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Differential Invariants of Non-degenerate Surfaces

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Abstract

This paper aims to prove that the set $\{g_{ij}(x), L_{ij}(x), i, j = 1, 2\}$ is a complete system of the $SM(3)$-invariants of a nondegenerate surface in R^3, where $\{g_{ij}(x)\}$ and $\{L_{ij}(x)\}$, $i, j = 1, 2$ are the sets of all coefficients of the first and second fundamental forms of a surface x in $\mathbb{R}^3$. A similar result was obtained for the group $M(3)$.

Keywords: Surface; Bonnet's Theorem; Differential Invariant

MSC 2010 No.: 53A07

1. Introduction

We first recall some preliminary information about differential geometry of curves and surfaces. Let U be a connected open subset of $\mathbb{R}^2$. A C^∞-mapping $x : U \to \mathbb{R}^3$ is called a surface in $\mathbb{R}^3$. It is known that a curve in $\mathbb{R}^3$ is uniquely determined by two local invariant quantities, namely curvature and torsion, as functions of arc length. Similarly, a surface in $\mathbb{R}^3$ is uniquely determined by certain local invariant quantities, the first and second fundamental forms. Let $I = Edu^2 + 2Fdudv + Gdv^2$

and $II = Ldu^2 + 2Mdudv + Ndv^2$ be the first and second fundamental forms of a surface x, respectively. If $E(u,v).G(u,v) - F^2(u,v) \neq 0$ for all $(u,v) \in U$, then the surface x is said to be regular. Let $H^*(2)$ be the set of all regular surfaces in $\mathbb{R}^3$. If $L(u,v) \neq 0$ and $N(u,v) \neq 0$ for all $(u,v) \in U$, then the surface x is said to be nondegenerate.

The groups $M(3)$ and $SM(3)$ are defined as $M(3) = \{F : \mathbb{R}^3 \longrightarrow \mathbb{R}^3 \mid Fx = gx + b,\ g \in O(3),\ b \in \mathbb{R}^3\}$, where $O(3)$ is the group of all real orthogonal 3×3-matrices, and $SM(3) = \{Fx = gx + b \in M(3) \mid g \in SO(3)\}$, where $SO(3) = \{g \in O(3) \mid \det(g) = 1\}$.

By Bonnet's theorem (Aminov (2001), Milman and Parker (1977), Kose et al. (2011)), if $x(u,v)$ and $y(u,v)$ are regular surfaces such that

$$\begin{aligned} E_x(u,v) = E_y(u,v), F_x(u,v) = F_y(u,v), G_x(u,v) = G_y(u,v), \\ L_x(u,v) = L_y(u,v), M_x(u,v) = M_y(u,v), N_x(u,v) = N_y(u,v), \end{aligned} \tag{1}$$

for all $(u,v) \in U$, then there exists an $F \in SM(3)$ such that $y(u,v) = Fx(u,v)$ for all $(u,v) \in U$.

Gürsoy and İncesu (2017) investigated the equivalence condition of compared two different control point system under the linear similarity transformations $LS(2)$ in $\mathbb{R}^2$ according to the invariant system of these control points. Aripov and Khadjiev (2007) found the complete system of global differential and integral invariants of a curve in Euclidean geometry.

In our study, we give other complete systems of $SM(3)$-invariants of nondegenerate surfaces and complete systems of $M(3)$-invariants of nondegenerate surfaces. There exists a complete system of differential invariants of surface $x(u,v)$ with four elements (see Alexeevskiy et al. (1990)). The original contribution of our work is to find a general system of invariants and to adapt the studies done for curves to surfaces.

Our paper is organized as follows. In Section 2, we give some elementary definitions and a proposition, which are used later on. In Section 3, we give the definitions of G-invariant differential field $\mathbb{R}^G$ and differential algebra of G-invariant differential polynomial functions $\mathbb{R}x, \Delta_d^{-1}{}^G$, where $x(u_1,u_2)$ is a surface in $\mathbb{R}^3$ and $G = SM(3)$ or $G = M(3)$, and where the function Δ_d is defined as follows:

$$\Delta_d = det\| < y_i, z_j > \|,\ i,j = 1,2,3,$$

and

$$y_1 = z_1 = \frac{\partial x}{\partial u_1}, y_2 = z_2 = \frac{\partial x}{\partial u_2}, y_3 = z_3 = \frac{\partial^2 x}{\partial u_d^2}.$$

Also, we obtain the generating system of the differential field $\mathbb{R}^G$ and the differential algebra $\mathbb{R}x, \Delta_d^{-1}{}^G$. In Section 4, we obtain the complete system of G-invariant differential rational functions of a nondegenerate surface for the groups $G = SM(3)$ and $G = M(3)$.

2. Coefficients of the second fundamental form

Let $< x,y >= x_1y_1 + x_2y_2 + x_3y_3$ be the inner product of two vectors in $\mathbb{R}^3$. Denote by $\det Gr(a_1,a_2)$ the determinant of the Gram matrix $\| < a_k, a_l > \|_{k,l=1}^2$ of the vectors $a_i \in \mathbb{R}^3$. Let

$\{e_1, e_2, e_3\}$ be the standard orthonormal basis in $\mathbb{R}^3$; let $\{a_1, a_2\}$ be a set of vectors in $\mathbb{R}^3$. We consider $a_1 = (a_{11}, a_{21}, a_{31})$, $a_2 = (a_{12}, a_{22}, a_{32})$ and $e = \{e_1, e_2, e_3\}$ as column vectors. Set $P_k := \det \|a_{ij}\|_{i\in\{1,2,3\}/\{k\}}^{j=1,2}$ and $A_k = (-1)^{1+k}P_k$ for $k = 1, 2, 3$. Let $[ea_1a_2] := A_1e_1 + A_2e_2 + A_3e_3 \in \mathbb{R}^3$. For any vectors b, a_1, a_2 in $\mathbb{R}^3$, denote $[ba_1a_2] := det|ba_1a_2|$. It follows from the Lagrange identity that if the vectors $a_1, a_2 \in \mathbb{R}^3$ are linearly independent, then the vector $\bar{n} = \dfrac{[ea_1a_2]}{\sqrt{detGr(a_1, a_2)}}$ is a unit vector and $< [ea_1a_2], a_j >= 0$ for all $j = 1, 2$. On the other hand, let the vectors $a_1, a_2 \in \mathbb{R}^3$ be linearly independent and $b \in \mathbb{R}^3$. Thus, it is obtained that $< \bar{n}, b >= \dfrac{[ba_1a_2]}{\sqrt{detGr(a_1, a_2)}}$.

Proposition 2.1.

Let x be a regular surface in $\mathbb{R}^3$. Then the coefficients of the second fundamental form of x are

$$\begin{aligned} L &= \left[\frac{\partial x}{\partial u_1^2}\frac{\partial x}{\partial u_1}\frac{\partial x}{\partial u_2}\right]\delta_x^{-\frac{1}{2}}, \\ M &= \left[\frac{\partial x}{\partial u_1 \partial u_2}\frac{\partial x}{\partial u_1}\frac{\partial x}{\partial u_2}\right]\delta_x^{-\frac{1}{2}}, \\ N &= \left[\frac{\partial x}{\partial u_2^2}\frac{\partial x}{\partial u_1}\frac{\partial x}{\partial u_2}\right]\delta_x^{-\frac{1}{2}}, \end{aligned} \tag{2}$$

where $\delta_x = detGr(y_1, y_2; z_1, z_2)$, and $y_1 = z_1 = \dfrac{\partial x}{\partial u_1}$, $y_2 = z_2 = \dfrac{\partial x}{\partial u_2}$.

Proof:

It follows from Kreyszig (1991). ■

3. Generating systems of some differential algebras of $M(3)$-invariant and $SM(3)$-invariant differential rational functions of the nondegenerate surfaces

Definition 3.1 (Kaplansky (1957)).

Let $x(u) = x(u_1, u_2)$ be a surface in $\mathbb{R}^3$. Let m_1, m_2 be non-negative integers, we set $x^{(0,0)} = x, x^{(m_1,m_2)} = \dfrac{\partial^{m_1+m_2}x}{\partial u_1^{m_1}\partial u_2^{m_2}}$. Any polynomial $p(x, x^{(1,0)}, x^{(0,1)}, x^{(1,1)}, ..., x^{(m_1,m_2)})$ of x and a finite number of partial derivatives of x with coefficients in $\mathbb{R}$ is called a differential polynomial of x and is denoted by $p\{x\}$.

The set of all differential polynomials of x will be denoted by $\mathbb{R}\{x\}$. It is a differential $\mathbb{R}$-algebra with respect to the derivations $\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}$. This differential $\mathbb{R}$-algebra is also an integral domain. The quotient field of it will be denoted by $\mathbb{R} < x >$. It is a differential field with respect to the derivations $\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}$. An element h of $\mathbb{R} < x >$ will be called a differential rational function of x and denoted by $h < x >$.

Let $x = x(u), y = y(u), \ldots z = z(u)$ be a finite number of surfaces in R^3 and $f_1, f_2, \ldots, f_m \in$

$\mathbb{R} < x, y, \ldots, z >$. A differential polynomial of $x, y, \ldots z, f_1, \ldots, f_m$ is similarly defined; it will be denoted by $p\{x, y, \ldots, z, f_1, \ldots, f_m\}$. The differential $\mathbb{R}$-algebra of all differential polynomials of $x, y, \ldots z, f_1, \ldots, f_m$ is denoted by $\mathbb{R}\{x, y, \ldots z, f_1, \ldots, f_m\}$. The differential field of all differential rational functions of $x, y, \ldots z, f_1, f_2, \ldots, f_m$ is denoted by $\mathbb{R} < x, y, \ldots z, f_1, \ldots, f_m >$.

Clearly, the set $Fx(u)$ is a surface in R^3 for any surface $x(u)$ in R^3 and $F \in M(3)$.

Definition 3.2.

A differential rational function $h < x, y, \ldots, z, f_1, \ldots, f_m >$ will be called G-invariant, where G is a subgroup of $M(3)$, if for all $g \in G$ we have

$$h < gx, gy, \ldots, gz, f_1 < gx, gy, \ldots, gz >, \ldots, f_m < gx, gy, \ldots, gz >>= \\ h < x, y, \ldots, z, f_1 < x, y, \ldots, z >, \ldots, f_m < x, y, \ldots, z >> .$$

The set of all G-invariant differential rational functions of the surfaces $x, y, \ldots, z$ and the functions $f_1, f_2 \ldots, f_m$ will be denoted by

$$\mathbb{R} < x, y, \ldots, z, f_1, \ldots, f_m >^G .$$

It is a differential subfield of $\mathbb{R} < x, y, \ldots, z, f_1, \ldots, f_m >$. The set of all G-invariant differential polynomial functions of $x, y, \ldots, z$ and $f_1, \ldots, f_m$ will be denoted by $\mathbb{R}\{x, y, \ldots, z, f_1, \ldots, f_m\}^G$. It is a differential subalgebra of the differential algebra $\mathbb{R}\{x, y, \ldots, z, f_1, \ldots, f_m\}$ and the differential field $\mathbb{R} < x, y, \ldots, z, f_1, \ldots, f_m >^G$.

Definition 3.3.

Let K be a differential subfield of $\mathbb{R} < x, y, \ldots, z >$. A subset S of K is a generating system of the differential field K if the smallest differential subfield of it containing S is K.

Definition 3.4.

Let $f_1, \ldots, f_m \in \mathbb{R} < x, y, \ldots, z >$ and K be a differential $\mathbb{R}$-subalgebra of $\mathbb{R}\{x, y, \ldots, z, f_1, \ldots, f_m\}$. A subset S of K will be called a generating system of the differential algebra K if the smallest differential subalgebra of it containing S is K.

Let $\mathbb{R}\left\{x, \Delta_d^{-1}\right\}^G$ be the differential algebra of all G-invariant differential polynomial functions of a surface x and the function Δ_d^{-1}, where $\Delta_d^{-1} := det\| < y_i, z_j > \|$, $i, j = 1, 2$ and $y_1 = z_1 = \dfrac{\partial x}{\partial u_1}$, $y_2 = z_2 = \dfrac{\partial x}{\partial u_2}$, $y_3 = z_3 = \dfrac{\partial^2 x}{\partial u_d^2}$. Note that the functions $< x^{(m_1,m_2)}, x^{(p_1,p_2)} >$ are $M(3)$-invariant. Hence the functions Δ_d and Δ_d^{-1} are $M(3)$-invariant. In what follows, $\Delta := \Delta_1$; we investigate properties of the differential algebra $\mathbb{R}\left\{x, \Delta_d^{-1}\right\}^G$ for $d = 1$; the other d being similar.

Theorem 3.5.

The set of elements

$$< \frac{\partial x}{\partial u_i}, \frac{\partial x}{\partial u_j} > \text{ for } 1 \leq i \leq j \leq 2; \;\; \Delta^{-1}; \;\; < \frac{\partial^2 x}{\partial u_1^2}, \frac{\partial^2 x}{\partial u_1 \partial u_r} > \text{ for } r = 1, 2, \tag{3}$$

is a generating system of the differential algebra $\mathbb{R}\left\{x, \Delta^{-1}\right\}^{M(3)}$.

Proof:

Let $\mathbb{R}\left\{\frac{\partial x}{\partial u_1}, \frac{\partial x}{\partial u_2}, \Delta^{-1}\right\}$ be the differential algebra of all differential polynomial functions of $\frac{\partial x}{\partial u_1}, \frac{\partial x}{\partial u_2}$, Δ^{-1} and $\mathbb{R}\left\{\frac{\partial x}{\partial u_1}, \frac{\partial x}{\partial u_2}, \Delta^{-1}\right\}^G$ be the differential algebra of all G-invariant differential polynomial functions of $\frac{\partial x}{\partial u_1}, \frac{\partial x}{\partial u_2}, \; \Delta^{-1}$.

Lemma 3.6.

$\mathbb{R}\left\{x, \Delta^{-1}\right\}^{M(3)} = \mathbb{R}\left\{\frac{\partial x}{\partial u_1}, \frac{\partial x}{\partial u_2}, \Delta^{-1}\right\}^{O(3)}$.

Proof:

It is similar to the proof of Lemma 1 in Khadjiev (2010). ■

Lemma 3.7.

The set

$$\left\{< x^{(m_1,m_2)}, x^{(p_1,p_2)} >\mid m_1 + m_2 \geq 1, p_1 + p_2 \geq 1, m_i, p_i \in \mathbb{N} \cup \{0\}\right\} \tag{4}$$

is a generating system of the differential algebra $\mathbb{R}\left\{\frac{\partial x}{\partial u_1}, \frac{\partial x}{\partial u_2}\right\}^{O(3)}$.

Proof:

It is similar to the proof of Lemma 3 in Khadjiev (2010). ■

Lemma 3.8.

The set

$$\left\{< x^{(m_1,m_2)}, x^{(p_1,p_2)} >, \Delta^{-1} \mid m_1 + m_2 \geq 1, p_1 + p_2 \geq 1, m_i, p_i \in \mathbb{N} \cup \{0\}\right\}$$

is a generating system of the differential algebra $\mathbb{R}\left\{\frac{\partial x}{\partial u_1}, \frac{\partial x}{\partial u_2}, \Delta^{-1}\right\}^{O(3)}$.

Proof:

Let $f \in \mathbb{R}\left\{\frac{\partial x}{\partial u_1}, \frac{\partial x}{\partial u_2}, \Delta^{-1}\right\}^{O(3)}$. Then f can be written in the form $f = \frac{h\left\{\frac{\partial x}{\partial u_1}, \frac{\partial x}{\partial u_2}\right\}}{\Delta(x)^m}$, where $h\left\{\frac{\partial x}{\partial u_1}, \frac{\partial x}{\partial u_2}\right\} \in \mathbb{R}\left\{\frac{\partial x}{\partial u_1}, \frac{\partial x}{\partial u_2}\right\}$ and $m \in \mathbb{N} \cup \{0\}$. Let $g \in O(3)$. Since f is $O(3)$-invariant, we have

$\frac{h\left\{\frac{\partial(gx)}{\partial u_1},\frac{\partial(gx)}{\partial u_2}\right\}}{\Delta(gx)^m}=\frac{h\left\{\frac{\partial x}{\partial u_1},\frac{\partial x}{\partial u_2}\right\}}{\Delta(x)^m}$. Since $\Delta(x)$ is $M(3)$-invariant, we have $h\left\{\frac{\partial(gx)}{\partial u_1},\frac{\partial(gx)}{\partial u_2}\right\}=h\left\{\frac{\partial x}{\partial u_1},\frac{\partial x}{\partial u_2}\right\}$ for all $g\in O(3)$ that is $h\in\mathbb{R}\left\{\frac{\partial x}{\partial u_1},\frac{\partial x}{\partial u_2}\right\}^{O(3)}$. Now Lemma 3.7 implies Lemma 3.8. ∎

Let $V:=\left\{<\frac{\partial x}{\partial u_i},\frac{\partial x}{\partial u_j}>\ \text{for}\ 1\le i\le j\le 2;\left(\frac{\partial^2 x}{\partial u_1^2},\frac{\partial^2 x}{\partial u_1\partial u_r}\right)\ \text{for}\ 1\le r\le 2\right\}$ and $\mathbb{R}\{V\}$ be the differential $\mathbb{R}$-subalgebra of $\mathbb{R}\left\{\frac{\partial x}{\partial u_1},\frac{\partial x}{\partial u_2},\Delta^{-1}\right\}^{O(3)}$ generated by V. Denote by $\mathbb{R}\left\{V,\Delta^{-1}\right\}$ the differential $\mathbb{R}$-subalgebra of $\mathbb{R}\left\{\frac{\partial x}{\partial u_1},\frac{\partial x}{\partial u_2},\Delta^{-1}\right\}^{O(3)}$ generated by elements of V and the function Δ^{-1}. According to Lemmas 3.6 and 3.8, for a proof of our theorem, it is enough to prove that $<x^{(m_1,m_2)},x^{(p_1,p_2)}>\in\mathbb{R}\left\{V,\Delta^{-1}\right\}$ for all $m_i,p_i\in\mathbb{N}\cup\{0\}$ such that $m_1+m_2\ge 1$ and $p_1+p_2\ge 1$.

Let $V_0:=\left\{<\frac{\partial x}{\partial u_i},\frac{\partial x}{\partial u_j}>|\ 1\le i\le j\le 2\right\}$, and $\mathbb{R}\{V_0\}$ be the differential $\mathbb{R}$-subalgebra of $\mathbb{R}\left\{\frac{\partial x}{\partial u_1},\frac{\partial x}{\partial u_2},\Delta^{-1}\right\}^{O(3)}$ generated by elements of V_0. Since $V_0\subset V$, it follows that $\mathbb{R}\{V_0\}\subset\mathbb{R}\{V\}$.

Lemma 3.9.

We have $<\frac{\partial^2}{\partial u_i\partial u_j},\frac{\partial x}{\partial u_l}>\in\mathbb{R}\{V_0\}$ for all $i,j,l\in\{1,2\}$.

Proof:

For all $i,j\in\{1,2\}$, we have $\frac{\partial}{\partial u_j}<\frac{\partial x}{\partial u_i},\frac{\partial x}{\partial u_i}>=2<\frac{\partial^2 x}{\partial u_i\partial u_j},\frac{\partial x}{\partial u_i}>$. This equality implies that $<\frac{\partial^2 x}{\partial u_i\partial u_j},\frac{\partial x}{\partial u_i}>\in\mathbb{R}\{V_0\}$ for all i,j such that $1\le i,j\le 2$. Using the fact that $<\frac{\partial^2 x}{\partial u_i\partial u_j},\frac{\partial x}{\partial u_i}>\in\mathbb{R}\{V_0\}$ and the equality $\frac{\partial}{\partial u_i}<\frac{\partial x}{\partial u_i},\frac{\partial x}{\partial u_j}>=<\frac{\partial^2 x}{\partial^2 u_i},\frac{\partial x}{\partial u_j}>+<\frac{\partial x}{\partial u_i},\frac{\partial^2 x}{\partial u_i\partial u_j}>$, we obtain $<\frac{\partial^2 x}{\partial u_i^2},\frac{\partial x}{\partial u_j}>\in\mathbb{R}\{V_0\}$ for all i,j such that $1\le i,j\le 2$. Assume that $i\neq j,i\neq l,j\neq l$. We have

$$\begin{cases}\frac{\partial}{\partial u_j}<\frac{\partial x}{\partial u_i},\frac{\partial x}{\partial u_l}>=<\frac{\partial^2 x}{\partial u_i\partial u_j},\frac{\partial x}{\partial u_l}>+<\frac{\partial x}{\partial u_i},\frac{\partial^2 x}{\partial u_j\partial u_l}>,\\ \frac{\partial}{\partial u_i}<\frac{\partial x}{\partial u_j},\frac{\partial x}{\partial u_l}>=<\frac{\partial^2 x}{\partial u_i\partial u_j},\frac{\partial x}{\partial u_l}>+<\frac{\partial x}{\partial u_j},\frac{\partial^2 x}{\partial u_i\partial u_l}>,\\ \frac{\partial}{\partial u_l}<\frac{\partial x}{\partial u_i},\frac{\partial x}{\partial u_j}>=<\frac{\partial^2 x}{\partial u_i\partial u_l},\frac{\partial x}{\partial u_j}>+<\frac{\partial x}{\partial u_i},\frac{\partial^2 x}{\partial u_j\partial u_l}>.\end{cases}\tag{5}$$

Put $\frac{\partial}{\partial u_j}<\frac{\partial x}{\partial u_i},\frac{\partial x}{\partial u_l}>=b_1$, $\frac{\partial}{\partial u_i}<\frac{\partial x}{\partial u_j},\frac{\partial x}{\partial u_l}>=b_2$, $\frac{\partial}{\partial u_l}<\frac{\partial x}{\partial u_i},\frac{\partial x}{\partial u_j}>=b_3$, $<\frac{\partial^2 x}{\partial u_i\partial u_j},\frac{\partial x}{\partial u_l}>=w_1$, $<\frac{\partial^2 x}{\partial u_j\partial u_l},\frac{\partial x}{\partial u_i}>=w_2$, $<\frac{\partial^2 x}{\partial u_i\partial u_l},\frac{\partial x}{\partial u_j}>=w_3$. Then the system Equation (4) takes the following form,

$$w_1+w_2=b_1,\ \ w_1+w_3=b_2,\ \ w_2+w_3=b_3.$$

As the system of equations for w_1,w_2,w_3, this system has the unique solution (w_1,w_2,w_3), where $w_1=\frac{1}{2}(b_1+b_2-b_3)\in\mathbb{R}\{V_0\},w_2=\frac{1}{2}(b_1+b_3-b_2)\in\mathbb{R}\{V_0\},w_3=\frac{1}{2}(b_2+b_3-b_1)\in\mathbb{R}\{V_0\}$. ∎

Lemma 3.10.

$\Delta\in\mathbb{R}\{V\}$.

Proof:

By the definition of Δ, we have $\Delta = \det \| < y_i, z_j > \|^3_{i,j=1}$. The definition of V implies that $< y_i, z_j > = < \frac{\partial x}{\partial u_i}, \frac{\partial x}{\partial u_j} > \in V$ for all $1 \leq i, j \leq 2$ and $< y_3, z_3 > = < \frac{\partial^2 x}{\partial u_1^2}, \frac{\partial^2 x}{\partial u_1^2} > \in V$. For $i = 3$ and any $1 \leq j \leq 2$ or $j = 3$ and any $1 \leq i \leq 2$, we have $< y_i, z_j > = < \frac{\partial^2 x}{\partial u_1^2}, \frac{\partial x}{\partial u_j} >$ or $< \frac{\partial^2 x}{\partial u_1^2}, \frac{\partial x}{\partial u_i} >$, respectively. By Lemma 3.9, $< \frac{\partial^2 x}{\partial u_1^2}, \frac{\partial x}{\partial u_i} > \in \mathbb{R}\{V\}$ for all $1 \leq i \leq 2$. Hence $\Delta \in \mathbb{R}\{V\}$. ■

Let $\det Gr(y_1, \ldots, y_m; z_1, \ldots, z_m) := \det || < y_i, z_j > ||^m_{i,j=1}$, where $y_1, y_2, \ldots, y_m, z_1, z_2 \ldots, z_m \in \mathbb{R}^3$.

Lemma 3.11 (Weyl (1946)).

For all vectors $y_1, y_2, y_3, y_4, z_1, z_2, z_3, z_4$ in R^3, we have $\det Gr(y_1, \ldots, y_4; z_1, \ldots, z_4) = \det || < y_i, z_j > ||^4_{i,j=1} = 0$.

Lemma 3.12.

Let $m_1, m_2 \in \mathbb{N} \cup \{0\}$ and $p_1, p_2 \in \mathbb{N} \cup \{0\}$ such that

$$m_1 + m_2 \geq 1, \;\; p_1 + p_2 \geq 1, \quad < x^{(m_1,m_2)}, \frac{\partial^2 x}{\partial u_1^2} >, < x^{(p_1,p_2)}, \frac{\partial^2 x}{\partial u_1^2} > \in \mathbb{R}\left\{V, \Delta^{-1}\right\},$$

and for any i such that $1 \leq i \leq 2$ we have

$$< x^{(m_1,m_2)}, \frac{\partial x}{\partial u_i} >, \;\; < x^{(p_1,p_2)}, \frac{\partial x}{\partial u_i} > \in \mathbb{R}\left\{V, \Delta^{-1}\right\}.$$

Then $< x^{(m_1,m_2)}, x^{(p_1,p_2)} > \in \mathbb{R}\left\{V, \Delta^{-1}\right\}$.

Proof:

Applying Lemma 3.11 to vectors

$$y_1 = z_1 = \frac{\partial x}{\partial u_1}, y_2 = z_2 = \frac{\partial x}{\partial u_2}, y_3 = z_3 = \frac{\partial^2 x}{\partial u_1^2}, y_4 = x^{(m_1,m_2)}, z_4 = x^{(p_1,p_2)}$$

we obtain the equality $\det A = 0$, where $A = \| < y_i, z_j > \|^4_{i,j=1}$. Denote by $D_{4|j}$ the cofactor of the element $< y_4, z_j >$ of the matrix A, where $j = 1, 2, 3, 4$. The equality $\det A = 0$ implies that

$$< y_4, z_1 > D_{4|1} + \cdots + < y_4, z_3 > D_{4|3} + < y_4, z_4 > D_{4|4} = 0. \tag{6}$$

Since $\Delta = D_{4|4} \neq 0$, Equation (6) implies that

$$\begin{aligned} < y_4, z_4 > = < x^{(m_1,m_2)}, x^{(p_1,p_2)} > = \\ -\left(< y_4, z_1 > D_{4|1} + \cdots + < y_4, z_3 > D_{4|3}\right) \Delta^{-1}. \end{aligned} \tag{7}$$

By the assumptions of our lemma, $< y_4, z_3 > = < x^{(m_1,m_2)}, \frac{\partial^2 x}{\partial u_1^2} > \in \mathbb{R}\left\{V, \Delta^{-1}\right\}$ and $< y_4, z_i > = < x^{(m_1,m_2)}, \frac{\partial x}{\partial u_i} > \in \mathbb{R}\left\{V, \Delta^{-1}\right\}$ for all i such that $1 \leq i \leq 2$. We prove that $D_{4|s} \in \mathbb{R}\left\{V, \Delta^{-1}\right\}$ for all s such that $1 \leq s \leq 3$. We have

$$D_{4|s} = (-1)^{4+s} detGr(y_1, y_2, y_3; z_1, \ldots, z_{s-1}, z_{s+1}, \ldots, z_4).$$

By the definition of V, we have $< y_3, z_3 >\in \mathbb{R}\{V\}$ and $< y_i, z_j >\in \mathbb{R}\{V\}$ for all i, j such that $1 \leq i, j \leq 2$. According to Lemma 3.9, we obtain $< y_3, z_j >\in \mathbb{R}\{V\}$ and $< y_i, z_3 >\in \mathbb{R}\{V\}$ for all i, j such that $1 \leq i, j \leq 2$. By the assumptions of our lemma, $< y_i, z_4 >=< \frac{\partial x}{\partial u_i}, x^{(p_1,p_2)} >\in \mathbb{R}\left\{V, \Delta^{-1}\right\}$ for all i such that $1 \leq i \leq 2$ and $< y_3, z_4 >=< \frac{\partial^2 x}{\partial u_1^2}, x^{(p_1,p_2)} >\in \mathbb{R}\left\{V, \Delta^{-1}\right\}$. Hence $D_{4|s} \in \mathbb{R}\left\{V, \Delta^{-1}\right\}$ for all s such that $1 \leq s \leq 3$ and Equation (7) implies that $< y_4, z_4 >\in \mathbb{R}\left\{V, \Delta^{-1}\right\}$. ■

Lemma 3.13.

$< \frac{\partial^2 x}{\partial u_1 \partial u_i}, \frac{\partial^2 x}{\partial u_1 \partial u_j} >\in \mathbb{R}\left\{V, \Delta^{-1}\right\}$ for all i, j such that $1 \leq i, j \leq 2$.

Proof:

By Lemma 3.9, $< \frac{\partial^2 x}{\partial u_1 \partial u_i}, \frac{\partial x}{\partial u_j} >\in \mathbb{R}\{V\}$ for all i, j such that $1 \leq i, j \leq 2$. By the definition of V, $< \frac{\partial^2 x}{\partial u_1 \partial u_i}, \frac{\partial^2 x}{\partial u_1^2} >\in V$ for all i such that $1 \leq i \leq 2$. Hence, using Lemma 3.12, we obtain $< \frac{\partial^2 x}{\partial u_1 \partial u_i}, \frac{\partial^2 x}{\partial u_1 \partial u_i} >\in \mathbb{R}\left\{V, \Delta^{-1}\right\}$ for all i, j such that $1 \leq i, j \leq 2$. ■

Lemma 3.14.

$< \frac{\partial^3 x}{\partial u_1^2 \partial u_i}, \frac{\partial x}{\partial u_j} >\in \mathbb{R}\left\{V, \Delta^{-1}\right\}$ for all i, j such that $1 \leq i, j \leq 2$.

Proof:

For all i, j such that $1 \leq i, j \leq 2$, we have the following equality

$$\frac{\partial}{\partial u_1} < \frac{\partial^2 x}{\partial u_1 \partial u_i}, \frac{\partial x}{\partial u_j} >=< \frac{\partial^3 x}{\partial u_1^2 \partial u_i}, \frac{\partial x}{\partial u_j} > + < \frac{\partial^2 x}{\partial u_1 \partial u_i}, \frac{\partial^2 x}{\partial u_1 \partial u_j} > . \tag{8}$$

By Lemma 3.9, $< \frac{\partial^2 x}{\partial u_1 \partial u_i}, \frac{\partial x}{\partial u_j} >\in \mathbb{R}\{V\}$. By Lemma 3.13, $< \frac{\partial^2 x}{\partial u_1 \partial u_i}, \frac{\partial^2 x}{\partial u_1 \partial u_j} >\in \mathbb{R}\left\{V, \Delta^{-1}\right\}$ for all i, j such that $1 \leq i, j \leq 2$. Hence Equation (8) implies that $< \frac{\partial^3 x}{\partial u_1^2 \partial u_i}, \frac{\partial x}{\partial u_j} >\in \mathbb{R}\left\{V, \Delta^{-1}\right\}$ for all such i, j. ■

Lemma 3.15.

$< \frac{\partial^2 x}{\partial u_1^2}, \frac{\partial^2 x}{\partial u_i \partial u_j} >\in \mathbb{R}\left\{V, \Delta^{-1}\right\}$ for all i, j such that $1 \leq i, j \leq 2$.

Proof:

For all i, j such that $1 \leq i, j \leq 2$, we have

$$\frac{\partial}{\partial u_i} < \frac{\partial^2 x}{\partial u_1^2}, \frac{\partial x}{\partial u_j} >=< \frac{\partial^3 x}{\partial u_1^2 \partial u_i}, \frac{\partial x}{\partial u_j} > + < \frac{\partial^2 x}{\partial u_1^2}, \frac{\partial^2 x}{\partial u_i \partial u_j} > . \tag{9}$$

By Lemma 3.9, we have $< \frac{\partial^2 x}{\partial u_1^2}, \frac{\partial x}{\partial u_j} >\in \mathbb{R}\{V\}$. By Lemma 3.14, it is obtained that $< \frac{\partial^3 x}{\partial u_1^2 \partial u_i}, \frac{\partial x}{\partial u_j} >\in \mathbb{R}\left\{V, \Delta^{-1}\right\}$ for all i, j such that $1 \leq i, j \leq 2$. Hence Equation (9) implies that $< \frac{\partial^2 x}{\partial u_1^2}, \frac{\partial^2 x}{\partial u_i \partial u_j} >\in \mathbb{R}\left\{V, \Delta^{-1}\right\}$ for all $1 \leq i, j \leq 2$. ■

Lemma 3.16.

For all i such that $1 \leq i \leq 2$, we have $< \frac{\partial^2 x}{\partial u_1^2}, \frac{\partial^3 x}{\partial u_1^2 \partial u_i} > \in \mathbb{R}\{V\}$.

Proof:

Since $< \frac{\partial^2 x}{\partial u_1^2}, \frac{\partial^2 x}{\partial u_1^2} > \in V$, the equality $\frac{\partial}{\partial u_i} < \frac{\partial^2 x}{\partial u_1^2}, \frac{\partial^2 x}{\partial u_1^2} >= 2 < \frac{\partial^2 x}{\partial u_1^2}, \frac{\partial^3 x}{\partial u_1^2 \partial u_i} >$ implies that $< \frac{\partial^2 x}{\partial u_1^2}, \frac{\partial^3 x}{\partial u_1^2 \partial u_i} > \in \mathbb{R}\{V\}$ for all $1 \leq i \leq 2$. ∎

Lemma 3.17.

$< x^{(m_1,m_2)}, \frac{\partial x}{\partial u_i} > \in \mathbb{R}\left\{V, \Delta^{-1}\right\}$, $< x^{(m_1,m_2)}, \frac{\partial^2 x}{\partial u_1^2} > \in \mathbb{R}\left\{V, \Delta^{-1}\right\}$ for all $m_1, m_2 \in \mathbb{N} \cup \{0\}$ such that $m_1 + m_2 \geq 1$.

Proof:

We use induction on $r = m_1 + m_2$. Let $r = 1$. Then $< x^{(m_1,m_2)}, \frac{\partial x}{\partial u_i} > \in V \subset \mathbb{R}\{V\}$ by the definition of V, and $< x^{(m_1,m_2)}, \frac{\partial^2 x}{\partial u_1^2} > \in \mathbb{R}\{V\}$ by Lemma 3.9. Hence our lemma holds for $r = 1$.

Assume that $< x^{(m_1,m_2)}, \frac{\partial x}{\partial u_i} > \in \mathbb{R}\left\{V, \Delta^{-1}\right\}$, $< x^{(m_1,m_2)}, \frac{\partial^2 x}{\partial u_1^2} > \in \mathbb{R}\left\{V, \Delta^{-1}\right\}$ for all i such that $1 \leq i \leq 2$ and $m_1, m_2 \in \mathbb{N} \cup \{0\}$ such that $m_1 + m_2 = r$. Let us prove these properties hold for $r+1$. By the inductive hypothesis, we have

$$< x^{(m_1,m_2,\ldots,m_n)}, \frac{\partial x}{\partial u_i} > \in \mathbb{R}\left\{V, \Delta^{-1}\right\} \quad \text{and} \quad < x^{(m_1,m_2)}, \frac{\partial^2 x}{\partial u_1^2} > \in \mathbb{R}\left\{V, \Delta^{-1}\right\} \tag{10}$$

for all i such that $1 \leq i \leq 2$ and $m_1, m_2 \in \mathbb{N} \cup \{0\}$ such that $m_1 + m_2 = r$. By Lemma 3.9, $< \frac{\partial^2 x}{\partial u_i \partial u_j}, \frac{\partial}{\partial u_l} x > \in \mathbb{R}\{V\}$ for all $i, j, l \in \{1, 2\}$, and by Lemma 3.15, $< \frac{\partial^2 x}{\partial u_1^2}, \frac{\partial^2 x}{\partial u_i \partial u_j} > \in \mathbb{R}\left\{V, \Delta^{-1}\right\}$ for all i, j such that $1 \leq i, j \leq 2$. Hence, applying Lemma 3.12 to differential polynomials $x^{(m_1,m_2)}$ and $\frac{\partial^2 x}{\partial u_i \partial u_j}$, we see that $< x^{(m_1,m_2)}, \frac{\partial^2 x}{\partial u_i \partial u_j} > \in \mathbb{R}\left\{V, \Delta^{-1}\right\}$ for all i, j such that $1 \leq i, j \leq 2$. We have

$$\frac{\partial}{\partial u_i} < x^{(m_1,m_2)}, \frac{\partial x}{\partial u_j} >=< \frac{\partial x^{(m_1,m_2)}}{\partial u_i}, \frac{\partial x}{\partial u_j} > + < x^{(m_1,m_2)}, \frac{\partial^2 x}{\partial u_i \partial u_j} > .$$

Since $< x^{(m_1,m_2)}, \frac{\partial x}{\partial u_i} > \in \mathbb{R}\left\{V, \Delta^{-1}\right\}$ and $< x^{(m_1,m_2)}, \frac{\partial^2 x}{\partial u_i \partial u_j} > \in \mathbb{R}\left\{V, \Delta^{-1}\right\}$, this equality implies that $< \frac{\partial x^{(m_1,m_2)}}{\partial u_i}, \frac{\partial x}{\partial u_j} > \in \mathbb{R}\left\{V, \Delta^{-1}\right\}$ for all i, j such that $1 \leq i, j \leq 2$.

By (10) and Lemma 3.14, $< \frac{\partial^3 x}{\partial u_1^2 \partial u_i}, \frac{\partial x}{\partial u_j} > \in \mathbb{R}\left\{V, \Delta^{-1}\right\}$ for all i, j such that $1 \leq i, j \leq 2$ and by Lemma 3.16, $< \frac{\partial^2 x}{\partial u_1^2}, \frac{\partial^3 x}{\partial u_1^2 \partial u_i} > \in \mathbb{R}\{V\}$ for all i such that $1 \leq i \leq 2$. Hence, applying Lemma 3.12 to $x^{(m_1,m_2)}$ and $\frac{\partial^3 x}{\partial u_1^2 \partial u_i}$, we see that $< x^{(m_1,m_2)}, \frac{\partial^3 x}{\partial u_1^2 \partial u_i} > \in \mathbb{R}\left\{V, \Delta^{-1}\right\}$. Since $< x^{(m_1,m_2)}, \frac{\partial^2 x}{\partial u_1^2} > \in \mathbb{R}\left\{V, \Delta^{-1}\right\}$ and $< x^{(m_1,m_2)}, \frac{\partial^3 x}{\partial u_1^2 \partial u_i} > \in \mathbb{R}\left\{V, \Delta^{-1}\right\}$, the equality

$$\frac{\partial}{\partial u_i} < x^{(m_1,m_2)}, \frac{\partial^2 x}{\partial u_1^2} >=< \frac{\partial x^{(m_1,m_2)}}{\partial u_i}, \frac{\partial^2 x}{\partial u_1^2} > + < x^{(m_1,m_2)}, \frac{\partial^3 x}{\partial u_1^2 \partial u_i} >$$

implies that $< \frac{\partial x^{(m_1,m_2)}}{\partial u_i}, \frac{\partial^2 x}{\partial u_1^2} > \in \mathbb{R}\left\{V, \Delta^{-1}\right\}$. Thus we have $< \frac{\partial x^{(m_1,m_2)}}{\partial u_i}, \frac{\partial x}{\partial u_j} > \in \mathbb{R}\left\{V, \Delta^{-1}\right\}$ and $< \frac{\partial x^{(m_1,m_2)}}{\partial u_i}, \frac{\partial^2 x}{\partial u_1^2} > \in \mathbb{R}\left\{V, \Delta^{-1}\right\}$ for all i, j such that $1 \leq i, j \leq 2$. The Lemma is proved. ∎

Lemma 3.18.

$< x^{(m_1,m_2)}, x^{(p_1,p_2)} >\in \mathbb{R}\left\{V, \Delta^{-1}\right\}$ for all $m_1, m_2, p_1, p_2 \in \mathbb{N}\cup\{0\}$ such that $m_1+m_2 \geq 1$, $p_1+p_2 \geq 1$, and $\mathbb{R}\left\{V, \Delta^{-1}\right\} = \mathbb{R}\left\{x, \Delta^{-1}\right\}^{M(3)}$.

Proof:

By Lemma 3.17 and Lemma 3.12, it is obtained that $< x^{(m_1,m_2)}, x^{(p_1,p_2)} >\in \mathbb{R}\left\{V, \Delta^{-1}\right\} \subset \mathbb{R}\left\{x, \Delta^{-1}\right\}^{M(3)}$ for all $m_1, m_2, p_1, p_2 \in \mathbb{N} \cup \{0\}$ such that $m_1 + m_2 \geq 1$ and $p_1 + p_2 \geq 1$. By Lemma 3.8, the system of all elements $< x^{(m_1,m_2)}, x^{(p_1,p_2)} >$, where $m_1, m_2, p_1, p_2 \in \mathbb{N} \cup \{0\}$, $m_1 + m_2 \geq 1$ and $p_1 + p_2 \geq 1$, is a generating system of $\mathbb{R}\left\{x, \Delta^{-1}\right\}^{M(3)}$ as an $\mathbb{R}$-algebra. Hence $\mathbb{R}\left\{V, \Delta^{-1}\right\} = \mathbb{R}\left\{x, \Delta^{-1}\right\}^{M(3)}$. ■

The proof of Theorem 3.5 is completed. ■

Theorem 3.19.

The set of elements

$$< \frac{\partial x}{\partial u_i}, \frac{\partial x}{\partial u_j} >, \text{ where } 1 \leq i \leq j \leq 2; \; < \frac{\partial^2 x}{\partial u_1^2}, \frac{\partial^2 x}{\partial u_1 \partial u_r} >, \text{ where } r = 1, 2, \tag{11}$$

is a generating system of the differential field $\mathbb{R} < x >^{M(3)}$.

Proof:

Let $\mathbb{R} < \frac{\partial x}{\partial u_1}, \frac{\partial x}{\partial u_2} >$ be the differential field of all differential rational functions of $\frac{\partial x}{\partial u_1}, \frac{\partial x}{\partial u_2}$ and $\mathbb{R} < \frac{\partial x}{\partial u_1}, \frac{\partial x}{\partial u_2} >^G$ be the differential field of all G-invariant differential rational functions of $\frac{\partial x}{\partial u_1}, \frac{\partial x}{\partial u_2}$.

Lemma 3.20.

$\mathbb{R} < x >^{M(3)} = \mathbb{R} < \frac{\partial x}{\partial u_1}, \frac{\partial x}{\partial u_2} >^{O(3)}$.

Proof:

It is similar to the proof of Lemma 1 in Khadjiev (2010). ■

Lemma 3.21.

Let $f \in \mathbb{R} < \frac{\partial x}{\partial u_1}, \frac{\partial x}{\partial u_2} >^{O(3)}$. Then there exist $O(3)$-invariant differential polynomials f_1, f_2 such that $f = f_1/f_2$.

Proof:

It is similar to the proof of Proposition 1 in Dieudonné and Carrell (1971). ■

Lemma 3.22.

The set

$$\left\{< x^{(m_1,m_2)}, x^{(p_1,p_2)} >\mid m_1+m_2 \geq 1, \;\; p_1+p_2 \geq 1, \;\; m_i, p_i \in \mathbb{N} \cup \{0\}\right\}$$

is a generating system of the differential field $\mathbb{R} < \frac{\partial x}{\partial u_1}, \frac{\partial x}{\partial u_2} >^{O(n+1)}$.

Proof:

It is similar to the proof of Lemma 3 in Khadjiev (2010). ■

Let V be the system (11). By Lemma 3.10, $\Delta \in \mathbb{R}\{V\} \subseteq \mathbb{R} < V >$. Hence

$$\mathbb{R}\left\{V, \Delta^{-1}\right\} \subseteq \mathbb{R} < V > \subseteq \mathbb{R} < \frac{\partial x}{\partial u_1}, \frac{\partial x}{\partial u_2} >^{O(3)} .$$

Lemmas 3.18 and 3.22 imply $\mathbb{R} < V >= \mathbb{R} < \frac{\partial x}{\partial u_1}, \frac{\partial x}{\partial u_2} >^{O(3)}$, so $\mathbb{R} < V >= \mathbb{R} < x >^{M(3)}$. The proof of Theorem 3.19 is completed. ■

For any set of vectors $\{a_1, a_2, a_3\}$ in R^3, where $a_j = (a_{1j}, a_{2j}, a_{3j})^T$ is a column vector, let $[a_1 a_2 a_3] := \det \|a_{ij}\|_{i,j=1}^3$. For any surface $x(u)$ in R^3, consider $\left[x^{(m_{11},m_{12})} x^{(m_{21},m_{22})} x^{(m_{31},m_{32})}\right]$ and set $\delta = \delta_x := \det Gr(y_1, y_2; z_1, z_2)$, where $y_1 = z_1 = \frac{\partial x}{\partial u_1}, y_2 = z_2 = \frac{\partial x}{\partial u_2}$.

Theorem 3.23.

The set of elements

$$\begin{cases} < \dfrac{\partial x}{\partial u_i}, \dfrac{\partial x}{\partial u_j} > \text{for } 1 \leq i \leq j \leq 2; < \dfrac{\partial^2 x}{\partial u_1^2}, \dfrac{\partial^2 x}{\partial u_1 \partial u_2} >; \\ \qquad \delta^{-1}, \; \Delta^{-1}, \; \left[\dfrac{\partial x}{\partial u_1} \quad \dfrac{\partial x}{\partial u_2} \quad \dfrac{\partial^2 x}{\partial u_1^2}\right], \end{cases} \tag{12}$$

is a generating system of the differential algebra $R\left\{x, \delta^{-1}, \Delta^{-1}\right\}^{SM(3)}$.

Lemma 3.24.

We have $\mathbb{R}\left\{x, \delta^{-1}, \Delta^{-1}\right\}^{SM(3)} = \mathbb{R}\left\{\frac{\partial x}{\partial u_1}, \frac{\partial x}{\partial u_2}, \delta^{-1}, \Delta^{-1}\right\}^{SO(3)}$.

Proof:

It is similar to the proof of Lemma 1 in Khadjiev (2010). ■

Lemma 3.25.

The set of elements

$$\delta^{-1}, \Delta^{-1}, \left[x^{(m_{11},m_{12})} x^{(m_{21},m_{22})} x^{(m_{31},m_{32})}\right], < x^{(p_1,p_2)}, x^{(q_1,q_2)} >, \tag{13}$$

where $m_{i1} + m_{i2} \geq 1$, $p_1 + p_2 \geq 1$, $q_1 + q_2 \geq 1$, is a generating system of the differential algebra $\mathbb{R}\left\{\frac{\partial x}{\partial u_1}, \frac{\partial x}{\partial u_2}, \delta^{-1}, \Delta^{-1}\right\}^{SO(3)}$.

Proof:

Let $\mathbb{R}[x^{(m_1,m_2)}, m_1+m_2 \geq 1]^{SO(3)}$ be the $\mathbb{R}$-algebra of all $SO(3)$-invariant polynomials of all $x^{(m_1,m_2)}$, where $m_1+m_2 \geq 1$. By the First Main Theorem for $SO(3)$ (see [Weyl (1946)], p.45), the system

$$\left[x^{(m_{11},m_{12})}x^{(m_{21},m_{22})}x^{(m_{31},m_{32})}\right], < x^{(p_1,p_2)}, x^{(q_1,q_2)} >,$$

where $m_{i1}+m_{i2} \geq 1, p_1+p_2 \geq 1, q_1+q_2 \geq 1$, is a generating system of $\mathbb{R}[x^{(m_1,m_2)}, m_1+m_2 \geq 1]^{SO(3)}$. This implies, as in Lemma 3.8, that the system Equation (13) is a generating system of $\mathbb{R}\left\{x, \delta^{-1}, \Delta^{-1}\right\}^{SM(3)}$. ∎

Denote by Z the set of elements

$$< \frac{\partial x}{\partial u_i}, \frac{\partial x}{\partial u_j} > \text{ for } 1 \leq i \leq j \leq 2; < \frac{\partial^2 x}{\partial u_1^2}, \frac{\partial^2 x}{\partial u_1 \partial u_2} >; \ \left[\frac{\partial x}{\partial u_1}\frac{\partial x}{\partial u_2}\frac{\partial^2 x}{\partial u_1^2}\right].$$

Then the system (12) has the form $\left\{Z, \delta^{-1}, \Delta^{-1}\right\}$. Let $\mathbb{R}\{Z\}$ be the differential subalgebra of $\mathbb{R}\left\{\frac{\partial x}{\partial u_1}, \frac{\partial x}{\partial u_2}, \delta^{-1}, \Delta^{-1}\right\}^{SO(3)}$ generated by the system Z. Denote by $\mathbb{R}\left\{Z, \delta^{-1}, \Delta^{-1}\right\}$ the differential $\mathbb{R}$-subalgebra of the differential algebra $\mathbb{R} < \frac{\partial x}{\partial u_1}, \frac{\partial x}{\partial u_2} >^{SO(3)}$ generated by the system $\left\{Z, \delta^{-1}, \Delta^{-1}\right\}$.

Lemma 3.26.

$\delta \in \mathbb{R}\{Z\}$.

Proof:

Since $< y_i, z_j > = < \frac{\partial x}{\partial u_i}, \frac{\partial x}{\partial u_j} > \in Z$ for all $1 \leq i, j \leq 2$, we see that $\delta \in \mathbb{R}\{Z\}$. ∎

Lemma 3.27 (Weyl (1946)).

The equality

$$[y_1 y_2 y_3][z_1 z_2 z_3] = \det || < y_i, z_j > ||_{i,j=1}^3$$

holds for all vectors $y_1, y_2, y_3, z_1, z_2, z_3$ in R^3.

Lemma 3.28.

$\Delta \in \mathbb{R}\{Z\}$.

Proof:

Applying Lemma 3.27 to vectors $y_1 = z_1 = \frac{\partial x}{\partial u_1}, y_2 = z_2 = \frac{\partial x}{\partial u_2}, y_3 = z_3 = \frac{\partial^2 x}{\partial u_1^2}$, we obtain

$$\left[\frac{\partial x}{\partial u_1} \quad \frac{\partial x}{\partial u_2} \quad \frac{\partial^2 x}{\partial u_1^2}\right]^2 = \det \| < y_i, z_j > \|_{i,j=1}^3 = \Delta. \tag{14}$$

Since $\left[\frac{\partial x}{\partial u_1} \quad \frac{\partial x}{\partial u_2} \quad \frac{\partial^2 x}{\partial u_1^2}\right] \in Z$, we have $\Delta \in \mathbb{R}\{Z\}$. ∎

By Lemmas 3.24 and 3.25, to prove Theorem 3.23 it suffices to prove that

$$< x^{(p_1,p_2)}, x^{(q_1,q_2)} >, \left[x^{(m_{11},m_{12})}x^{(m_{21},m_{22})}x^{(m_{31},m_{32})}\right] \in \mathbb{R}\left\{Z, \delta^{-1}, \Delta^{-1}\right\},$$

for all $m_{ij}, p_i, q_i \in \mathbb{N} \cup \{0\}$ such that $m_{i1} + m_{i2} \geq 1, p_1 + p_2 \geq 1, q_1 + q_2 \geq 1$.

Lemma 3.29.

$< \frac{\partial^2 x}{\partial u_1^2}, \frac{\partial^2 x}{\partial u_1^2} > \in \mathbb{R}\left\{Z, \delta^{-1}\right\}$ and $V \subset \mathbb{R}\left\{Z, \delta^{-1}\right\}$, where V is the system used in the proof of Theorem 3.5.

Proof:

Denote by $D_{3|j}$, where $j = 1, 2, 3$, the cofactor of the element $< y_3, z_j >$ of the matrix $\|< y_i, z_j >\|_{i,j=1}^3$ in Equation (14). Then Equation (14) implies the equality

$$\Delta =< y_3, z_1 > D_{3|1}+ < y_3, z_2 > D_{3|2}+ < y_3, z_2 > D_{3|2}+ < y_3, z_3 > D_{3|3}. \tag{15}$$

Since $\delta = D_{3|3} \neq 0$, Equation (15) implies that

$$< y_3, z_3 >=< \frac{\partial^2 x}{\partial u_1^2}, \frac{\partial^2 x}{\partial u_1^2} >= \Delta\delta^{-1}- < y_3, z_1 > D_{3|1}\delta^{-1}- < y_3, z_2 > D_{3|2}\delta^{-1}. \tag{16}$$

Since $V_0 \subset Z$, by Lemma 3.9, $< y_3, z_j >=< \frac{\partial^2 x}{\partial u_1^2}, \frac{\partial x}{\partial u_j} > \in \mathbb{R}\{V_0\} \subset \mathbb{R}\{Z\}$ for all $j = 1, \ldots 2$. We prove that $D_{3|s} \in \mathbb{R}\{Z\}$ for all $s = 1, 2$. Since

$$D_{3|s} = (-1)^{3+s} \det Gr(y_1, y_2; z_1, \ldots, z_{s-1}, z_{s+1}, \ldots, z_3),$$

the elements of $D_{3|s}$ have the forms $< y_i, z_j >$, where $i, j \leq 2$, and $< y_k, z_3 >$ for $k \leq 2$. By the definition of Z, $< y_i, z_j > \in Z \subset \mathbb{R}\{Z\}$ for all $i, j \leq 2$. By Lemma 3.9, for all $k \leq 2$ we have $< y_k, z_3 >=< \frac{\partial x}{\partial u_k}, \frac{\partial^2 x}{\partial u_1^2} > \in \mathbb{R}\{V_0\} \subset \mathbb{R}\{Z\}$. Hence Equation (16) implies that $< y_3, z_3 > \in \mathbb{R}\left\{Z, \delta^{-1}\right\}$. Since $V \subset Z \cup \{< y_3, z_3 >\}$, we obtain $V \subset \mathbb{R}\left\{Z, \delta^{-1}\right\}$. ■

Lemma 3.30.

We have $< x^{(p_1,p_2)}, x^{(r_1,r_2)} > \in \mathbb{R}\left\{Z, \delta^{-1}, \Delta^{-1}\right\}$ for all $p_i, r_i \in \mathbb{N} \cup \{0\}$ such that $p_1 + p_2 \geq 1$ and $r_1 + r_2 \geq 1$.

Proof:

By Lemma 3.29, $V \subseteq \mathbb{R}\left\{Z, \delta^{-1}\right\}$. $\mathbb{R}\left\{V, \Delta^{-1}\right\} \subseteq \mathbb{R}\left\{Z, \delta^{-1}, \Delta^{-1}\right\}$ is obtained. Hence, by Lemma 3.18, $< x^{(p_1,p_2)}, x^{(r_1,r_2)} > \in \mathbb{R}\left\{Z, \delta^{-1}, \Delta^{-1}\right\}$ for all $p_i, r_i \in \mathbb{N} \cup \{0\}$ such that $p_1 + p_2 \geq 1$ and $r_1 + r_2 \geq 1$. ■

Lemma 3.31.

$\left[x^{(m_{11},m_{12})}x^{(m_{21},m_{22})}x^{(m_{31},m_{32})}\right] \in \mathbb{R}\left\{Z, \delta^{-1}, \Delta^{-1}\right\}$ for all $m_{ij} \in \mathbb{N} \cup \{0\}$ such that $m_{i1} + m_{i2} \geq 1$ and $i = 1, 2, 3$.

Proof:

Applying Lemma 3.27 to vectors $y_1 = \frac{\partial x}{\partial u_1}, y_2 = \frac{\partial x}{\partial u_2}, \cdots, y_3 = \frac{\partial^2 x}{\partial u_1^2}$;

$z_1 = x^{(m_{11},m_{12})}, z_2 = x^{(m_{21},m_{22})}, z_3 = x^{(m_{31},m_{32})}$, we obtain

$$[y_1y_2y_3][z_1z_2z_3] = \det || < y_i, z_j > ||_{i,j=1}^3 \tag{17}$$

By Equation (14), $\Delta = [y_1y_2y_3]^2$. Using this equality and Equation (17), we obtain

$$[z_1z_2z_3] = \Delta^{-1}[y_1y_2y_3] \det || < y_i, z_j > ||_{i,j=1}^3 \tag{18}$$

By Lemma 3.30, $< y_i, z_j > \in \mathbb{R}\left\{Z, \delta^{-1}\Delta^{-1}\right\}$ for all $1 \leq i, j \leq 3$. Equation (18) implies, since $[y_1y_2y_3] \in Z \subset \mathbb{R}\left\{Z, \delta^{-1}\Delta^{-1}\right\}$, that $[z_1z_2z_3] \in \mathbb{R}\left\{Z, \delta^{-1}\Delta^{-1}\right\}$. ■

Let us finish the proof of our theorem. By Lemma 3.30, $< x^{(p_1,p_2)}, x^{(r_1,r_2)} > \in \mathbb{R}\left\{Z, \delta^{-1}, \Delta^{-1}\right\}$ for all $p_i, r_i \in \mathbb{N} \cup \{0\}$ such that $p_1 + p_2 \geq 1$ and $r_1 + r_2 \geq 1$. By Lemma 3.31, $\left[x^{(m_{11},m_{12})}x^{(m_{21},m_{22})}x^{(m_{31},m_{32})}\right] \in \mathbb{R}\left\{Z, \delta^{-1}, \Delta^{-1}\right\}$ for all $m_{ij} \in \mathbb{N} \cup \{0\}$ such that $m_{i1} + m_{i2} \geq 1$, where $i = 1, 2, 3$. Hence Lemmas 3.24 and 3.25 imply that $\mathbb{R}\left\{Z, \delta^{-1}, \Delta^{-1}\right\} = \mathbb{R}\left\{x(u), \Delta^{-1}\right\}^{SM(3)}$. The proof of Theorem 3.23 is completed. ■

Theorem 3.32.

The set of elements

$$< \frac{\partial x}{\partial u_i}, \frac{\partial x}{\partial u_j} > \quad \text{for } 1 \leq i \leq j \leq 2; \quad < \frac{\partial^2 x}{\partial u_1^2}, \frac{\partial^2 x}{\partial u_1 \partial u_2} >; \; \left[\frac{\partial x}{\partial u_1}\frac{\partial x}{\partial u_2} \quad \frac{\partial^2 x}{\partial u_1^2}\right] \tag{19}$$

is a generating system of the differential field $\mathbb{R} < x(u) >^{SM(3)}$.

Proof:

Use the following lemmas.

Lemma 3.33.

$\mathbb{R} < x >^{SM(3)} = \mathbb{R} < \frac{\partial x}{\partial u_1}, \frac{\partial x}{\partial u_2} >^{SO(3)}$.

Proof:

It is similar to the proof of Lemma 1 in Khadjiev (2010). ■

Lemma 3.34.

Let $f \in \mathbb{R} < \frac{\partial x}{\partial u_1}, \frac{\partial x}{\partial u_2} >^{SO(3)}$. Then there exist $SO(3)$-invariant differential polynomials f_1, f_2 such that $f = f_1/f_2$.

Proof:

It is similar to the proof of Proposition 1 in Dieudonné and Carrell (1971) ■

Lemma 3.35.

The set of all elements

$$\left[x^{(m_{11},m_{12})}x^{(m_{21},m_{22})}x^{(m_{31},m_{32})}\right], < x^{(p_1,p_2)}, x^{(q_1,q_2)} >, \quad (20)$$

where $m_{i1} + m_{i2} \geq 1, p_1 + p_2 \geq 1, q_1 + q_2 \geq 1$, is a generating system of the differential field $\mathbb{R} < \frac{\partial x}{\partial u_1}, \frac{\partial x}{\partial u_2} >^{SO(3)}$.

Proof:

Let $B := \mathbb{R}[x^{(m_1,m_2)} \mid m_1 + m_2 \geq 1]^{SO(3)}$. By the First Main Theorem for $SO(3)$ (see Weyl (1946)), the system Equation (20) is a generating system of B. Lemma 3.34 implies that Equation (20) is a generating system of $\mathbb{R} < \frac{\partial x}{\partial u_1}, \frac{\partial x}{\partial u_2} >^{SO(3)}$. ■

Let Z be the system Equation (19). By Lemmas 3.26 and 3.28, $\delta, \Delta \in \mathbb{R}\{Z\} \subseteq \mathbb{R} < Z >$. Hence $\mathbb{R}\left\{Z, \delta^{-1}, \Delta^{-1}\right\} \subseteq \mathbb{R} < Z > \subseteq \mathbb{R} < \frac{\partial x}{\partial u_1}, \frac{\partial x}{\partial u_2} >^{SO(3)}$. Lemmas 3.30, 3.31 and 3.35 imply that $\mathbb{R} < Z > = \mathbb{R} < \frac{\partial x}{\partial u_1}, \frac{\partial x}{\partial u_2} >^{SO(3)}$. Using Lemma 3.33, we get $\mathbb{R} < Z > = \mathbb{R} < x >^{SM(3)}$. The proof of Theorem 3.32 is now completed. ■

Theorem 3.36.

The set (where $i, j, s = 1, 2,\ 1 \leq i \leq j \leq 2$)

$$< \frac{\partial x}{\partial u_i}, \frac{\partial x}{\partial u_j} >,\ \delta^{-1},\ \Delta^{-1},\ \left[\frac{\partial^2 x}{\partial u_1 \partial u_s}\frac{\partial x}{\partial u_1}\frac{\partial x}{\partial u_2}\right], \quad (21)$$

is a generating system of the differential algebra $\mathbb{R}\left\{x, \delta^{-1}, \Delta^{-1}\right\}^{SM(3)}$.

Proof:

Let

$$W := \left\{< \frac{\partial x}{\partial u_i}, \frac{\partial x}{\partial u_j} >, \left[\frac{\partial^2 x}{\partial u_1 \partial u_s}\frac{\partial x}{\partial u_1}\frac{\partial x}{\partial u_2}\right] \mid i, j, s = 1, 2, 1 \leq i \leq j \leq 2\right\}.$$

Denote by $\mathbb{R}\{W\}$ the differential $\mathbb{R}$-subalgebra of $\mathbb{R} < \frac{\partial x}{\partial u_1}, \frac{\partial x}{\partial u_2} >^{O(3)}$ generated by elements of W and by $\mathbb{R}\left\{W, \delta^{-1}, \Delta^{-1}\right\}$ the differential $\mathbb{R}$-subalgebra of $\mathbb{R} < \frac{\partial x}{\partial u_1}, \frac{\partial x}{\partial u_2} >^{O(3)}$ generated by functions δ^{-1}, Δ^{-1}, and elements of W.

Consider the set $V_0 = \left\{< \frac{\partial x}{\partial u_i}, \frac{\partial x}{\partial u_j} >, 1 \leq i \leq j \leq 2\right\}$ from the proof of Theorem 3.5. Since $V_0 \subset W$ and $\delta \in \mathbb{R}\{V_0\}$, we have $\delta \in \mathbb{R}\{V_0\} \subset \mathbb{R}\{W\}$ and $\mathbb{R}\left\{V_0, \delta^{-1}\right\} \subset \mathbb{R}\left\{W, \delta^{-1}, \Delta^{-1}\right\}$.

By Lemma 3.9, $< \frac{\partial^2 x}{\partial u_i \partial u_j}, \frac{\partial x}{\partial u_l} x > \in \mathbb{R}\{V_0\}$ for all $i, j, l \in \{1, 2\}$. Hence

$$< \frac{\partial^2 x}{\partial u_i \partial u_j}, \frac{\partial x}{\partial u_l} x > \in \mathbb{R}\{W\} \quad \text{for all } i, j, l \in \{1, 2\}. \quad (22)$$

Lemma 3.37.

$< \frac{\partial^2 x}{\partial u_1^2}, \frac{\partial^2 x}{\partial u_1 \partial u_i} > \in \mathbb{R}\left\{W, \delta^{-1}, \Delta^{-1}\right\}$ for all $i = 1, 2$.

Proof:

Applying Lemma 3.27 to vectors $y_1 = \frac{\partial x}{\partial u_1},\ y_2 = \frac{\partial x}{\partial u_2},\ y_3 = \frac{\partial^2 x}{\partial u_1^2}$; $z_1 = \frac{\partial x}{\partial u_1},\ z_2 = \frac{\partial x}{\partial u_2},\ z_3 = \frac{\partial^2 x}{\partial u_1 \partial u_i}$, we obtain

$$[y_1 y_2 y_3][z_1 z_2 z_3] = \det || < y_i, z_j > ||_{i,j=1}^3. \tag{23}$$

For $j = 1, 2, 3$, let $D_{3|j}$ be the cofactor of the element $< y_3, z_j >$ of the matrix $\|< y_i, z_j >\|_{i,j}^3$. Equation (23) implies that

$$[y_1 y_2 y_3][z_1 z_2 z_3] =< y_3, z_1 > D_{3|1}+ < y_3, z_2 > D_{3|2}+ < y_3, z_3 > D_{3|3}.$$

This equality implies the equality

$$< y_3, z_3 > D_{3|3} = [y_1 y_2 y_3][z_1 z_2 z_3]- < y_3, z_1 > D_{3|1}- < y_3, z_2 > D_{3|2}. \tag{24}$$

Since $D_{3|3} = \delta$, Equation (24) implies that

$$< y_3, z_3 >= \delta^{-1}([y_1 y_2 y_3]\, [z_1 z_2 z_3]- < y_3, z_1 > D_{3|1}- < y_3, z_2) D_{3|2}). \tag{25}$$

Since $[y_1 y_2 y_3] \in \mathbb{R}\{W\}$ and $[z_1 z_2 z_3] \in \mathbb{R}\{W\}$, we obtain $[y_1 y_2 y_3]\, [z_1 z_2 z_3] \in \mathbb{R}\{W\}$. By Lemma 3.9, $< y_3, z_i >=< \frac{\partial^2 x}{\partial u_1^2}, \frac{\partial x}{\partial u_i} >\in \mathbb{R}\{V_0\}$ for all $i = 1, 2$. Hence $< y_3, z_i >=< \frac{\partial^2 x}{\partial u_1^2}, \frac{\partial x}{\partial u_i} >\in \mathbb{R}\{W\}$ for all $i = 1, 2$.

We prove that $D_{3|s} \in \mathbb{R}\left\{W, \delta^{-1}, \Delta^{-1}\right\}$ for all $s = 1, 2$. Since

$$D_{3|s} = (-1)^{(3+s)} \det Gr(y_1, y_2; z_1, \ldots, z_{s-1}, z_{s+1}, \ldots, z_3),$$

elements of $D_{3|s}$ have the forms $< y_i, z_j >$, where $i, j \leq 2$, and $< y_k, z_3 >$, where $k \leq 2$. By the definition of W, $< y_i, z_j >\in W \subset \mathbb{R}\{W\}$ for all $i, j \leq 2$. By Lemma 3.9,

$$< y_k, z_3 >=< \frac{\partial x}{\partial u_k}, \frac{\partial^2 x}{\partial u_1 \partial u_i} >\in \mathbb{R}\{V_0\} \subset \mathbb{R}\{W\},$$

for all $k \leq 2$. Hence Equation (25) implies that $< y_3, z_3 >\in \mathbb{R}\left\{W, \delta^{-1}, \Delta^{-1}\right\}$. ■

Lemma 3.37 implies that $Z \subset \mathbb{R}\left\{W, \delta^{-1}, \Delta^{-1}\right\}$, where Z is the system (19). By Theorem 3.23 $\mathbb{R}\left\{Z, \delta^{-1}, \Delta^{-1}\right\} = \mathbb{R}\left\{x, \delta^{-1}, \Delta^{-1}\right\}^{SM(3)}$. Hence $\mathbb{R}\left\{W, \delta^{-1}, \Delta^{-1}\right\} = \mathbb{R}\left\{x, \delta^{-1}, \Delta^{-1}\right\}^{SM(3)}$. The proof of Theorem 3.36 is completed. ■

Theorem 3.38.

The set of elements

$$< \frac{\partial x}{\partial u_i}, \frac{\partial x}{\partial u_j} >, \text{ where } 1 \leq i \leq j \leq 2; \ \left[\frac{\partial^2 x}{\partial u_1 \partial u_j} \frac{\partial x}{\partial u_1} \frac{\partial x}{\partial u_2}\right], \tag{26}$$

is a generating system of the differential field $\mathbb{R} < x >^{SM(3)}$.

Proof:

Let W be the system Equation (26). Since $\Delta = \left[\frac{\partial^2 x}{\partial u_1^2} \frac{\partial x}{\partial u_1} \frac{\partial x}{\partial u_2}\right]^2$, we have $\Delta \in \mathbb{R}\{W\}$. Hence $\Delta^{-1} \in \mathbb{R} < W >$. Since $\delta \in \mathbb{R}\{W\}$, we obtain that $\delta^{-1} \in \mathbb{R} < W >$. So $\mathbb{R}\left\{W, \delta^{-1}, \Delta^{-1}\right\} \subseteq \mathbb{R} < W >$.

Lemma 3.37 implies that

$$Z \subset \mathbb{R}\left\{W, \delta^{-1}, \Delta^{-1}\right\} \subseteq \mathbb{R} < W > \subseteq \mathbb{R} < x >^{SM(3)} .$$

In the proof of Theorem 3.32, it is proved that $\mathbb{R} < Z >= \mathbb{R} < x >^{SM(3)}$. Hence, using the equality $\mathbb{R} < Z >= \mathbb{R} < x >^{SM(3)}$, we obtain $\mathbb{R} < W >= \mathbb{R} < x >^{SM(3)}$. Theorem 3.38 is proved. ∎

Proposition 3.39.

Let $d \in \{1,2\}$ and $x : U \longrightarrow \mathbb{R}^3$ be a d-nondegenerate surface. Then x is a regular surface and $\delta_x(u) > 0$ for all $u \in U$.

Proof:

Let x be a d-nondegenerate surface. Then $L_{dd}(x(u)) \neq 0$ for all $u \in U$. This implies that $[a_1(x)a_2(x)a_3(x)] \neq 0$, where $\{a_i(x) \mid i = 1,2,3\}$ is the set of column vectors, $a_i(x) = \frac{\partial x}{\partial u_i}$ for $1 \leq i \leq 2$ and $a_3(x) = \frac{\partial^2 x}{\partial u_d^2}$. Hence the vectors $a_1(x), a_2(x), a_3(x)$ are linearly independent for all $u \in U$. Then $a_1(x), a_2(x)$ are also linearly independent. This implies that $\det \| < a_i(x), a_j(x) > \|_{i,j=1}^2 = \delta_x(u) \neq 0$ for all $u \in U$. In this case, it is known that $\delta_x(u) > 0$. ∎

Let $\{g_{ij}(x), L_{ij}(x) \mid i,j = 1,2\}$ be the set of all coefficients of the first and second fundamental forms of a surface $x(u)$ in R^3. Assume that $x(u)$ is a d-nondegenerate surface in R^3. Then $\Delta_d \neq 0$ for all $u \in U$. Hence the function Δ_d^{-1} exists. By Proposition 3.39, $\delta_x(u) > 0$. Hence the function $\delta_x(u)^{-\frac{1}{2}}$ exists.

Theorem 3.40.

Let $d \in \{1,2\}$ and $x : U \longrightarrow \mathbb{R}^3$ be a d-nondegenerate surface in R^3. Then the set $\left\{g_{ij}(x), \Delta_d^{-1}, \delta^{-\frac{1}{2}}, L_{dr}(x) \mid i,j,r = 1,2;\ i \leq j\right\}$ is a generating system of the differential algebra $\mathbb{R}\left\{g_{ij}(x), \Delta_d^{-1}, \delta^{-\frac{1}{2}}, L_{ij}(x) \mid i,j = 1,2; i \leq j\right\}$.

Proof:

For $d = 1$, let

$$W_1 := \{g_{ij}(x), L_{1r}(x) \mid i,j,r = 1,2;\ i \leq j\} \quad \text{and} \quad \mathbb{R}\left\{W_1, \Delta^{-1}, \delta^{-\frac{1}{2}}\right\},$$

be the differential $\mathbb{R}$-subalgebra of $\mathbb{R}\left\{g_{ij}(x), \Delta^{-1}, \delta^{-\frac{1}{2}}, L_{ij}(x) \mid i,j = 1,2; i \leq j\right\}$ generated by elements of the system W_1 and functions $\Delta^{-1}, \delta^{-\frac{1}{2}}$.

Using Equation (2), we obtain $\left[\frac{\partial^2 x}{\partial u_1 \partial u_j} \frac{\partial x}{\partial u_1} \frac{\partial x}{\partial u_2}\right] = \delta^{-\frac{1}{2}} L_{1j}(x)$ for all $j = 1,2$. Hence we have $\left[\frac{\partial^2 x}{\partial u_1 \partial u_j} \frac{\partial x}{\partial u_1} \frac{\partial x}{\partial u_2}\right] \in \mathbb{R}\left\{W_1, \Delta^{-1}, \delta^{-\frac{1}{2}}\right\}$ for all $j = 1,2$. This implies $W \subseteq \mathbb{R}\left\{W_1, \Delta^{-1}, \delta^{-\frac{1}{2}}\right\}$, where W is the system Equation (26). Hence $\left\{W, \Delta^{-1}, \delta^{-1}\right\} \subseteq \mathbb{R}\left\{W_1, \Delta^{-1}, \delta^{-\frac{1}{2}}\right\}$. By Theorem 3.36 $\left[\frac{\partial^2 x}{\partial u_i \partial u_j} \frac{\partial x}{\partial u_1} \frac{\partial x}{\partial u_2}\right] \in \left\{W, \Delta^{-1}, \delta^{-1}\right\} \subseteq \mathbb{R}\left\{W_1, \Delta^{-1}, \delta^{-\frac{1}{2}}\right\}$ for all $i,j = 1,2$. Equation (2) implies that $L_{ij} = \delta^{-\frac{1}{2}} \left[\frac{\partial^2 x}{\partial u_i \partial u_j} \frac{\partial x}{\partial u_1} \frac{\partial x}{\partial u_2}\right] \in \mathbb{R}\left\{W_1, \Delta^{-1}, \delta^{-\frac{1}{2}}\right\}$ for all $i,j = 1,2$. Hence $\mathbb{R}\left\{W_1, \Delta^{-1}, \delta^{-\frac{1}{2}}\right\} =$

$\mathbb{R}\left\{g_{ij}(x), \Delta^{-1}, \delta^{-\frac{1}{2}}, L_{ij}(x) \mid i,j=1,2; i \leq j\right\}$. The proof of Theorem 3.40 is completed. ■

4. Complete systems of G-invariants of surfaces

Let G be any subgroup of $M(3)$.

Definition 4.1.

Two surfaces $x, y : U \longrightarrow \mathbb{R}^3$ in R^3 will be called *G-equivalent* if there exists $F \in G$ such that $y(u) = Fx(u)$ for all $u \in U$. In this case, it will be denoted by $x \overset{G}{\sim} y$.

In this section, $A(x) := \|a_1(x)a_2(x)a_3(x)\|$ is the matrix with column vectors $a_i(x) = \frac{\partial x}{\partial u_i}$ for all i such that $1 \leq i \leq 2$, and $a_3(x) = \frac{\partial^2 x}{\partial u_1^2}$. Denote $[a_1(x)a_2(x)a_3(x)] := \det A(x)$.

Any 1-nondegenerate surface in $\mathbb{R}^3$ will be briefly called a *nondegenerate* surface. Let x be a nondegenerate surface in R^3. Since x is a nondegenerate surface, we have $\Delta_x = [a_1(x)a_2(x)a_3(x)]^2 \neq 0$ for all $u \in U$. Hence $[a_1(x)a_2(x)a_3(x)] \neq 0$ for all $u \in U$ and $A(x)^{-1}$ is well-defined.

Theorem 4.2.

Let $x, y : U \longrightarrow \mathbb{R}^3$ be nondegenerate surfaces in R^3.

(1). Let $x \overset{M(3)}{\sim} y$. Then for all i, j, s such that $1 \leq i, j, s \leq 2$ and for all $u \in U$, we have

$$< \frac{\partial x}{\partial u_i}, \frac{\partial x}{\partial u_j} >=< \frac{\partial y}{\partial u_i}, \frac{\partial y}{\partial u_j} >, < \frac{\partial^2 x}{\partial u_1^2}, \frac{\partial^2 x}{\partial u_1 \partial u_s} >=< \frac{\partial^2 y}{\partial u_1^2}, \frac{\partial^2 y}{\partial u_1 \partial u_s} > . \tag{27}$$

(2). Conversely, assume that equalities Equation (27) hold. Then $x \overset{M(3)}{\sim} y$. Moreover, the unique $g \in O(3)$ and the unique $b \in \mathbb{R}^3$ exist such that $y(u) = gx(u) + b$ for all $u \in U$. Explicitly: $g = A(y)A(x)^{-1}$ and $b = y - A(y)A(x)^{-1}x$.

Proof:

(1). Assume that $x \overset{M(3)}{\sim} y$. The functions $< \frac{\partial x}{\partial u_i}, \frac{\partial x}{\partial u_j} >$ and $< \frac{\partial^2 x}{\partial u_1^2}, \frac{\partial^2 x}{\partial u_1 \partial u_s} >$ are $M(3)$-invariant, so equalities Equation (27) hold.

(2). Assume that equalities Equation (27) hold. Equation (27) and Lemma 3.10 imply that $\Delta_x(u) = \Delta_y(u)$ for all $u \in U$. Since x, y are nondegenerate surfaces, it follows that $\Delta_x(u) \neq 0$ and $\Delta_y(u) \neq 0$ for all $u \in U$. Hence $\Delta_x(u)^{-1} = \Delta_y(u)^{-1}$ for all $u \in U$. Let V be the system used in the proof of Theorem 3.5 and $f\{x\} \in \mathbb{R}\left\{V, \Delta^{-1}\right\}$. Then Theorem 3.5, Equation (27) and the equality $\Delta_x(u)^{-1} = \Delta_y(u)^{-1}$ imply that

$$f\{x(u)\} = f\{y(u)\} \quad \text{for all } u \in U. \tag{28}$$

For any s such that $1 \leq s \leq 2$, we set $\frac{\partial A(x)}{\partial u_s} := \left\| \frac{\partial a_1(x)}{\partial u_s} \frac{\partial a_2(x)}{\partial u_s} \frac{\partial a_3(x)}{\partial u_s} \right\|$. Consider the matrix $A(x)^{-1} \frac{\partial A(x)}{\partial u_s} = \left\| p_{ij}^s(x) \right\|$.

Lemma 4.3.

$p^s_{ij}(x) \in \mathbb{R}\left\{V, \Delta^{-1}\right\}$ for all i, j, s such that $1 \leq i, j \leq 3, 1 \leq s \leq 2$.

Proof:

The equality $A(x)^{-1}\frac{\partial A(x)}{\partial u_s} = \left\|p^s_{ij}(x)\right\|$ implies that $A(x)\left\|p^s_{ij}(x)\right\| = \frac{\partial A(x)}{\partial u_s}$. Since x is a nondegenerate surface, we have $\Delta_x(u) = (\det A(x)(u))^2 \neq 0$ for all $u \in U$. Since $\det A(x)(u) \neq 0$, the system $A(x)\left\|p^s_{ij}\{x\}\right\| = \frac{\partial A(x)}{\partial u_s}$ of linear equations has the following solution,

$$p^s_{ij}(x) = \left[a_1(x)\dots a_{i-1}(x)\frac{\partial a_j(x)}{\partial u_s}a_{i+1}(x)\dots a_3(x))\right][a_1(x)a_2(x)a_3(x)]^{-1},$$

where i, j, s such that $1 \leq i, j \leq 3$ and $1 \leq s \leq 2$. This equality implies that

$$p^s_{ij}(x) = \left[a_1(x)\dots a_{i-1}(x)\frac{\partial a_j(x)}{\partial u_s}a_{i+1}(x)\dots a_3(x))\right][a_1(x)a_2(x)a_3(x)]\Delta^{-1}, \tag{29}$$

for all i, j, s such that $1 \leq i, j \leq 3$ and $1 \leq s \leq 2$. Using Lemma 3.27 and Theorem 3.5, we obtain

$$\left[a_1(x)\dots a_{i-1}(x)\frac{\partial a_j(x)}{\partial u_s}a_{i+1}(x)\dots a_3(x))\right][a_1(x)a_2(x)a_3(x)] \in \mathbb{R}\left\{V, \Delta^{-1}\right\}.$$

Since $\Delta^{-1} \in \mathbb{R}\left\{V, \Delta^{-1}\right\}$, it follows that Equation (29) implies that $p^s_{ij}(x) \in \mathbb{R}\left\{V, \Delta^{-1}\right\}$ for all i, j, s such that $1 \leq i, j \leq 3$ and $1 \leq s \leq 2$. ■

Lemma 4.4.

$A(x(u))^{-1}\frac{\partial A(x(u))}{\partial u_s} = A(y(u))^{-1}\frac{\partial A(y(u))}{\partial u_s}$ for all s such that $1 \leq s \leq 2$ and $u \in U$.

Proof:

Using Equations (27), (28) and Lemma 4.3, we obtain $p^s_{ij}(x(u)) = p^s_{ij}(y(u))$ for all $u \in U$ and i, j, s such that $1 \leq i, j \leq 3$ and $1 \leq s \leq 2$. Hence the equality $A(x)^{-1}\frac{\partial A(x)}{\partial u_s} = \left\|p^s_{ij}(x)\right\|$ implies that $A(x(u))^{-1}\frac{\partial A(x(u))}{\partial u_s} = A(y(u))^{-1}\frac{\partial A(y(u))}{\partial u_s}$ for all s such that $1 \leq s \leq 2$ and $u \in U$. ■

Now we complete the proof of our theorem. We have the following equality

$$\begin{aligned}\frac{\partial(A(y)A(x)^{-1})}{\partial u_s} &= \frac{\partial A(y)}{\partial u_s}A(x)^{-1} + A(y)\frac{\partial A(x)^{-1}}{\partial u_s}\\ &= \frac{\partial A(y)}{\partial u_s}A(x)^{-1} - A(y)A(x)^{-1}\frac{\partial A(x)}{\partial u_s}A(x)^{-1}\\ &= A(y)\left(A(y)^{-1}\frac{\partial A(y)}{\partial u_s} - A(x)^{-1}\frac{\partial A(x)}{\partial u_s}\right)A(x)^{-1}\end{aligned}$$

for all s such that $1 \leq s \leq 2$ and $u \in U$. Using this equality and the equality in Lemma 4.4, we see that $\frac{\partial(A(y)A(x)^{-1})}{\partial u_s} = 0$ for all s such that $1 \leq s \leq 2$. Since U is a connected open subset of $\mathbb{R}^2$, using this equality for all s such that $1 \leq s \leq 2$, we see that $A(y(u))A(x(u))^{-1}$ does not depend on $u \in U$. Put $g = A(y)A(x)^{-1}$. Because $\det A_x(u) \neq 0$ and $\det A_y(u) \neq 0$ for all $u \in U$, we have $\det g \neq 0$ and $A(y) = gA(x)$ for all $u \in U$.

Let us prove that $g \in O(3)$. The equality $A(x)^\top A(x) = \|< a_i(x), a_j(x) >\|_{i,j=1}^3$, Lemma 3.17 and Equation (28) imply that $A(x)^\top A(x) = A(y)^\top A(y)$. This and the equality $A(y) = gA(x)$ imply that $g^\top g = I$, where I is the unit matrix. Hence $g \in O(3)$.

The equality $A_y(u) = gA_x(u)$ implies that $\frac{\partial y(u)}{\partial u_s} = g\frac{\partial x(u)}{\partial u_s}$ for all s such that $1 \leq s \leq 2$ and $u \in U$. These equalities imply existence of a vector $b \in \mathbb{R}^3$ such that $y(u) = gx(u) + b$ for all $u \in U$.

Let $y(u) = Dx(u) + c$ for certain $c \in \mathbb{R}^3$ and $D \in O(3)$ and all $u \in U$. Then $\frac{\partial y(u)}{\partial u_i} = D\frac{\partial x(u)}{\partial u_i}$ for all $i = 1, 2$ and $u \in U$. Using these equalities, we see that $A(y(u)) = DA(x(u))$ for all $u \in U$. Hence $D = A(y)A(x)^{-1} = g$. The uniqueness of g is proved. The equalities $y(u) = Dx(u) + c$ and $D = A(y)A(x)^{-1}$ imply that $c = y - A(y)A(x)^{-1}x = b$. Proof of Theorem 4.2 is completed. ∎

Theorem 4.2 means that the system Equation (11) is a complete system of $M(2)$-invariants on the set of all nondegenerate surfaces in R^3.

Theorem 4.5.

Let $x, y : U \longrightarrow \mathbb{R}^3$ be nondegenerate surfaces in $\mathbb{R}^3$. The following are true

(1). Let $x \overset{SM(3)}{\sim} y$. Then for all i, j such that $1 \leq i, j \leq 2$ and any $u \in U$, we have

$$\begin{cases} < \frac{\partial x}{\partial u_i}, \frac{\partial x}{\partial u_j} >=< \frac{\partial y}{\partial u_i}, \frac{\partial y}{\partial u_j} >, < \frac{\partial^2 x}{\partial u_1^2}, \frac{\partial^2 x}{\partial u_1 \partial u_2} >=< \frac{\partial^2 y}{\partial u_1^2}, \frac{\partial^2 y}{\partial u_1 \partial u_2} >, \\ \left[\frac{\partial x}{\partial u_1} \quad \frac{\partial x}{\partial u_2} \quad \frac{\partial^2 x}{\partial u_1^2}\right] = \left[\frac{\partial y}{\partial u_1} \quad \frac{\partial y}{\partial u_2} \quad \frac{\partial^2 y}{\partial u_1^2}\right]. \end{cases} \tag{30}$$

(2). Conversely, assume that equalities Equation (30) hold. Then $x \overset{SM(3)}{\sim} y$. Moreover, the unique $g \in SO(3)$ and the unique $b \in \mathbb{R}^3$ exist such that $y = gx + b$. Explicitly, we have $g = A(y)A(x)^{-1}$ and $b = y - A(y)A(x)^{-1}x$.

Proof:

(1). Assume that $x \overset{SM(3)}{\sim} y$. The functions $< \frac{\partial x}{\partial u_i}, \frac{\partial x}{\partial u_j} >$, $< \frac{\partial^2 x}{\partial u_1^2}, \frac{\partial^2 x}{\partial u_1 \partial u_s} >$ and $\left[\frac{\partial x}{\partial u_1} \quad \frac{\partial x}{\partial u_2} \quad \frac{\partial^2 x}{\partial u_1^2}\right]$ are $SM(3)$-invariant, so equalities Equation (30) hold.

(2). Assume that equalities Equation (30) hold. Let Z be the system Equation (19), $\mathbb{R}\{Z\}$ be the differential $\mathbb{R}$-subalgebra in Theorem 3.23. Let $\delta = \delta_x := \det Gr(v_1, v_2; z_1, z_2)$ where $v_1 = z_1 = \frac{\partial x}{\partial u_1}, v_2 = z_2 = \frac{\partial x}{\partial u_2}$. By Lemma 3.26 and Lemma 3.28, $\delta_x, \Delta_x \in \mathbb{R}\{Z\}$. Hence Equation (30) implies that $\delta_x = \delta_y, \Delta_x = \Delta_y$ for all $u \in U$. Since $x(u), y(u)$ are nondegenerate surfaces, we have $\Delta_x(u) \neq 0$ and $\Delta_y(u) \neq 0$ for all $u \in U$. By Proposition 3.39, $\delta_x(u) > 0$ and $\delta_y(u) > 0$ for all $u \in U$.

The equalities $\delta_x = \delta_y$ and $\Delta_x = \Delta_y$ for all $u \in U$ and Proposition 3.39 imply that $\delta_x^{-1} = \delta_y^{-1}$ and $\Delta_x^{-1} = \Delta_y^{-1}$ for all $u \in U$. Let $f\{x\} \in \mathbb{R}\left\{Z, \delta^{-1}, \Delta^{-1}\right\}$, where $\mathbb{R}\left\{Z, \delta^{-1}, \Delta^{-1}\right\}$ is the differential algebra used in the proof of Theorem 3.23. Then equalities $\delta_x^{-1} = \delta_y^{-1}$ and $\Delta_x^{-1} = \Delta_y^{-1}$ and Equation (30) imply $f(x) = f(y)$ for all $u \in U$. Using Lemma 3.29, Equation (30) and the equality $f(x) = f(y)$, we obtain equalities Equation (27). Hence by Theorem 4.2 there exist the unique $g \in O(3)$

and $b \in \mathbb{R}^3$ such that $y(u) = gx(u) + b$ for all $u \in U$. This equality and Equation (30) imply that

$$\left[\frac{\partial x}{\partial u_1} \quad \frac{\partial x}{\partial u_2} \quad \frac{\partial^2 x}{\partial u_1^2}\right] = \det(g)\left[\frac{\partial y}{\partial u_1} \quad \frac{\partial y}{\partial u_2} \quad \frac{\partial^2 y}{\partial u_1^2}\right].$$

Since $\Delta_x(u) = \left[\frac{\partial x}{\partial u_1} \quad \frac{\partial x}{\partial u_2} \quad \frac{\partial^2 x}{\partial u_1^2}\right]^2 \neq 0$ for all $u \in U$, we see that $\det(g) = 1$. By Theorem 4.2, $g = A(y)A(x)^{-1}$ and $b = y - A(y)A(x)^{-1}x$. The proof of Theorem 4.5 is completed. ■

Theorem 4.5 means that the system Equation (19) is a complete system of $SM(2)$-invariants on the set of all nondegenerate surfaces in $\mathbb{R}^3$.

Theorem 4.6.

Let $d \in \{1, 2\}$ and $x, y : U \longrightarrow \mathbb{R}^3$ be d-nondegenerate surfaces in $\mathbb{R}^3$.

(1). Assume that $x \overset{SM(3)}{\sim} y$. Then for all i, j, s such that $1 \leq i, j, s \leq 2$, where $i \leq j$, and all $u \in U$, we have

$$g_{ij}(x) = g_{ij}(y), \qquad L_{ds}(x) = L_{ds}(y). \tag{31}$$

(2). Conversely, assume that equalities Equation (31) hold. Then $x \overset{SM(3)}{\sim} y$. Moreover, the unique $g \in SO(3)$ and $b \in \mathbb{R}^3$ exist such that $y = gx + b$. Here $g = A(y)A(x)^{-1}$ and $b = y - A(y)A(x)^{-1}x$.

Proof:

(1). Assume that $x \overset{SM(3)}{\sim} y$. The functions $g_{ij}(x)$ and $L_{ds}(x)$, are $SM(3)$-invariant for all $1 \leq i, j, s \leq 2$. So equalities (31) hold.

(2). Assume that equalities (31) hold. We prove the theorem for the case $d = 1$. The case $d = 2$ is similar. Let W_1 be the set and $\mathbb{R}\{W_1\}$ be the differential $\mathbb{R}$-algebra defined in the proof of Theorem 3.40. Let $\delta = \delta_x$ be the function used in the proof of Theorem 3.23. Since $\delta = \det \|g_{ij}\|^2_{i,j=1}$, we have $\delta \in \mathbb{R}\{W_1\}$. Using Equation (2), we obtain $\Delta = \delta(L_{11})^2$. Hence $\Delta \in \mathbb{R}\{W_1\}$. Since $x(u), y(u)$ are nondegenerate surfaces, we have $\Delta_x(u) \neq 0$ and $\Delta_y(u) \neq 0$ for all $u \in U$. By Proposition 3.39, $\delta_x(u) > 0$ and $\delta_y(u) > 0$.

Let $\mathbb{R}\left\{W_1, \delta^{-\frac{1}{2}}, \Delta^{-1}\right\}$ be the differential algebra used in the proof of Theorem 3.40. By Theorem 3.40, $L_{ij} \in \mathbb{R}\left\{W_1, \delta^{-\frac{1}{2}}, \Delta^{-1}\right\}$ for all $i, j = 1, 2$. Using Equation (2), we obtain $\left[\frac{\partial^2 x}{\partial u_i \partial u_j} \frac{\partial x}{\partial u_1} \frac{\partial x}{\partial u_2}\right] = \delta^{-\frac{1}{2}} L_{ij} \in \mathbb{R}\left\{W_1, \delta^{-\frac{1}{2}}, \Delta^{-1}\right\}$ for all $i, j = 1, \ldots n$. This implies $W \subset \mathbb{R}\left\{W_1, \delta^{-\frac{1}{2}}, \Delta^{-1}\right\}$), where W is the set defined in the proof of Theorem 3.36. Hence $\mathbb{R}\{W, \delta^{-1}, \Delta^{-1}\} \subseteq \mathbb{R}\left\{W_1, \delta^{-\frac{1}{2}}, \Delta^{-1}\right\}$. Lemma 3.37 implies that $Z \subset \mathbb{R}\{W, \delta^{-1}, \Delta^{-1}\}$, where Z is the system Equation (19). Hence $\mathbb{R}\{Z\} \subseteq \mathbb{R}\left\{W_1, \delta^{-\frac{1}{2}}, \Delta^{-1}\right\}$.

The equalities $\delta_x = \delta_y$ and $\Delta_x = \Delta_y$ for all $u \in U$ imply that $\delta_x^{-1} = \delta_y^{-1}$ and $\Delta_x^{-1} = \Delta_y^{-1}$ for all $u \in U$. Let $f\{x\} \in \mathbb{R}\{Z\} \subseteq \mathbb{R}\left\{W_1, \delta^{-\frac{1}{2}}, \Delta^{-1}\right\}$. Then equalities $\delta_x^{-1} = \delta_y^{-1}$, $\Delta_x^{-1} = \Delta_y^{-1}$, and Equation (31) imply that

$$f\{x(u)\} = f\{y(u)\}, \tag{32}$$

for all $u \in U$. Since $\mathbb{R}\{Z\} \subseteq \mathbb{R}\left\{W_1, \delta^{-\frac{1}{2}}, \Delta^{-1}\right\})$, Equation (32) implies Equation (30). Then, by Theorem 4.5, $x \overset{SM(3)}{\sim} y$. Moreover, by Theorem 4.5, the unique $g \in SO(3)$ and $b \in \mathbb{R}^3$ exist such that $y = gx + b$, namely $g = A(y)A(x)^{-1}$ and $b = y - A(y)A(x)^{-1}x$. Proof of Theorem 4.6 is completed. ∎

5. Conclusion

The algebra of Euclidean differential invariants for a nondegenerate surface is generated by the mean curvature through invariant differentiation. There are some applications on it. For example, equivalence and signatures of submanifolds, characterization of moduli spaces, invariant differential equations, invariant variational problems etc. Also, the method of moving frame gives the local solution of this problem (Olver (2009)).

In this paper, we give another complete systems of $SM(3)$-invariants of nondegenerate surfaces and complete systems of $M(3)$-invariants of nondegenerate surfaces. There exists a complete system of differential invariants of surface $x(u, v)$ with four elements (see Alexeevskiy et al. (1990)). We give the definitions of G-invariant differential field $\mathbb{R}^G$ and differential algebra of G-invariant differential polynomial functions $\mathbb{R}x, \Delta_d^{-1G}$, where x is a surface in $\mathbb{R}^3$ and $G = SM(3)$ or $G = M(3)$. In here, the function Δ_d is defined as follows:

$$\Delta_d = det\| < y_i, z_j > \|,\ i, j = 1, 2, 3,$$

and

$$y_1 = z_1 = \frac{\partial x}{\partial u_1}, y_2 = z_2 = \frac{\partial x}{\partial u_2}, y_3 = z_3 = \frac{\partial^2 x}{\partial x_d^2}.$$

Also, we obtain generating system of the differential field $\mathbb{R}^G$ and the differential algebra $\mathbb{R}x, \Delta_d^{-1G}$. We obtain the complete system of G-invariant differential rational functions of a nondegenerate surface for the groups $G = SM(3)$ and $G = M(3)$.

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