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Effect of Aggregation Function in MOMA-Plus Method For Obtaining Pareto Optimal Solutions

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Abstract

In this work, we have proposed some variants of MOMA-Plus method that we have numerically tested for the resolution of nonlinear multiobjective optimization problems. This MOMA-Plus method and variants differ from each other by the choice of aggregation functions in order to reduce the number of objective functions. The theoretical results allowing us to use these aggregation functions to transform multiobjective optimization problems into single objective optimization problems are proved by two theorems. This study has highlighted the advantages of each aggregation function according to the type of Pareto front of the optimization problem. Six benchmarks test problems have been solved in this work by each of these methods and a comparative study was carried out through the performance indicators which are the differentiation with Pareto front, the convergence to the Pareto front and distributivity on the Pareto front. This allowed us to classify these methods on these benchmarks by using the Graphical Analysis for Interactive Assistance (GAIA) method.

Keywords: Aggregation function; MOMA-Plus method; Multiobjective optimization; Pareto optimal solution; GAIA method

MSC 2020 No.: 90C30, 65K05, 49M37

1. Introduction

Multiobjective optimization consists in simultaneously minimizing or maximizing several objective functions subject to or no constraints. These types of optimization problem, contrary to single-objective optimization, provide several solutions called Pareto optimal solutions or best compromise solutions. Here, the non-uniqueness of the solutions is due to the conflicting nature of the objective functions. It is important to note that multiobjective optimization participates in the decision aid in several fields of life (Gebreel (2022); Poda et al. (2019); Kaur et al. (2018); Dennis and Woods (1985); Poda et al. (2018)) such as economy, management, civil engineering, electronics, artificial intelligence, big data, etc.

The complexity of the decision space for most of these types of optimization problems had led to the implementation of numerous method (Albayrak et al. (2019); Miettinen (1999)). Despite these many methods in the literature, it is literally difficult to find one which is best for a given problem on the speed, the convergence and the distributivity. Because of these difficulties for solving these types of problems, most researchers try to transform these multiobjective optimization problems into single-objective optimization problems. This is generally made possible thanks to the use of aggregation functions. The aggregation functions which are well known and mostly used are: the weighted sum (Gass and Saaty (1955)) which will be noted by S_1 in the following, the weighted distance of Tchebychev (Kaliszweski (1987); Klamroth and Tind (2007)) which will be noted by S_2 in the following, the augmented weighted distance of Tchebychev (Bowman (1976)) which will be noted by S_3 in the following, the Keeney-Raiffa approach (Yann and Siarry (2002)) which will be noted by S_4 in the following, the distance to objective approach (Miettinen (1999)) which will be noted by S_5 in the following, and the ε-constraint approach (Miettinen (1999)) which will be noted by S_6 in the following. It is these six aggregation functions, which are used in the following for building the different variants of MOMA-Plus method.

The MOMA-Plus method on which this work is based is a method using a scalar approach for solving multiobjective optimization problems. In previous works, the weighted sum was used to solve linear problems (Somé et al. (2013)), the Tchebychev weighted distance for nonlinear problems (Som et al. (2020); Somé et al. (2013)), the linear compromise function for linear combinatorial optimization problems (Poda et al. (2019) and Poda et al. (2018)) and the quadratic compromise function for nonlinear combinatorial optimization problems. The first two aggregation functions have also contributed to resolution of the single-objective and multiobjective fuzzy optimization problems (Compaoré et al. (2017); Compaoré et al. (2018)).

In the literature, there is no study about the effect of the scalarization function in multiobjective problem resolution. For this work, the effect of previous aggregation functions in the MOMA-Plus method will be measured through the assessment of the performance indexes, which are mainly convergence, diversity and differentiation. This last parameter is also an innovation of this work. It allows us to estimate the ratio of the obtained Pareto optimal solutions compared to the expected Pareto optimal solutions. The idea behind the assessment of these performance indexes is to arrive at a comparative analysis of the different aggregation functions that will be used in this work according to some well known test problems. Moreover, we will use a GAIA multicriteria method

(Kaur et al. (2018)) to classify these aggregation functions. In this work, we studied the effect of choosing the aggregation function which enables the transition of many Pareto solutions search to unique optimal solution search. The theoretical and analytical results of these procedures are presented through six test problems (Deb et al. (2002)).

For the best presentation of this work, we have subdivided it in three sections. The first section is devoted to the introduction, the second section will be devoted to present the main results about MOMA-Plus and variants, and the third section will be devoted to the conclusion.

2. MOMA-Plus and variants

Generally, the multiobjective optimization problem is formulated as follows:

$$\begin{array}{ll} \min & \big(f_1(x), f_2(x), \cdots, f_p(x)\big), \\ \text{subject to} & \begin{cases} g_i(x) \leq 0;\ i = 1, \cdots, m, \\ x \in \mathbb{R}^n, \end{cases} \end{array} \tag{1}$$

where $f = (f_1, f_2, \cdots, f_p)$ and $f_i,\ i = \overline{1,p}$ are objective functions, $g_i, i = \overline{1,m}$ are the constraints link to f minimization, and $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$ is a decision variable. Note $D = \big\{x \in \mathbb{R}^n /\ g_1(x) \leq 0, \cdots, g_m(x) \leq 0\big\}$ the decision space and $Y = f(D)$ the objective functions space.

2.1. Theoretical results

2.1.1. *Principle*

MOMA-Plus is a method that enables us to solve a multiobjective optimization problem with several variables by transforming it into a single objective optimization problem of a single variable, without constraints, following the steps:

◇ **Step 1**: aggregation of objective functions;
◇ **Step 2**: penalization of constraint functions;
◇ **Step 3**: reduction of several variables into a single variable;
◇ **Step 4**: optimization of single objective function;
◇ **Step 5**: configuration of obtained solution to the solution of initial problem.

For more details on MOMA-Plus, the reader can consult the following papers: Som et al. (2020), Poda et al. (2019), Poda et al. (2018), Compaoré et al. (2017), Somé et al. (2017), Somé et al. (2013), and Somé et al. (2011).

In the present work, the major change occurred at the step of the objective functions aggregation. Indeed, we will use successively the aggregation functions described in Section 2. The impact of these aggregation functions will be studied both theoretically and numerically. The steps outlined in our study are similar to the initial MOMA-Plus steps.

2.1.2. *Diagram*

The presentation of MOMA-Plus modified can be done in five essential steps. Its flow diagram can be presented as follows:

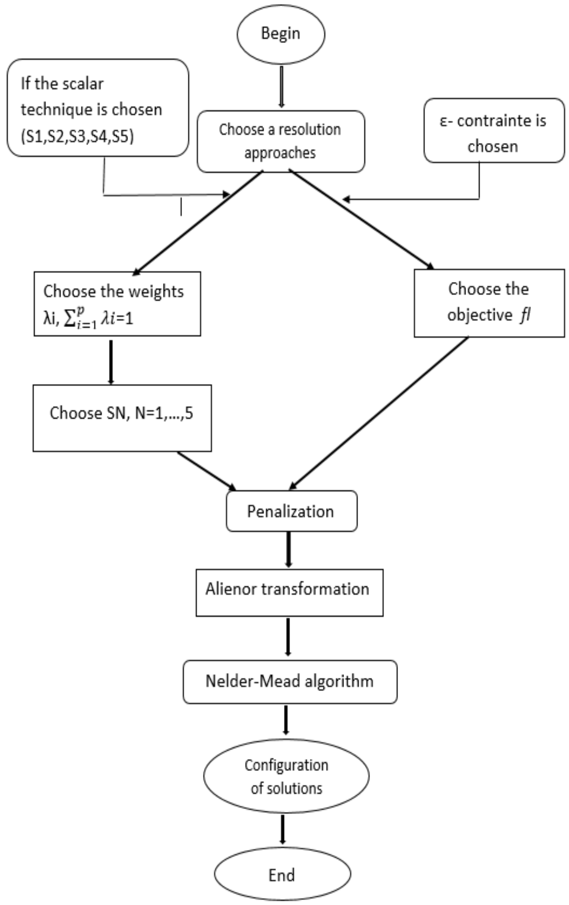

Figure 1. MOMA-Plus modified flow diagram

2.1.3. *Description*

A. Scalarization

According to one of these previous aggregation function, the initial problem can be formulated as follows:

$$\begin{aligned} &\min \quad S_N(x), \quad N \in \{1,2,3,4,5,6\}, \\ &\text{subject to} \begin{cases} g_i(x) \leq 0, \ i = 1, \cdots, m, \\ x \in \mathbb{R}^n. \end{cases} \end{aligned} \tag{2}$$

This brings us to the following theorem.

Theorem 2.1.

Any optimal solution to problem (2) is an optimal Pareto solution to problem (1) and reciprocally.

Proof:

We have chosen to do the proof for $N = 2$.

Let x^* be the optimal solution of the problem (2) and $I = \{1; ...; p\}$. Assume that x^* is not a Pareto optimal solution of the problem (1), then $\exists y \in D,\ \forall j \in I : f_j(y) \leq f_j(x^*)$, and for at least one $k \in I, f_k(y) < f_k(x^*)$. Then, $\exists y \in D,\ \forall j \in I, f_j(y) - \overline{z}_j \leq f_j(x^*) - \overline{z}_j$ and for at least one $k \in I, f_k(y) - \overline{z}_k < f_k(x^*) - \overline{z}_k$. So, $\exists y \in D, \forall j \in I, |f_j(y) - \overline{z}_j| \leq |f_j(x^*) - \overline{z}_j|$ and for at least one $k \in I, |f_k(y) - \overline{z}_k| < |f_k(x^*) - \overline{z}_k|$ because $f_k(x) - \overline{z}_j > 0$. It follows that $\exists y \in D, \forall j \in I, \lambda_j |f_j(y) - \overline{z}_j| \leq \lambda_j |f_j(x^*) - \overline{z}_j|$ and for at least one $k \in I, \lambda_j |f_k(y) - \overline{z}_k| < \lambda_j |f_k(x^*) - \overline{z}_k|$ because $\lambda_j \geq 0$. Therefore, $\exists y \in D, \max\limits_{j \in I} \{\lambda_j |f_j(y) - \overline{z}_j\}| \leq \max\limits_{j \in I} \{\lambda_j |f_j(x^*) - \overline{z}_j|\}$. So, $\exists J \subset I, \max\limits_{j \in J} \{\lambda_j |f_j(y) - z_j^*|\} < \max\limits_{j \in J} \{\lambda_j |f_j(x^*) - z_j^*|\}$. Then, $\exists J \subset I, S_2(f_j(y), \lambda_j, \overline{z}_j) < S_2(f_j(x^*), \lambda_j, \overline{z}_j), \forall j \in J$. Thus, $S_2(f(y), \lambda, \overline{z}) < S_2(f(x^*), \lambda, \overline{z})$. Hence, that is absurd because x^* is the optimal solution of the problem (2).

Reciprocally, if x^* is a Pareto optimal solution of the problem (1), then there is no $x \in D$ such as $f_j(x) \leq f_j(x^*),\ \forall j \in I$ and for at least one $k \in I$ such as $f_k(x) < f_k(x^*)$. Then, there is no $x \in D,\ \forall j \in I, f_j(x) - \overline{z}_j \leq f_j(x^*) - \overline{z}_j$ and for at least one $k \in I, f_k(x) - \overline{z}_k < f_k(x^*) - \overline{z}_k$. So, there is no $x \in D,\ \forall j \in I, |f_j(x) - \overline{z}_j| \leq |f_j(x^*) - \overline{z}_j|$ and for at least one $k \in I, |f_k(y) - \overline{z}_k| < |f_k(x^*) - \overline{z}_k|$ because $f_k(x) - \overline{z}_j > 0$. This implies that there is no $x \in D, \forall j \in I, \lambda_j |f_j(x) - \overline{z}_j| \leq \lambda_j |f_j(x^*) - \overline{z}_j|$ and for at least one $k \in I, \lambda_j |f_k(x) - \overline{z}_k| < \lambda_j |f_k(x^*) - \overline{z}_k|$ because $\lambda_j \geq 0$. Then, there is no $x \in D, \max\limits_{j \in I} \{\lambda_j |f_j(x) - \overline{z}_j\}| \leq \max\limits_{j \in I} \{\lambda_j |f_j(x^*) - \overline{z}_j|\}$. So, there is no $x \in D, S_2(f(x), \lambda, \overline{z}) \leq S_2(f(x^*), \lambda, \overline{z})$. Consequently, x^* is an optimal solution of the problem (2). ∎

B. Penalization Approach

In the literature, there exists many penalization functions (Hassan et al. (2020); Poda et al. (2018); Compaoré et al. (2017)) allowing to transform the former problem (2) with constraints to a problem without constraints. The penalization function uses by MOMA-Plus method is formulated as follows:

- For $N = 1, \cdots, 6$ let us take:

$$L_N(x) = S_N(x) + \sigma_N \sum_{i=1}^{m} (g_i(x) + |g_i(x)|), \tag{3}$$

  with

$$\sigma_N \geq \frac{K_N - S_N(x)}{\sum\limits_{i=1}^{m} g_i(x)} \text{ where } K_N = \max_{x \in D} \{S_N(x)\}.$$

That allows us to transform problem (2) to a single-objective optimization problem without constraints as follows:

$$Glob. \min_{x\in[a,b]^n} \; L_N(x). \tag{4}$$

The following theorem characterizes the global optimality of the problem (4).

Theorem 2.2.

Any global minimum of problem (4), for N fixed, is a global minimum of problem (2).

Proof:

For fixed $N = 2$, considering x^* as a point which realizes a global minimum problem (4), therefore $\forall x \in D : L_2(x^*) \leq L_2(x)$. Therefore, $S_2(f(x^*), \lambda, \overline{z}) + \sigma_2 \sum_{i=1}^{m}(g_i(x^*) + |g_i(x^*)|) \leq S_2(f(x), \lambda, \overline{z}) + \sigma_2 \sum_{i=1}^{m}(g_i(x) + |g_i(x)|)$. Thereafter, we obtain $S_2(f(x^*), \lambda, \overline{z}) \leq S_2(f(x), \lambda, \overline{z}),$ because by definition $\forall x \in D,\ g_i(x) \leq 0$. And finally, $\exists J \subset I, S_2(f(x^*), \lambda, \overline{z}) < S_2(f(x), \lambda, \overline{z})$. Hence, x^* is a point which realizes a global minimum of problem (2). ∎

C. Alienor transformation

Alienor transformation is a transformation that allows to reduce the number of the variables of the problem based on the $\alpha-$dense curve. Here we use the Alienor transformation to transform problems in single variable problems. The Alienor transformation that we use here is as follows (Konfé et al. (2005)):

$$x_i = \chi_i(\theta) = \frac{1}{2}\Big[(b_i - a_i)\cos(\omega_i\theta + \phi_i) + a_i + b_i\Big],\ i = 1, \cdots, n, \tag{5}$$

where ω_i and ϕ_i are slowly increasing sequences; $\theta \in [0; \theta_{max}]$ with $\theta_{max} = \dfrac{(b-a)\theta^1 + (b+a)}{2}$ and $\theta^1 = \dfrac{2\pi - \phi_1}{\omega_1}$; $a = \min_{i=\overline{1,n}} a_i$ and $b = \max_{i=\overline{1,n}} b_i$.

In Cherruault (1989), it is proved that all points $x \in \mathbb{R}^n$ can be approached by at least one point of $\mathbb{R}$ using the Alienor transformation χ . For the variants of Alienor transformation, one can consult the following papers by Benneoula and Cherruault (2005) and Cherruault and Mora (2005).

The application of the relation (5) to the problem (4) gives us the single objective problem without constraints with a unique variable. Hence, this formulation is:

$$\min_{\theta\in[0;\theta_{max}]} \Gamma_N(\theta), \tag{6}$$

with $\Gamma_N(\theta) = L_N(\chi(\theta))$, $N \in \{1, 2, 3, 4, 5, 6\}$ and $\chi(\theta) = \big(\chi_1(\theta), \chi_2(\theta), \cdots, \chi_n(\theta)\big)$.

All solution of the problem (4) can be approach by a global minimum of the problem (6) (Cherruault (1989)).

D. Resolution and solutions configuration

The problem (6) is single objective and one variable. Then, the Nelder-Mead simplex algorithm is appropriate to solve it. After determining the optimal solution of the problem (6), we re-use the Alienor transformation presented above to reconstitute the Pareto optimal solution of the initial problem (1).

In the rest of this paper, following the choice of the aggregation function, we will note the methods as following: SP_MOMA-Plus when the weighted sum is used; DP_MOMA-Plus when the Tchebychev weighted distance is used; DP^+_MOMA-Plus when the augmented Tchebychev distance is used; L2_MOMA-Plus when the distance to objective approach is used; KR_MOMA-Plus when the Keeney-Raiffa approach is used; EP_MOMA-Plus when the ϵ-constraint approach is used.

2.1.4. *Algorithm*

Table 1. MODIFIED MOMA-PLUS ALGORITHM

Algorithm 2 MODIFIED MOMA-PLUS ALGORITHM

(1) Begin

(2) If scalar method is chosen, do

 (a) For j from 1 to p do
 Enter the aggregation function $S_N;\ N = 1, 2, \cdots, 5$
 (b) $g(x) \longleftarrow g_1(x) + |g_1(x)|$;
 For i from 2 to m do
 $g(x) \longleftarrow g(x) + g_i(x) + |g_i(x)|$; ("g is the constraints functions of the problem (1)")
 End for
 $L_N(x) \longleftarrow S_N(x) + \sigma_N * g(x)$;("Penalization")
 (c) For i from 1 to n do
 $x_i \longleftarrow \chi_i(\theta)$;("Alienor transformation")
 End for
 $\Gamma_N(\theta) \longleftarrow L_N(\chi_1(\theta), \chi_2(\theta), \cdots, \chi_n(\theta))$;
 (d) $\theta^* \longleftarrow Arg\min(\Gamma_N(\theta))$;("Using algorithm (7) to find the optimum")
 (e) For i from 1 to n,
 $x_i \longleftarrow \chi_i(\theta^*)$; (" allow to determine the solution of the initial problem")
 End for
 (f) End for
 (g) Display the solution x of the problem which is one of the best compromise corresponding to fixed λ_k

(3) End if

(4) else

(5) if ε- constraint is chosen, then choose an objective function $f_l,\ l \in \{1, \cdots, p\}$;

 (a) While $\varepsilon_j \leq \max_{j \neq l} f_j,\ j = \{1; 2, \cdots; p\}$
 repeat
 (b) $h(x) \longleftarrow f_1(x) - \varepsilon_1 + |f_1(x) - \varepsilon_1|$;
 (c) $g(x) \longleftarrow g_1(x) + |g_1(x)|$;
 For j from 2 to p and $j \neq l$ do
 $h(x) \longleftarrow h(x) + f_j(x) - \varepsilon_j + |f_j(x) - \varepsilon_j|$;
 End for
 For i from 2 to m do
 $g(x) \longleftarrow g(x) + g_i(x) + |g_i(x)|$;("g is the constraints functions of the chosen problem")
 End for
 (d) $H(x) \longleftarrow h(x) + g(x)$;
 $L_6(x) \longleftarrow f_l(x) + \sigma_6 * H(x)$;("Penalization")
 (e) For i from 1 to n do
 $x_i \longleftarrow \chi_i(\theta)$;("Alienor transformation")
 End for
 $\Gamma_6(\theta) \longleftarrow L_6(\chi_1(\theta), \chi_2(\theta), \cdots, \chi_n(\theta))$;
 (f) $\theta^* \longleftarrow Arg\min(\Gamma_6(\theta))$;("Using algorithm (7) to find the optimum")
 (g) For i from 1 to n,
 $x_i \longleftarrow \chi_i(\theta^*)$; ("allow to determine the solution of the initial problem")
 End for
 (h) End while
 (i) Display the solution x of the problem which is one of the best compromise corresponding to fixed λ_k

(6) End if

(7) End

2.2. Numerical results

2.2.1. *Test problems*

Here are the test problems treated, which are proposed by Deb et al. (2002):

Table 2. Test Problems

Notation	Test problems	n	Decision space	Pareto optimal solutions characteristic
T_1	$\begin{cases} Minf_1(x_1,x_2)=x_1 \\ Minf_2(x_1,x_2)=\frac{1+x_2}{x_1} \\ 0.1\leqslant x_1\leqslant 1 \\ 0\leqslant x_2\leqslant 5 \end{cases}$	2	$x_i \in [0;1]$	Convex
T_2	$\begin{cases} Minf_1(x)=x_1 \\ Minf_2(x)=g(x)\times(1-(\frac{f_1}{g})^2) \\ g(x)=1+\frac{9}{n-1}\times\sum_{i=2}^{n}x_i \\ x=(x_1,x_2,\cdots,x_n)\in[0.1]^n \end{cases}$	30	$x_i \in [0;1]$	Not convex
T_3	$\begin{cases} Minf_1(x)=x_1 \\ Minf_2(x)=g(x)\times\sqrt{1-\frac{f_1}{g}} \\ g(x)=1+\frac{9}{n-1}\times\sum_{i=2}^{n}x_i \\ x=(x_1,x_2,\cdots,x_n)\in[0.1]^n \end{cases}$	30	$x_i \in [0;1]$	Not convex
T_4	$\begin{cases} Minf_1(x)=x_1 \\ Minf_2(x)=g\left(1-\sqrt{\frac{f_1(x)}{g}}\right) \\ g(x)=1+\frac{9}{n-1}\times\sum_{i=2}^{n}x_i \\ x=(x_1,x_2,\cdots,x_n)\in[0.1]^n \end{cases}$	30	$x_i \in [0;1]$	Convex
T_5	$\begin{cases} Minf_1(x)=x_1 \\ Minf_2(x)=g(x)\times h(x) \\ g(x)=1+\frac{9}{n-1}\times\sum_{i=2}^{n}x_i \\ x=(x_1,x_2,\cdots,x_n)\in[0.1]^n \\ h(x)=1-\sqrt{\frac{f_1(x)}{g(x)}}-\frac{f_1(x)}{g(x)}sin(10\pi f_1(x)) \end{cases}$	30	$x_i \in [0;1]$	Multi-modal and discontinuous
T_6	$\begin{cases} Min\ f_1(x)=x_1 \\ Min\ f_2(x)=g\left(1-\sqrt{\frac{f_1(x)}{g}}\right) \\ g=1+10\,(n-1)+\sum_{i=1}^{n}(x_i-10cos(4\pi x_i)) \\ x_i\in[0;1] \end{cases}$	30	$x_i \in [0;1]$	Multimodal and discontinue

The numerical solutions obtained by using different variants are plotted at the same time in the same figure with the analytical solutions. These figures are given in the Appendix section.

2.2.2. *Performance study*

The built MOMA-Plus variants were compared according to the three performance indexes mentioned above.

The differentiation. We note it by α in this work. It will be calculated using the following formula:

$$\alpha = \frac{\mu_1 - \mu}{\mu_1}, \tag{7}$$

where μ is the number of the Pareto optimal solution obtained and μ_1 is the number of Pareto optimal solutions waited. In other words, μ_1 correspond to the number of set of weights used. So, in this work, this number is fixed to $\mu_1 = 56$.

Table 3. Differentiation index values

α	T_1	T_2	T_3	T_4	T_5	T_6
SP_MOMA-Plus	0.4286	0.5000	0.6071	0.3393	0.4107	0.3393
DP_MOMA-Plus	0.0000	0.0357	0.0000	0.0000	0.0893	0.0179
DP^+_MOMA-Plus	0.4107	0.0357	0.0000	0.0000	0.0893	0.0125
KR_MOMA-Plus	0.4107	0.0714	0.2679	0.0000	0.3036	0.0357
L2_MOMA-Plus	0.0000	0.0536	0.2679	0.0000	0.3214	0.0179
EP_MOMA-Plus	0.0702	0.0000	0.0000	0.0000	0.1053	0.0000

Hence, Table 4 presents a ranking of variants on the test problems following the differentiation indexes.

Table 4. Ranking of the values of differentiation index

Ranks	T_1	T_2	T_3	T_4	T_5	T_6
SP_MOMA-Plus	6	6	6	6	6	6
DP_MOMA-Plus	1	2	1	1	1	3
DP^+_MOMA-Plus	4	2	1	1	1	2
KR_MOMA-Plus	4	5	4	1	4	5
L2_MOMA-Plus	1	4	4	1	5	3
EP_MOMA-Plus	3	1	1	1	3	1

The convergence. Noted by τ this is calculated by the following relation (Deb et al. (2002)),

$$\tau = \frac{\sqrt{\sum_{i=1}^{\mu} d_i^2}}{\mu}, \tag{8}$$

where μ is the size of the obtained solutions set and d_i is the euclidean distance between the obtained solution i to the analytical front. This also enables us to evaluate the accuracy of obtained results through the method compared to solutions of analytical front. The convergence index values obtained by using MOMA-Plus variant on all test problems are given in Table 5.

Table 5. Convergence index values

τ	T_1	T_2	T_3	T_4	T_5	T_6
SP_MOMA-Plus	0.0537	0.0049	0.0059	0.0063	0.0065	0.0174
DP_MOMA-Plus	0.0691	0.0042	0.0046	0.0137	0.0599	0.1154
DP$^+$_MOMA-Plus	0.0525	0.0036	0.0047	0.0044	0.0065	0.0126
KR_MOMA-Plus	0.0149	0.0033	0.0021	0.0038	0.0103	0.0081
L2_MOMA-Plus	0.0320	0.0033	0.0025	0.0044	0.0113	0.0131
EP_MOMA-Plus	0.0851	0.0037	0.0051	0.0048	0.0165	0.0070

Hence, Table 6 presents a ranking of variants on the test problems following the convergence indexes.

Table 6. Ranking of the values of convergence index

Ranks	T_1	T_2	T_3	T_4	T_5	T_6
SP_MOMA-Plus	4	6	6	5	1	5
DP_MOMA-Plus	5	5	3	6	6	6
DP$^+$_MOMA-Plus	3	3	4	2	1	3
KR_MOMA-Plus	1	1	1	1	3	2
L2_MOMA-Plus	2	1	2	2	4	4
EP_MOMA-Plus	6	4	5	4	5	1

The distributivity. We have noted it by β and it is calculated by the formula (Deb et al. (2002)):

$$\beta = \frac{d_f + d_l + \sum_{i=1}^{\mu-1} |d_i - \overline{d}|}{d_f + d_l + (\mu - 1)\overline{d}}, \tag{9}$$

where μ is the size of the obtained solutions, d_i is the Euclidean distance between the obtained solution i and the analytical solution, $\overline{d}$ is the average of distances and d_f and d_l are respectively the difference between the extreme analytical solution and the extreme obtained solution. It enables us to evaluate the distributivity of solutions on the Pareto front. The distributivity index obtained by the MOMA-Plus variants on all test problems are given in Table 7.

Table 7. Distributivity index values

β	T_1	T_2	T_3	T_4	T_5	T_6
SP_MOMA-Plus	0.9678	0.9630	0.9977	0.9756	0.9691	0.9819
DP_MOMA-Plus	1.1833	0.0309	0.9820	0.3483	0.9835	0.9818
DP^+_MOMA-Plus	0.9688	0.9818	0.9819	0.9823	0.9691	0.9819
KR_MOMA-Plus	0.9818	0.9819	0.9818	0.9826	0.9822	0.9823
L2_MOMA-Plus	0.9818	0.9818	0.9818	0.9819	0.9824	0.9819
EP_MOMA-Plus	0.9812	0.9824	0.9823	0.9826	0.9803	0.9825

Hence, Table 8 presents a ranking of variants on the test problems following the distributivity indexes.

Table 8. Ranking of the values of distributivity index

Ranks	T_1	T_2	T_3	T_4	T_5	T_6
SP_MOMA-Plus	1	2	6	2	1	2
DP_MOMA-Plus	6	1	4	1	6	1
DP^+_MOMA-Plus	2	3	3	4	1	2
KR_MOMA-Plus	4	5	1	5	4	5
L2_MOMA-Plus	4	3	1	3	5	2
EP_MOMA-Plus	3	6	5	5	3	6

2.2.3. *Comments*

The analysis of these three (3) performance indexes will help us to do a ranking of these MOMA-Plus method variants. Considering that these performance indexes have the same importance, we can deduce the following table which presented the global ranking.

Table 9. Ranking concerning performance index

Global Rank	Rank α	Rank γ	Rank Δ
SP_MOMA-Plus	36	27	14
DP_MOMA-Plus	09	31	19
DP^+_MOMA-Plus	11	16	15
KR_MOMA-Plus	23	09	24
L2_MOMA-Plus	18	15	18
EP_MOMA-Plus	10	25	28

In Table 9, for each variant, the global rank is calculated. This global rank is the sum of obtained ranks on all test problems. So, by applying the GAIA method ((Kaur et al. (2018))) on these data, we obtain the graphical classification as presents in Figure 2.

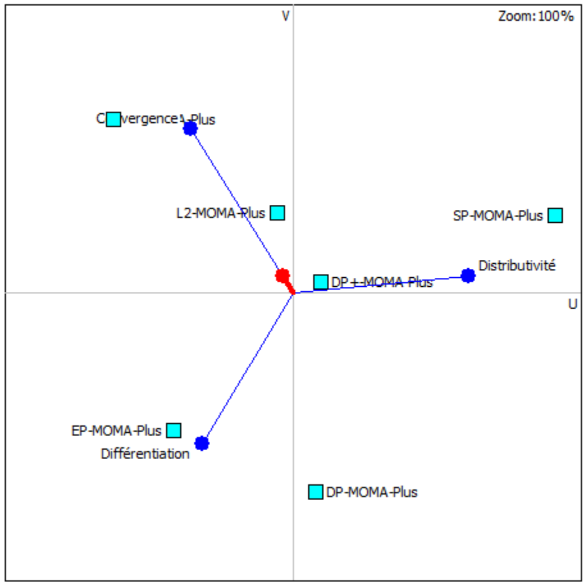

Figure 2. GAIA evaluation

It should be noted that the weightings of the evaluation criteria are identical. The red line defines the direction toward the compromise according to the priorities that are the criteria, defined by differentiation, convergence and diversity. In Figure 38, variant $L2\_MOMA - Plus$ is the best approach, as it is in the direction and close to the red axis.

It should be noted that the weighting of the evaluation criteria is identical. The red line defines the direction towards the compromise according to the priorities that are the criteria, defined by differentiation, convergence and diversity.

In the figure, the variant $L2\_MOMA - Plus$ is the best approach because it is in the direction and close to the red axis. This would mean that the L_2 function is the best scalarization function that improves the performance of MOMA-Plus on these three (3) criteria.

Indeed, the action closest to this line is the one that is close to the requirements defined by each of the criteria defined in the decision process. However, a descriptive analysis of the single-criteria flows gives us the following interpretations:

- The variant $EP\_MOMA - Plus$ is respectively closer to the differentiation axis. This means that $EP\_MOMA - Plus$ is the approach that has less repeating solutions in the Pareto solutions set.
- The variant $DP^+ - MOMA - Plus$ is in the direction of the distributivity criterion axis, The same is true for the $SP - MOMA - Plus$ approach, which is also close to this axis. This would mean that the weighted sum and the augmented weighted distance of Tchebychev are

the aggregation functions that improve the MOMA-Plus method on the distributivity of Pareto solutions.

- The $KR - MOMA - Plus$ variant is in the direction of the convergence axis, compared to the other approaches. If the analysis were to focus on convergence, the $KR - MOMA - Plus$ approach would be the most appropriate. Thus, the Kenney-Raiffa function is the scalarization function that improves the convergence of the Pareto solutions of the MOMA-Plus method.

3. Conclusion

In this work, we have established six possibilities of using other techniques for aggregating the objective functions of a multiobjective optimization problem for the MOMA-Plus method. To accomplish this, two theorems were established to prove the theoretical foundations for using these aggregation functions, which have the role to transform the optimization problems with multiple objectives to single-objective optimization problem. These results allowed us to resolve only the equivalent single objective optimization problem and deduce the results of the initial multiobjective optimization problem. Moreover, we successfully tested six benchmarks with the MOMA-Plus method and variants. The numerical results on these six problems enabled us to help the decision makers or MOMA-Plus method users to choose the better variant according to the initial problem. Furthermore, their preferences about the qualities such as good differentiation, good distributivity, good convergence can be considered. So, on these six test problems, we can say that $L2$_MOMA-Plus is the best variant of MOMA-Plus according to the using of GAIA method.

In future work, we will be interested in the study of the complexity of MOMA-Plus and variants algorithms, the comparative study with others method on the performances and the application of these methods to resolve a real life problem.

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Appendix

Table 10. For the problem T_1

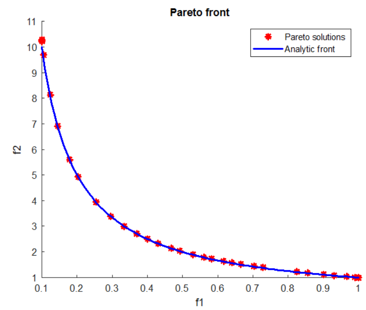


Figure 3. Solutions by SP_MOMA-Plus

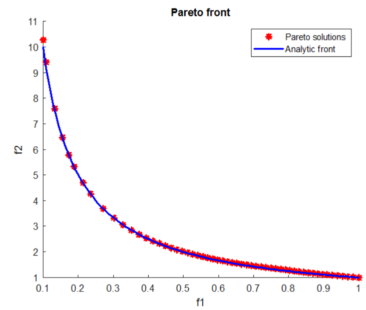


Figure 4. Solutions by DP_MOMA-Plus

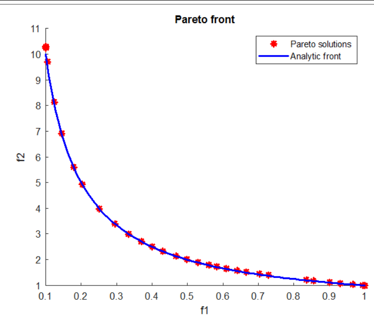


Figure 5. Solutions by DP$^+$_MOMA-Plus

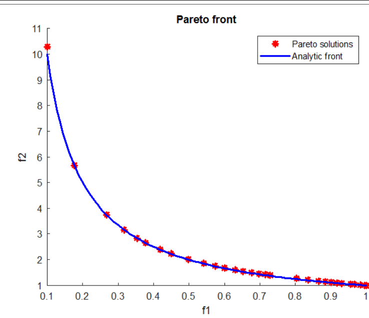


Figure 6. Solutions by KR_MOMA-Plus

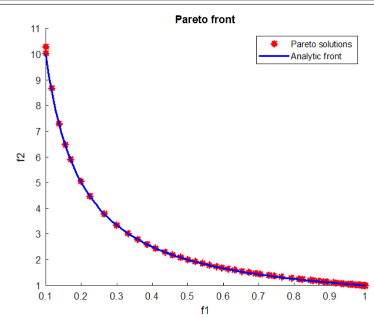


Figure 7. Solutions by L2_MOMA-Plus

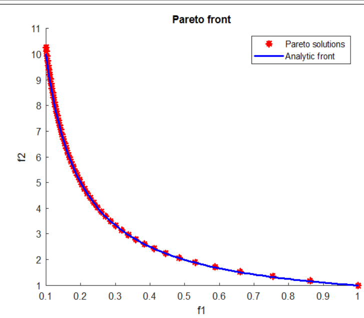


Figure 8. Solutions by EP_MOMA-Plus

Table 11. For the problem T_2

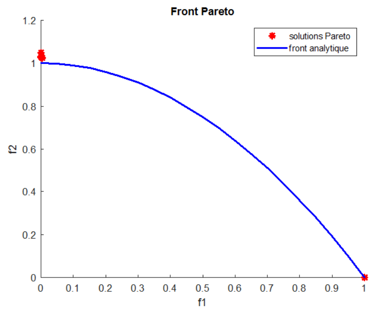
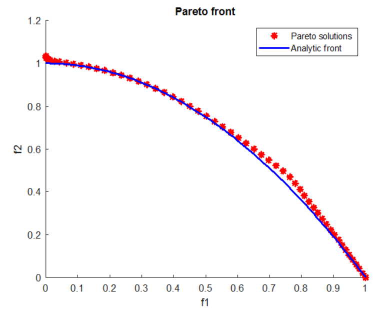
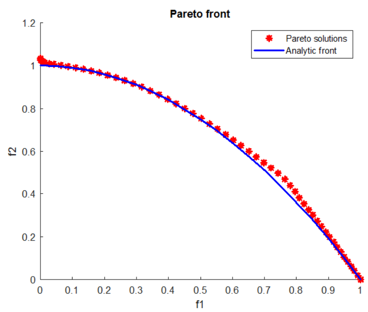
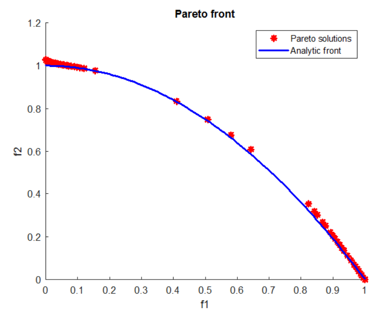
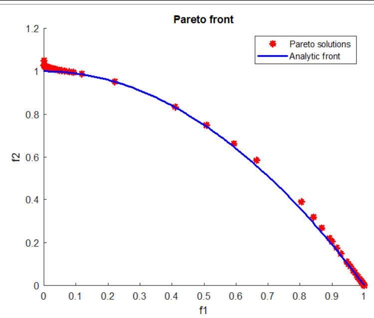
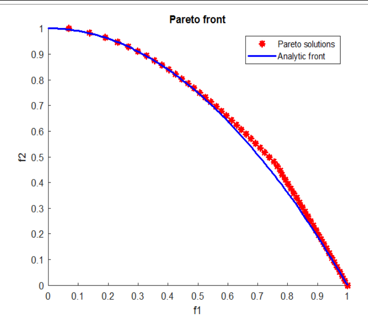
 **Figure 9.** Solutions by SP_MOMA-Plus	 **Figure 10.** Solutions by DP_MOMA-Plus
 **Figure 11.** Solutions by DP^{+}_MOMA-Plus	 **Figure 12.** Solutions by KR_MOMA-Plus
 **Figure 13.** Solutions by L2_MOMA-Plus	 **Figure 14.** Solutions by EP_MOMA-Plus

Table 12. For the problem T_3

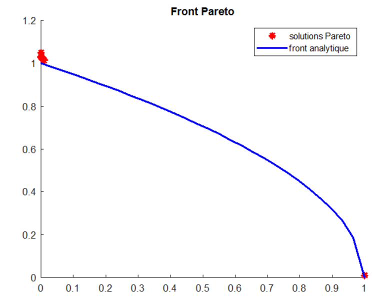


Figure 15. Solutions by SP_MOMA-Plus

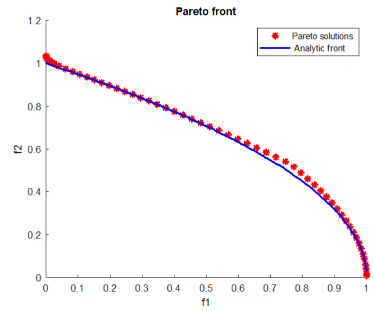


Figure 16. Solutions by DP_MOMA-Plus

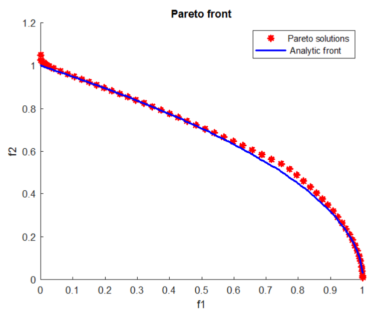


Figure 17. Solutions by DP$^+$_MOMA-Plus

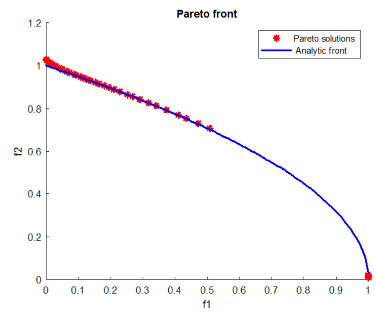


Figure 18. Solutions by KR_MOMA-Plus

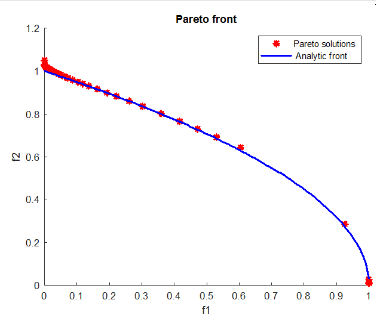


Figure 19. Solutions by L2_MOMA-Plus

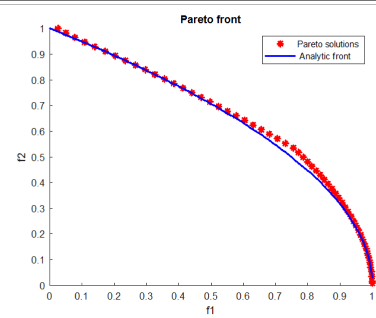


Figure 20. Solutions by EP_MOMA-Plus

Table 13. For the problem T_4

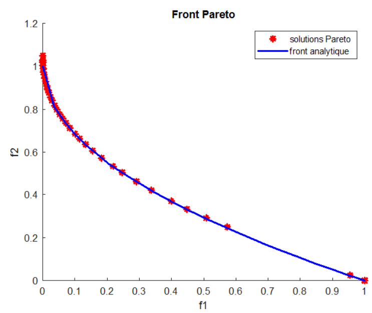


Figure 21. Solutions by SP_MOMA-Plus

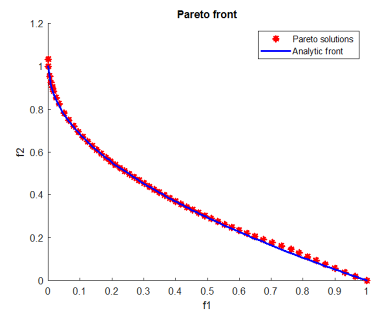


Figure 22. Solutions by DP_MOMA-Plus

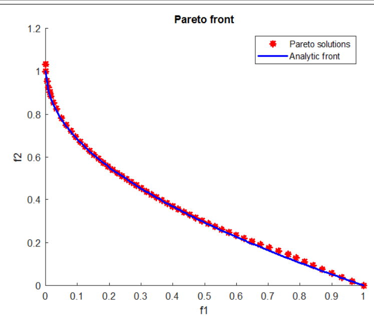


Figure 23. Solutions by DP$^+$_MOMA-Plus

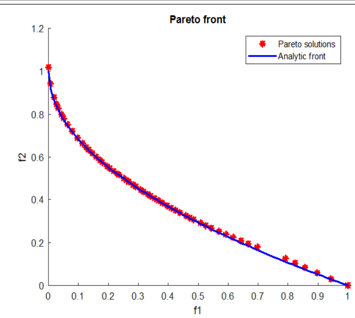


Figure 24. Solutions by KR_MOMA-Plus

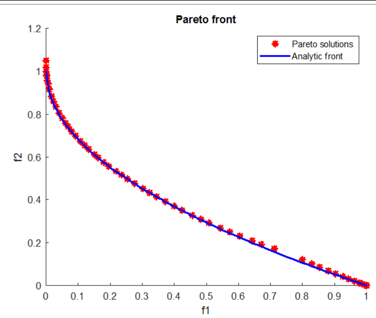


Figure 25. Solutions by L2_MOMA-Plus

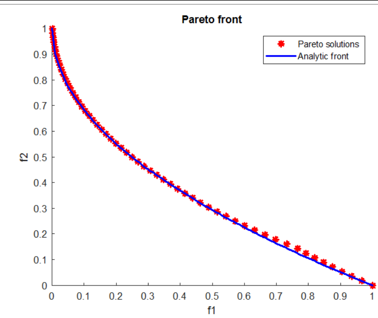


Figure 26. Solutions by EP_MOMA-Plus

Table 14. For the problem T_5

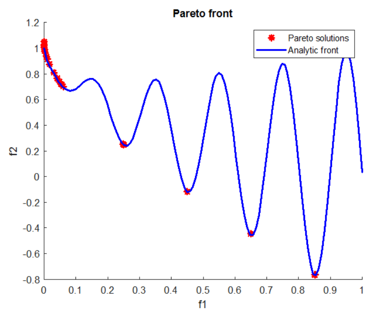


Figure 27. Solutions by SP_MOMA-Plus

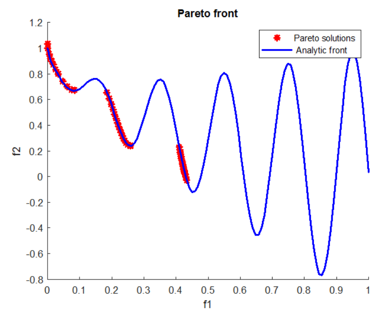


Figure 28. Solutions by DP_MOMA-Plus

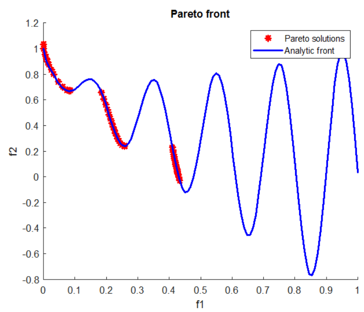


Figure 29. Solutions by DP^{+}_MOMA-Plus

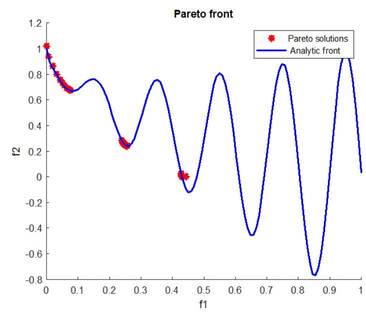


Figure 30. Solutions by KR_MOMA-Plus

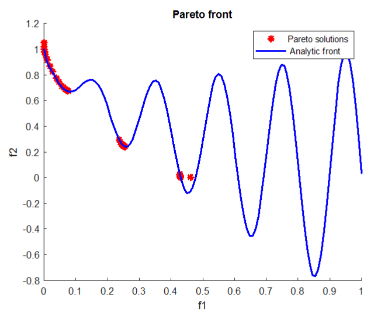


Figure 31. Solutions by L2_MOMA

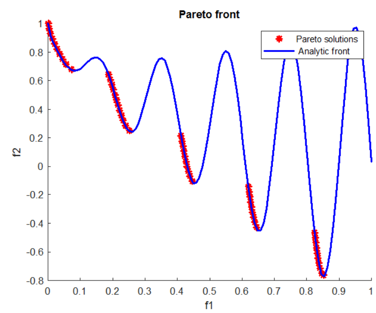


Figure 32. Solutions by EP_MOMA-Plus

Table 15. For the problem T_6

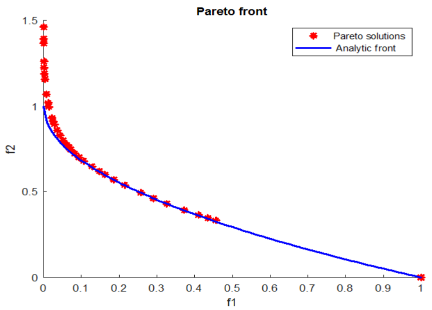


Figure 33. Solutions by SP_MOMA-Plus

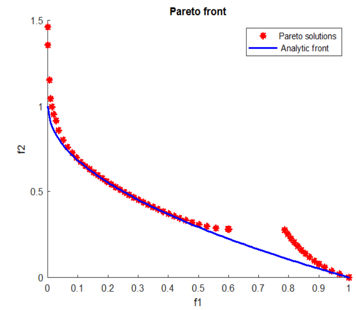


Figure 34. Solutions by DP_MOMA-Plus

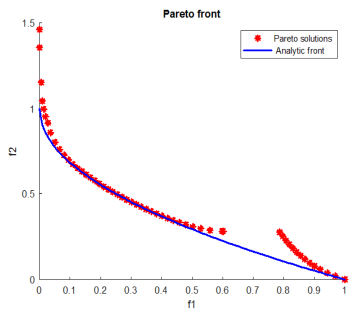


Figure 35. Solutions by DP$^+$_MOMA-Plus

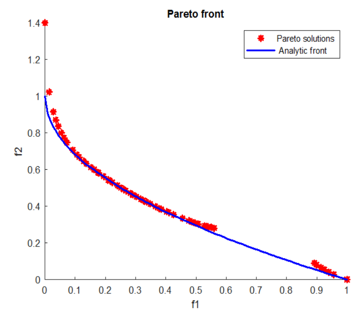


Figure 36. Solutions by KR_MOMA-Plus

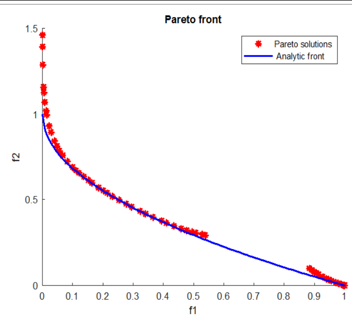


Figure 37. Solutions by L2_MOMA-Plus

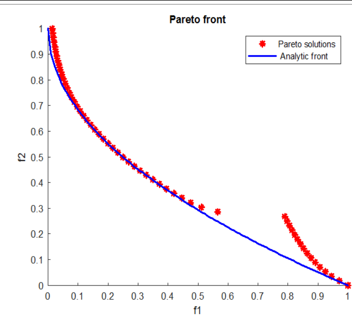


Figure 38. Solutions by EP_MOMA-Plus