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Mehtap Lafci Büyükkahraman
Uşak University, mehtap.lafci@usak.edu.tr

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On the Mackey-Glass Model with Piecewise Constant Argument

Mehtap Lafci Büyükkahraman

Department of Mathematics
Uşak University
64200, Uşak
TURKEY
mehtap.lafci@usak.edu.tr

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Abstract

In this paper, we deal with the Mackey-Glass model with piecewise constant argument. Because the corresponding difference equation is the difference solution of the equation, the difference equation can clearly predict the dynamic behavior of the equation. So, we look at how the difference equation behaves. We study the asymptotic stability of the equilibrium point of the difference equation and it is obtained that this point is a repeller under some conditions. Also, it is shown that every oscillatory solution of the difference equation has semi-cycles of length at least two, and every oscillatory solution of the difference equation is attracted to the equilibrium point. Finally, some examples are presented to support the results.

Keywords: Mackey-Glass model; Piecewise constant argument; Asymptotic stability; Oscillation

MSC 2010 No.: 34K11, 39A21

1. Introduction

Mathematical models have a major advantage as they can be quickly applied and validated. The well-known Mackey–Glass model, which is one of such models, was introduced by Mackey-Glass (1977) to show the population concentration of cells in the blood. After that, the delay differential equation model was looked into by Liz et al. (2002), Berezansky-Braverman (2006), Su et al. (2011), Wu et al. (2011), Junges-Gallas (2012), Berezansky et al. (2012; 2013), and Amil et al. (2015). Also, Berezansky-Braverman (2017) obtained sufficient conditions for local and global stability of the positive equilibrium of Mackey–Glass equation with two delays. Moreover, Kubyshkin-Moryakova (2019) considered bifurcations of periodic solutions of Mackey–Glass equation. Recently, Prepbrazhenskaya (2020) has studied the dynamics of the generalized Mackey–Glass equation with two delays using the large parameter method. And, Tan (2020) has showed attractivity of all non-oscillating solutions about the positive equilibrium point and the global asymptotical stability of the trivial equilibrium point of non-autonomous Mackey-Glass model with two variable delays. Amdouni et al. (2021) have investigated the boundedness, existence, uniqueness and global stability of the weighted piecewise pseudo almost automorphic solutions of the new class of Mackey–Glass model.

The first mathematical model with a piecewise constant argument was constructed by Busenberg and Cooke (1982). After such model, these types of equations have been intensively studied by many authors (Aftabizadeh-Wiener (1985), Aftabizadeh et al. (1987), Akhmet (2011), Bereketoglu et al. (2011), Chiu-Li (2019), Cooke-Wiener (1984; 1991), Küpper-Yuan (2002), Pinto (2009), Rong-Jialin (1997), Shah-Wiener (1983) and references cited therein).

Some populations require the qualities of both differential and difference equations in both continuous and discrete time situations, which calls into question the usage of piecewise constant arguments. Some works on modeling population dynamics in terms of the time step are shown by Gopalsamy et al. (1991), Gopalsamy-Liu (1998), Gurcan et al. (2014), Karakoç (2017) and Öztepe (2021). Such equations possess the structure of continuous systems in intervals. Continuity of the solution of an equation at a point linking any two consecutive intervals implies a recurrence relation for the values of the solution at such points. Mathematically, differential equations with piecewise constant arguments are described as hybrid dynamical systems, and hence, this type of equation shows the properties of both differential and difference equations. So, we consider a Mackey-Glass model with piecewise constant argument. As we know, there is no work on the model which are important as a model. This is the main reason why we choose to study the model.

The organization of the paper is as follows. The Mackey-Glass model with piecewise constant argument is discussed in Section 2 along with the definition of the model's solution. Section 3 presents the main results. Section 4 illustrates our results by numerically simulating examples of the model.

2. Preliminaries

In this paper, we consider the differential equation with piecewise constant argument,

$$\frac{dc}{dt} = \frac{\lambda a^m c([t-1])}{a^m + c^m([t-1])} - gc(t), \tag{1}$$

where λ, a, m, g are positive constants, $[t-1]$ is the delay and $[\cdot]$ denotes the greatest integer function. The equation (1) is obtained from the following delay model for the blood cell population suggested by Mackey-Glass (1977),

$$\frac{dc}{dt} = \frac{\lambda a^m c(t-\tau)}{a^m + c^m(t-\tau)} - gc(t),$$

where $c(t)$ is the concentration of cells (the population species) in the circulating blood with units $cells/mm^3$. The cells are lost at gc rate proportional to their concentration, and g has dimension (day^{-1}). After the reduction in cells in the blood stream, there is a delay before the marrow releases further cells to replenish the deficiency. The flux of cells into the blood stream depends on the cell concentration at an earlier time $c(t-\tau)$, where τ is the delay which is a positive constant (see Arino et al. (2007)).

Let us give the following definition of the solution of Equation (1).

Definition 2.1.

A function c defined on the set $\{-1\} \cup [0, \infty)$ is a solution of Eqation (1) if

i) $c(t)$ is continuous on $(0, \infty)$,
ii) $c(t)$ is differentiable and satisfies (1) for any $t \in (0, \infty)$, with the possible exception of the points $[t]$ in $(0, \infty)$, where one-sided derivatives exist.

Because of the biological meaning of the model, we'll only look at positive solutions. So, we investigate Equation (1) with the initial conditions

$$c(-1) = c_{-1} > 0,\ c(0) = c_0 > 0. \tag{2}$$

As a result, it can be simply demonstrated all solutions of Equation (1) with the initial conditions (2) are positive using the method of steps. Difference equations are used to explore differential equations with piecewise constant arguments. So, we apply the results of such equations.

3. Main Results

This section focuses on solving Equation (1) in terms of the corresponding difference equation and analyzing the behavior of these solutions.

Theorem 3.1.

Equation (1) has a unique solution $c(t)$ on $\{-1\} \cup [0, \infty)$ with the initial conditions

$$c(-1) = c_{-1},\ c(0) = c_0. \tag{3}$$

Also, for $n \leq t < n+1,\ n \in \mathbb{N},\ c$ has the form

$$c(t) = c(n)\mathrm{e}^{-g(t-n)} + \frac{\lambda a^m c(n-1)}{a^m + c^m(n-1)} \left(\frac{1 - \mathrm{e}^{-g(t-n)}}{g} \right), \tag{4}$$

where the sequence $c(n)$ is the unique solution of the difference equation

$$c(n+1) = \mathrm{e}^{-g} c(n) + \left(\frac{1 - \mathrm{e}^{-g}}{g} \right) \frac{\lambda a^m c(n-1)}{a^m + c^m(n-1)}, \tag{5}$$

with the initial conditions (3) .

Proof:

Let $c_n(t) \equiv c(t)$ be a solution of (1) for $n \leq t < n+1$. So, Equation (1) becomes a linear differential equation

$$\frac{dc}{dt} + gc(t) = \frac{\lambda a^m c(n-1)}{a^m + c^m(n-1)}. \tag{6}$$

The solution of Equation (6) is obtained as

$$c_n(t) = c(n)\mathrm{e}^{-g(t-n)} + \frac{\lambda a^m c(n-1)}{a^m + c^m(n-1)} \left(\frac{1 - \mathrm{e}^{-g(t-n)}}{g} \right). \tag{7}$$

On the other hand, from (7), the solution $c_{n-1}(t)$ on $n-1 \leq t < n$ can be stated as

$$c_{n-1}(t) = c(n-1)\mathrm{e}^{-g(t-n+1)} + \frac{\lambda a^m c(n-2)}{a^m + c^m(n-2)} \left(\frac{1 - \mathrm{e}^{-g(t-n+1)}}{g} \right).$$

The difference equation (5) is then obtained by applying the continuity of the solutions at $t = n$. In virtue of the initial conditions (3), it is possible to acquire a unique solution to the difference equation (5). As a result, the unique solution of (1) with the initial conditions (3) becomes (4). ■

Remark 3.1.

The difference equation (5) can clearly determine the dynamic behavior of Equation (1) because the difference equation (5) is the difference solution of Equation (1).

Now, let's take a look at how Equation (5) behaves. As a result, we must first establish equilibrium of (5).

Equation (5) always has the trivial equilibrium, denoted by $E_0 = 0$, and a unique biologically meaningful equilibrium, denoted by

$$E^* = a\left(\frac{\lambda}{g} - 1\right)^{1/m}, \tag{8}$$

which are also the critical points of Equation (1).

Because of the biological meaningful equilibrium that is positive equilibrium E^* throughout this paper, we assume that $\frac{\lambda}{g} > 1$.

Theorem 3.2.

The equilibrium E^* given by (8) is locally asymptotically stable if and only if

$$-\frac{\mathrm{e}^g}{\mathrm{e}^g - 1} < 1 - m + \frac{gm}{\lambda}. \tag{9}$$

Proof:

The corresponding linearized equation of Equation (5) about E^* is

$$x_{n+1} = \mathrm{e}^{-g} x_n + (1 - \mathrm{e}^{-g}) \left(1 - m + \frac{gm}{\lambda}\right) x_{n-1}, \quad n = 0, 1, 2, \dots. \tag{10}$$

From Linearized Stability Theorem in Gibbons et al. (2002), it is obtained that E^* is locally asymptotically stable if and only if the condition (9) is true. ∎

Theorem 3.3.

The equilibrium E^* given by (8) is a repeller if and only if the condition

$$1 - m + \frac{gm}{\lambda} < -\frac{\mathrm{e}^g}{\mathrm{e}^g - 1} \quad \text{or} \quad 1 - m + \frac{gm}{\lambda} > \frac{\mathrm{e}^g + 1}{\mathrm{e}^g - 1}, \tag{11}$$

holds.

Proof:

Using the linearized equation (10), from Linearized Stability Theorem in Gibbons et al. (2002), it is known that the positive equilibrium point is a repeller if and only if the following cases hold:

(i) $\left|(1 - \mathrm{e}^{-g}) \left(1 - m + \frac{gm}{\lambda}\right)\right| > 1,$

(ii) $\mathrm{e}^{-g} < \left|1 - (1 - \mathrm{e}^{-g}) \left(1 - m + \frac{gm}{\lambda}\right)\right|.$

In view of (i) and (ii), the positive equilibrium point of Equation (5) is a repeller if and only if (11) holds. ∎

Theorem 3.4.

If

$$a^m + (1 - m) c_{n-1}^m < 0, \quad n \in \mathbb{N}, \tag{12}$$

then every oscillatory solution of Equation (5) has semi-cycles of length at least two.

Proof:

We rewrite the right hand side of Equation (5) as

$$f(x, y) = \mathrm{e}^{-g} x + \left(\frac{1 - \mathrm{e}^{-g}}{g}\right) \frac{\lambda a^m y}{a^m + y^m}. \tag{13}$$

The first derivatives of (13) with respect to x and y are

$$\frac{\partial f}{\partial x} = \mathrm{e}^{-g} \quad \text{and} \quad \frac{\partial f}{\partial y} = \lambda a^m \left(\frac{1 - \mathrm{e}^{-g}}{g}\right) \left(\frac{a^m + (1 - m) y^m}{(a^m + y^m)^2}\right),$$

which are positive and negative, respectively, because of $\mathrm{e}^{-g} > 0$ and the condition (12). So, we get that f is increasing in x and decreasing in y and by Theorem 1.7.4 in Kulenovic-Ladas (2001), every oscillatory solution of Equation (5) has semi-cycles of length at least two. ∎

Theorem 3.5.

If

$$a^m + (1-m)c_{n-1}^m < 0, \quad n \in \mathbb{N},$$

then every solution of Equation (5) is attracted to the equilibrium E^*.

Proof:

Applying Theorem 2.4.1 in Kocic-Ladas (1993) to Equation (5) under the initial conditions (3) for $k = 1$, with

$$\alpha = \mathrm{e}^{-g} \in [0,1) \quad \text{and} \quad F(u) = \lambda a^m \left(\frac{1-\mathrm{e}^{-g}}{g}\right) \frac{u}{a^m + u^m},$$

and calculating the derivative according to u of F, we get

$$\frac{\partial F}{\partial u} = \lambda a^m \left(\frac{1-\mathrm{e}^{-g}}{g}\right) \frac{a^m + (1-m)u^m}{(a^m + u^m)^2},$$

which is negative. Thus, the function F is decreasing. Moreover, in this theorem the $L,\ U$ system can be written as

$$U = \left(\frac{\lambda a^m}{g}\right) \frac{L}{(a^m + L^m)^2},$$
$$L = \left(\frac{\lambda a^m}{g}\right) \frac{U}{(a^m + U^m)^2}.$$

Hence, this system has exactly one solution $\{L, U\}$. As a result, every solution of Equation (5) is attracted to E^*. ∎

4. Numerical Examples

In this section, we will give some examples to illustrate our results.

Example 4.1.

As a special case of Equation (1), we consider the following equation:

$$\frac{dc}{dt} = \frac{3c([t-1])}{1 + c([t-1])} - 2c(t). \tag{14}$$

The difference equation is

$$c(n+1) = \mathrm{e}^{-2}c(n) + \frac{3}{2}(1-\mathrm{e}^{-2})\frac{c(n-1)}{1 + c(n-1)}, \tag{15}$$

with the initial conditions

$$c(-1) = 0.1, \quad c(0) = 0.1.$$

The corresponding linearized equation of Equation (15) about E^* is

$$x_{n+1} = \mathrm{e}^{-2}x_n + \frac{2}{3}(1-\mathrm{e}^{-2})x_{n-1}, n = 0, 1, 2, \ldots.$$

Since all hypotheses of Theorem 3.2 are satisfied for $\lambda = 3,\ a = 1,\ m = 1,\ g = 2$, the equilibrium $E^* = 0.5$ is locally asymptotically stable in Figure 1.

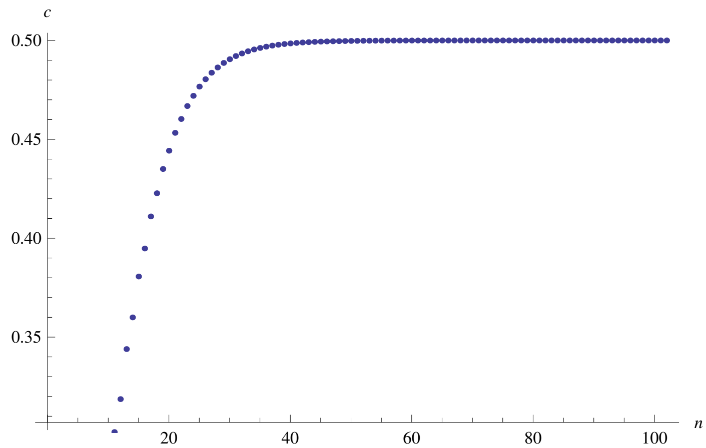

Figure 1. The equilibrium $E^* = 0.5$ of the difference equation (15) is locally asymptotically stable in $n \in [0, 100]$ with $c(-1) = 0.1,\ c(0) = 0.1.$ and $\lambda = 3,\ a = 1,\ m = 1,\ g = 2$ in Equation (14).

Example 4.2.

Using $\lambda = 0.2,\ a = 0.1,\ m = 10,\ g = 0.1$ in Equation (1), we get

$$\frac{dc}{dt} = \frac{0.2c([t-1])}{1 + 10^{10}c^{10}([t-1])} - 0.1c(t),$$

and the corresponding difference equation is

$$c(n+1) = \mathrm{e}^{-0.1}c(n) + \frac{2(1-\mathrm{e}^{-0.1})c(n-1)}{1 + 10^{10}c^{10}(n-1)}, \tag{16}$$

with the initial conditions

$$c(-1) = 0.1,\ c(0) = 0.1.$$

The corresponding linearized equation of Equation (16) about E^* is

$$x_{n+1} = \mathrm{e}^{-0.1}x_n - 4(1-\mathrm{e}^{-0.1})x_{n-1}, \quad n = 0, 1, 2, \ldots.$$

All hypotheses of Theorem 3.2 are satisfied for $\lambda = 0.2,\ a = 0.1,\ m = 10,\ g = 0.1$. So, the equilibrium $E^* = 0.1$ is locally asymptotically stable in Figure 2.

Remark 4.1.

The parameters values $\lambda = 0.2,\ a = 0.1,\ m = 10,\ g = 0.1$ and the initial condition on c of 0.1 are the same as Figure 1 given in Mackey-Glass (1977).

Example 4.3.

Taking $\lambda = 2,\ a = 0.1,\ m = 6,\ g = 1$ in Equation (1), we obtain

$$\frac{dc}{dt} = \frac{2c([t-1])}{1 + 10^6c^6([t-1])} - c(t),$$

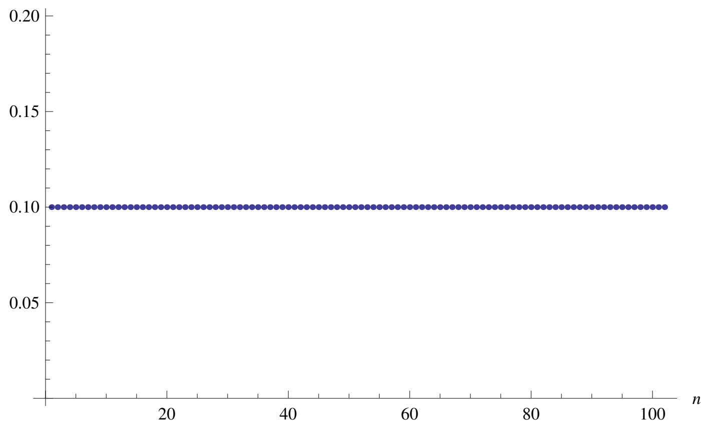

Figure 2. The equilibrium $E^* = 0.1$ of the difference equation (16) is locally asymptotically stable in $n \in [0, 100]$ with $c(-1) = 0.1,\ c(0) = 0.1$ and $\lambda = 0.2,\ a = 0.1,\ m = 10,\ g = 0.1$.

and the corresponding difference equation is

$$c(n+1) = \frac{1}{\mathrm{e}}c(n) + 2\left(\frac{\mathrm{e}-1}{e}\right)\frac{c(n-1)}{1+10^6c^6(n-1)}, \tag{17}$$

with the initial conditions

$$c(-1) = 1,\ c(0) = 1.$$

The corresponding linearized equation of Equation (17) about $E^* = 0.1$ is

$$x_{n+1} = \frac{1}{\mathrm{e}}x_n - 2\left(\frac{\mathrm{e}-1}{\mathrm{e}}\right)x_{n-1}, \quad n = 0, 1, 2, \ldots.$$

Because the first condition of (11) is satisfied, $E^* = 0.1$ in Figure 3 is a repeller.

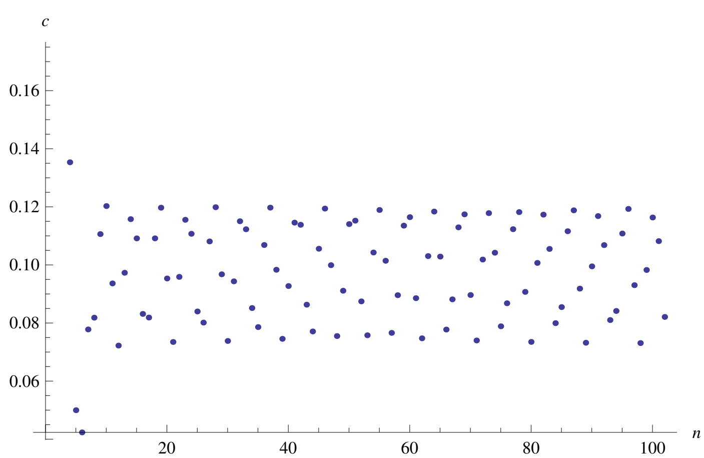

Figure 3. Every solution of the difference equation (17) is a repeller in $n \in [0, 100]$ with $c(-1) = 1,\ c(0) = 1$ and $\lambda = 2,\ a = 0.1,\ m = 6,\ g = 1$.

Example 4.4.

Considering $\lambda = 0.2,\ a = 0.1,\ m = 10,\ g = 0.1$ in Equation (1), we find

$$\frac{dc}{dt} = \frac{0.2c([t-1])}{1+10^{10}c^{10}([t-1])} - 0.1c(t),$$

and the corresponding difference equation is

$$c(n+1) = \mathrm{e}^{-0.1}c(n) + \frac{2(1-\mathrm{e}^{-0.1})c(n-1)}{1+10^{10}c^{10}(n-1)}, \tag{18}$$

with the initial conditions

$$c(-1) = 1.1,\ c(0) = 1.$$

The hypotheses of Theorem 3.5 are satisfied for $\lambda = 0.2,\ a = 0.1,\ m = 10,\ g = 0.1$, and then every solution of Equation (18) is attracted to $E^* = 0.1$ in Figure 4.

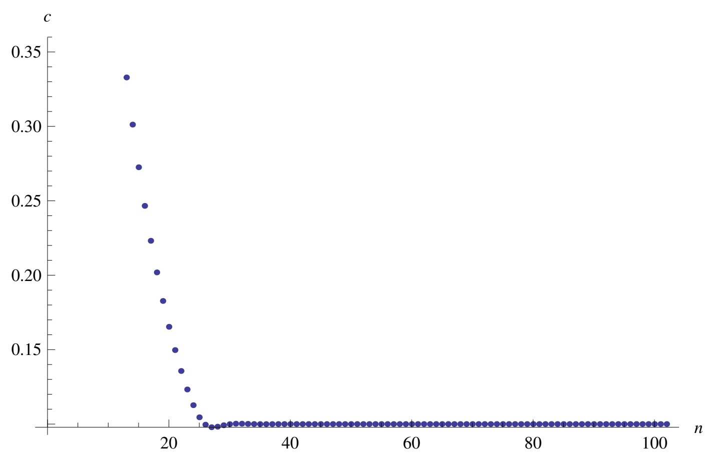

Figure 4. Every solution of the difference equation (18) is attracted to $E^* = 0.1$ in $n \in [0, 100]$ with $c(-1) = 1.1,\ c(0) = 1$ and $\lambda = 0.2,\ a = 0.1,\ m = 10,\ g = 0.1$,

5. Conclusion

In this work, we have introduced a Mackey-Glass Model with piecewise constant argument. Since the differential equation studied here can be written in terms of the solution of the corresponding difference equation, we consider the asymptotic stability of the equilibrium point of the corresponding difference equation and it is obtained that this point is a repeller. Moreover, it is shown that every oscillatory solution of the difference equation has semi-cycles of length at least two, and every solution of the difference equation is attracted the equilibrium point. The first figure and the second figure demonstrate that the equilibrium $E^* = 0.5$ and $E^* = 0.1$ are locally asymptotically stable in $n \in [0, 100]$, respectively. Figure 3 shows that $E^* = 0.1$ of the difference equation (17) is a repeller and every solution of the difference equation (18) attracted to $E^* = 0.1$ in $n \in [0, 100]$ in Figure 4.

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