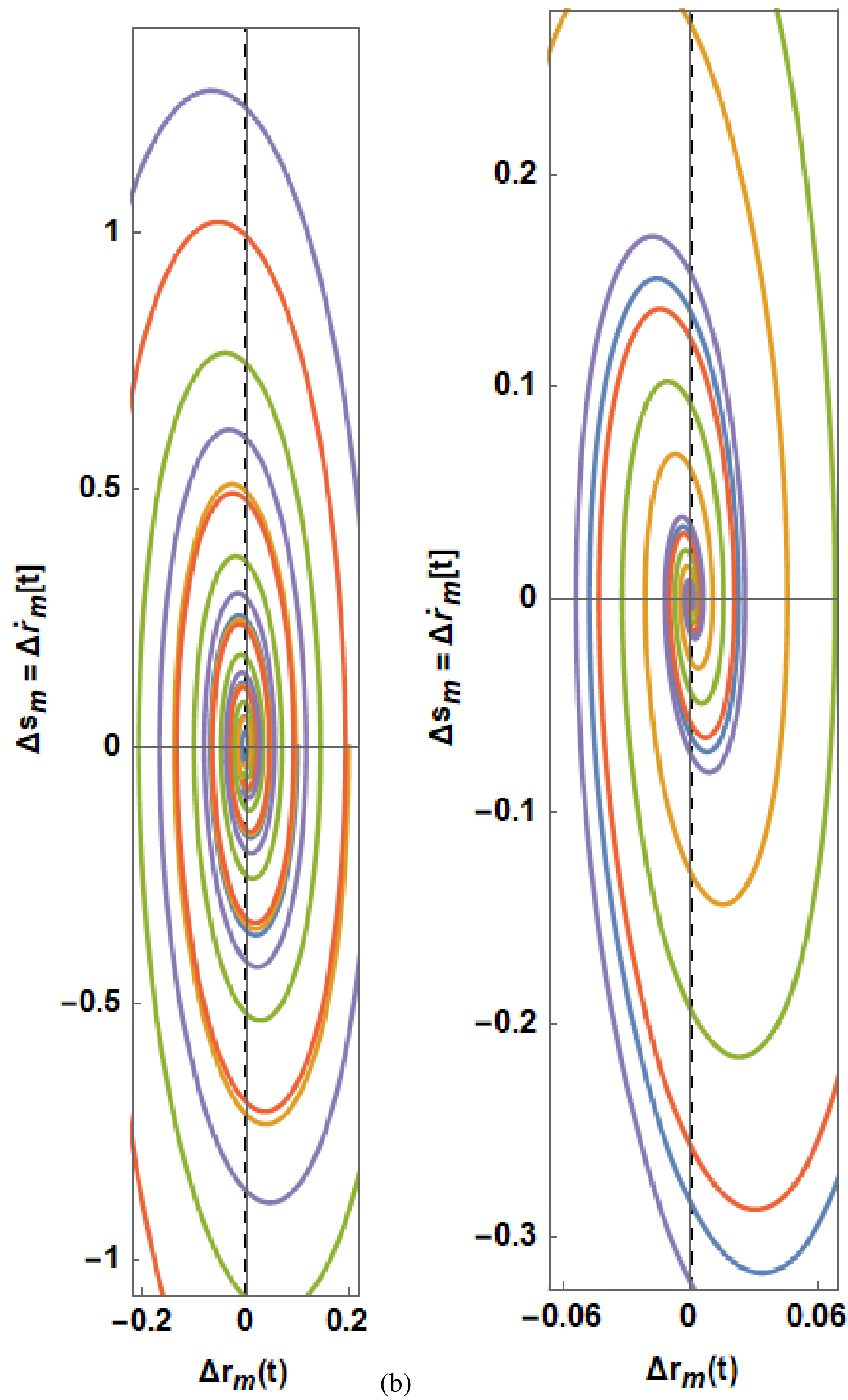
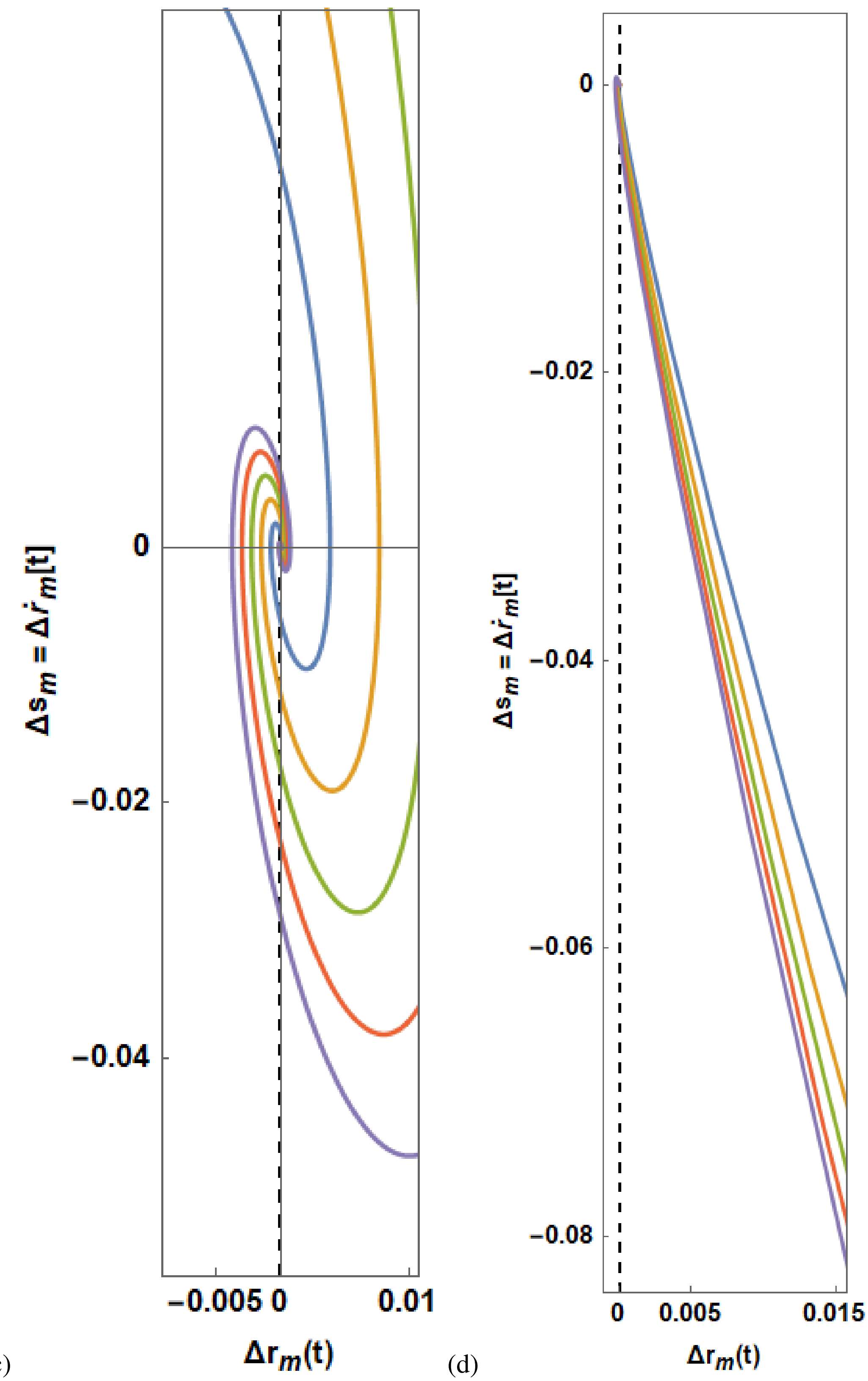


Figure 9. Resonant curves for different values of B



(a) (b) Figure Contd.



(c) (d)

Figure 10. Phase Portrait

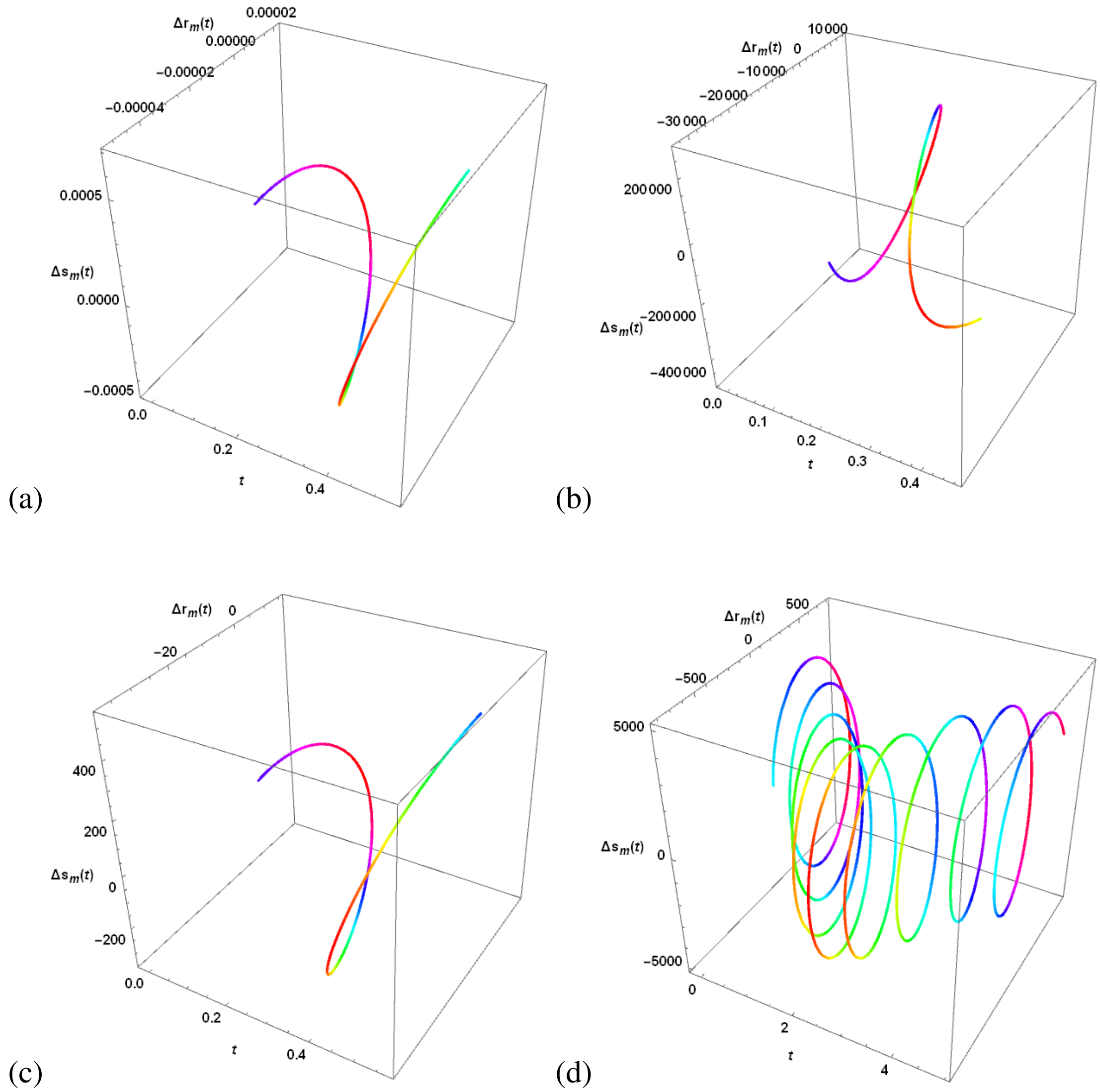


Figure 11. Phase Space