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Active Control of a Forced Mindlin-type Beam

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Abstract

In this study, optimal dynamic response control of a forced Mindlin-type beam is studied. The beam under consideration, which consists of central host layer and two piezoelectric patch actuators bonded on perfectly to both sides of the beam. It is assumed that the beam is subject to the forcing function, initially at rest and undeformed. Hence, a forced Mindlin-type beam is considered for active vibration control. For this aim, well-posedness and controllability of the system are presented. Performance index functional to be minimized by using minimum level of control voltage consists of a weighted quadratic functions of displacement and velocity of the beam and also includes a quadratic functional of the control function as a penalty term. In order to obtain the optimal control function, an adjoint variable satisfying the adjoint equation corresponding to state equation is defined. A maximum principle is introduced and optimal control function is obtained by means of maximum principle. It is not sensible to use the Linear Quadratic Regulator and Linear Quadratic Gaussian methods to solve the control problem in this paper since the equation under consideration also includes Heaviside function and its spatial derivatives due to existence of piezoelectric patch actuators. Therefore, maximum principle is employed in the present paper Also, by using maximum principle, control problem is reduced to solving a system of partial differential equations including state, adjoint variables, which are linked by initial, boundary and terminal conditions. The solution of this system is obtained by using MATLAB. Numerical results are presented in tables and graphical forms to demonstrate the effectiveness and capability of the introduced control algorithm.

Keywords: Well-posedness; Micro-structure; Optimality; Control; Vibration

MSC 2010 No.: 35Q93, 49J20, 49N05, 74H45

1. Introduction

When the dimensions of a structure become comparable to the size of its material micro-structure, size and micro-structural effects are observed (Polyzos and Fotiadis, 2012). Ignoring the micro-structural effects in a structure leads to underestimation of the structural behavior and lack of accurate results. By becoming aware of this, in 1964 and 1965 Mindlin, who followed the studies of Hencky (1947), Uflyand (1948) and Timoshenko and Goodier (1934), proposed an enhanced general elastic theory to describe linear elastic behavior of isotropic materials with microstructural effects in Mindlin (1964) and Mindlin (1965), respectively, and also he named these as first and second gradient elastic theories of Mindlin. He succeeded by considering the potential energy density as a quadratic form not only of strains but also of gradient of strains and the kinetic energy density as a quadratic form of both velocities and gradient of velocities Polyzos and Fotiadis (2012). Since then, many works dealing with strain gradient elastic theories, derived either from lattice models or homogenization approaches, have appeared in the literature Polyzos and Fotiadis (2012). They are elegant but they are not ability to reproduce entirely the equation of motion as well as the classical and nonclassical boundary conditions appearing in Mindlin theory. In Polyzos and Fotiadis (2012), an equation of motion in one dimension confirming the first and second strain gradient elastic theories of Mindlin are derived. For more information and details about the statements above and Mindlin theory see (Cheung and Zhou (2003), Endo (2015), Gbadeyan and Dada (2006), Mindlin (1964), Mindlin (1965), Polyzos and Fotiadis (2012)). On the other hand, for well modeled structures, vibration control is very important since it increases the lifespan of the structures. Therefore, control of undesirable vibrations is active research area and it has gained much attention by including smart material technology to structures. Piezoelectric actuators among the smart materials are more preferred due to their large band-with, their mechanical simplicity and ability to produce force acting against vibrations in the structures. In using a piezoelectric material as an actuator, converse piezoelectric effect activated by an electric field, is employed for inducing mechanical stresses or strains which, in turn, are transformed into control forces or moments by a suitable structural arrangement Hurlebaus and Gaul (2006). The references, Banks et al. (1996) and Preumont (2002), can provide the general overview about the piezoelectric actuators and smart materials.

Particulary, in this paper, dynamic response control of a forced Mindlin-type beam is considered. In order to achieve the control of the system, well-posedness and controllability of the system are presented. The performance index functional reflects the dynamic response of the beam and it is chosen a sum of the quadratic functional of the displacement and velocity of the beam. Also, a quadratic functional of the control voltage, which is to be applied the piezoelectric actuator, is added to performance index functional as a penalty term. Optimal control function is obtained by deriving a maximum principle (e.g. Kucuk et al. (2015); Yildirim et al. (2017); Rastegar et al. (2013); Kucuk et al. (2014); Yildirim et al. (2016)). The solution of the problem is obtained by means of MATLAB. In order to show the effectiveness and capability of the piezoelectric dynamic response control of the beam, two examples are illustrated. The original contribution of the paper to literature is that in order to obtain the active dynamic response control of a forced Mindlin-type beam, maximum principle is firstly employed in this paper. This paper is organized as follows. In the next section mathematical formulation of the vibrating beam is given and existence and uniqueness of solution to the system under consideration is presented. In Section 3, optimal control problem is defined and adjoint system corresponding to the beam system is introduced and also maximum principle is derived. In the last section, for showing the correctness and validity of the proposed control algorithm, numerical results are presented.

2. Mathematical Formulation of the Control Problem

Let us consider a three layer beam, which consists of central host layer and two piezoelectric patch actuators bonded on perfectly to both sides of the beam. It is assumed that the beam is subject to the forcing function, initially at rest and undeformed. Then, the beam model, which confirming the first and second strain gradient elastic theories of Mindlin, is given by following Polyzos and Fotiadis (2012),

$$w_{xx} - \frac{13\ell^2}{12} w_{xxxx} + \frac{\ell^4}{72} w_{xxxxxx} - \frac{1}{c^2} w_{tt} + \frac{1}{c^2}\frac{\ell^2}{3}\frac{\rho'}{\rho} w_{ttxx} = \mathcal{F}(x,t) + \mathcal{CH}, \tag{1}$$

where x is space variable, t is time variable, $(x,t) \in \Omega = [0,\ell] \times [0,t_f]$, ℓ is the length of the beam, t_f is the fixed terminal time, $w(x,t)$ is the displacement at $L^2(\Omega)$, $c^2 = \frac{E}{\rho}$, E is Youngs modulus, ρ is line density, $\rho' \equiv \rho$ is the density of the micro-structural cells, $\mathcal{F}(x,t) = f_1(x)f_2(t)$ is the forcing function (external excitation) with $f_1(x)$ showing up the distribution of the external excitation over the beam, $\mathcal{CH} = C_v(t)(H''(x-x_1) - H''(x-x_2))$, in which $C_v(t)$ is the control voltage function to be applied the piezoelectric patch actuators, $H(x)$ is Heaviside function, (x_1, x_2) is the location of piezoelectric actuators bonded on the both sides of the beam. Equation (1) is subject to the following boundary conditions,

$$w(x,t) = w_{xx}(x,t) = w_{xxxx}(x,t) = 0 \quad \text{at} \quad x = 0 \quad \text{and} \quad x = \ell, \tag{2}$$

and initial conditions,

$$w(x,t) = w_0(x), \quad w_t(x,t) = w_1(x) \quad \text{at} \quad t = 0. \tag{3}$$

Before dealing with the control of the system, let us show in the next section that the system defined by Equations (1)-(3) has a unique solution and also the system is controllable.

2.1. Well-posedness and Controllability of the System

Let us assume that the following assumptions are valid for the system defined by Equations (1)-(3),

$$w_0(x) \in H^1(0,\ell), \quad w_1(x) \in L^2(0,\ell), \quad w, \quad \frac{\partial^n w}{\partial x^n} \in L^2(\Omega), \quad n = 1,2,...,6, \tag{4a}$$

$$\frac{\partial^j w}{\partial x^n \partial t^m} \in L^2(0,\ell), \quad j = n+m, \quad n = 0,1,2, \qquad m = 0,1,2, \tag{4b}$$

$$C_v(t) \in \mathcal{C}_{ad} = \{C_v(t) \mid C_v(t) \in L^2(0,t_f), \mid C_v(t) \mid < \infty\}, \tag{4c}$$

in which $H^1(0,\ell) = \{w_0(x) \in L^2(0,\ell) : \frac{\partial w_0}{\partial x} \in L^2(0,\ell)\}$ and $L^2(\Omega)$ is the Hilbert space of real-valued square-integrable functions on the domain Ω in the Lebesque sense with usual inner product and norm defined by

$$< \rho, \varrho >_{\Omega}= \iint\limits_{\Omega} \rho \varrho d\Omega, \quad \| \rho \|^2 = < \rho, \rho > .$$

Then, with Equation (4), the system defined by Equations (1)-(3) satisfies the Cauchy-Kovalevsky theorem and the system has a solution (Zachmaonoglou and Thoe, 1986). By the following lemma, let us achieve the uniqueness of the solution.

Lemma 2.1.

The system defined by Equations(1)-(3) has a unique solution.

Proof:

Let us assume that w_1 and w_2 are two solutions to the system under the same conditions. Then the difference $u = w_1 - w_2$ satisfies the following homogeneous initial conditions,

$$u(x,t) = 0, \quad u_t(x,t) = 0 \quad \text{at} \quad t = 0, \tag{5}$$

and boundary conditions,

$$u(x,t) = u_{xx}(x,t) = u_{xxxx}(x,t) = 0 \quad \text{at} \quad x = 0 \quad \text{and} \quad x = \ell, \tag{6}$$

and equation of motion as follows,

$$u_{xx} - \frac{13\ell^2}{12} u_{xxxx} + \frac{\ell^4}{72} u_{xxxxxx} - \frac{1}{c^2} u_{tt} + \frac{1}{c^2} \frac{\ell^2}{3} \frac{\rho'}{\rho} u_{ttxx} = 0. \tag{7}$$

Let us show that u is identically equal to zero. Then, introduce the following energy integral,

$$E(t) = \frac{1}{2} \int\limits_0^{\ell} \left\{ \frac{\partial^2}{\partial x^2}(u^2) - \frac{13}{12}\ell^2 \frac{\partial^4}{\partial x^4}(u^2) + \frac{\ell^4}{72} \frac{\partial^6}{\partial x^6}(u^2) - \frac{1}{c^2}(u_t^2) + \frac{1}{c^2}\frac{\ell^2}{3}\frac{\rho'}{\rho}\frac{\partial^2}{\partial x^2}(u_t^2) \right\} dx, \tag{8}$$

and show that $E(t)$ is independent of t. Differentiating $E(t)$ with respect to t, it is easy to see following equality,

$$\frac{dE(t)}{dt} = \int\limits_0^{\ell} \left\{ \frac{\partial^2}{\partial x^2}(uu_t) - \frac{13}{12}\ell^2 \frac{\partial^4}{\partial x^4}(uu_t) + \frac{\ell^4}{72} \frac{\partial^6}{\partial x^6}(uu_t) - \frac{1}{c^2}(u_t u_{tt}) + \frac{1}{c^2}\frac{\ell^2}{3}\frac{\rho'}{\rho}\frac{\partial^2}{\partial x^2}(u_t u_{tt}) \right\} dx. \tag{9}$$

Integrating by parts and using homogeneous boundary conditions shown by Equation (6), Equation (9) becomes

$$\frac{dE(t)}{dt} = \int\limits_0^{\ell} \left\{ u_{xx} - \frac{13\ell^2}{12} u_{xxxx} + \frac{\ell^4}{72} u_{xxxxxx} - \frac{1}{c^2} u_{tt} + \frac{1}{c^2}\frac{\ell^2}{3}\frac{\rho'}{\rho} u_{ttxx} \right\} u_t dx. \tag{10}$$

Because of the right-hand side of Equation (7), we obtain

$$\frac{dE(t)}{dt} = 0, \quad \text{that is,} \quad E(t) = \text{constant}.$$

Taking the initial conditions given by Equation (5) into consideration, it follows that

$$E(0)=\frac{1}{2}\int_0^\ell \left\{\frac{\partial^2}{\partial x^2}(u^2)-\frac{13}{12}\ell^2\frac{\partial^4}{\partial x^4}(u^2)+\frac{\ell^4}{72}\frac{\partial^6}{\partial x^6}(u^2)-\frac{1}{c^2}(u_t^2)+\frac{1}{c^2}\frac{\ell^2}{3}\frac{\rho'}{\rho}\frac{\partial^2}{\partial x^2}(u_t^2)\right\}\Big|_{t=0} dx=0.$$

Then it is concluded that $u(x,t)$ is identically zero and $u=w_1-w_2=0 \Rightarrow w_1=w_2$. Namely, the system defined by Equations (1)-(3) has a unique solution. ∎

Also, in order to show the Equations (1)-(3) have a unique solution, the equation under consideration can be reduced to ordinary differential equation by means of Galerkin expansion and by considering second order Picard-Lindelof existence-uniqueness theorem, well-posedness of Equations (1)-(3) can be shown. Note that the existence and uniqueness of the solution to Equations (1)-(3) is shown by the previous lemma. By considering Lemma 2.1, it is concluded that in order to preserve the uniqueness of the solution to the system $w(x,t)$, corresponding control voltage $C_v(t)$ function must be unique. Then, it is said that the system under consideration has a unique solution $w(x,t)$ and a unique control function $C_v(t)$. In this case, system defined by Equations (1)-(3) is named as observable. Hilbert Uniqueness method showed that observable is equal to the controllable (Guliyev and Jabbarova (2010), Pedersen (1999)). Namely, the system under consideration is controllable.

3. Problem Definition and Maximum Principle

3.1. Problem Definition

The aim of the control problem defined here is to determine the control voltage function optimally to suppress the undesirable vibrations in the beam. The dynamic response of the beam is modeled as a performance index functional including quadratic functional of the displacement and velocity at a predetermined terminal time t_f and also consisting of the control voltage function accumulating in the control process $[0,t_f]$ as a penalty term. The performance index functional of the beam is defined as follows,

$$\mathcal{J}(C_v)=\mu_1\int_0^\ell w^2(x,t_f)dx+\mu_2\int_0^\ell w_t^2(x,t_f)dx+\mu_3\int_0^{t_f} C_v^2(t)dt, \tag{11}$$

where $\mu_i \geq 0$ for $i=1,2$ such that $\mu_1+\mu_2\neq 0;\quad \mu_3\geq 0$ are weighting constants. The first two terms on left-hand side of Equation (11) are the contribution of the modified energy of the beam, and the last term represents the control effort that accumulates over $[0,t_f]$. Speaking clearly, the objective of the optimal control problem is to find the optimal control voltage function $C_v(t)$ which minimizes the performance index functional at a predetermined terminal time with a minimum expenditure of the control voltage. The set of admissible control functions is defined as follows,

$$\mathcal{C}_{ad}=\{C_v(t)\mid C_v(t)\in L^2(0,t_f), \mid C_v(t)\mid<\infty\},$$

and control of the beam can be stated as follows,

$$\mathcal{J}(C_v^\circ)=\min_{C_v\in\mathcal{C}_{ad}}\mathcal{J}(C_v), \tag{12}$$

subject to Equations (1)-(3). In the next section, a necessary condition to be satisfied by the optimal control function is derived by means of maximum principle in terms of Hamiltonian functional. In Barnes (1971), it is presented that under some convexity assumption, which are satisfied by Equation (11), on performance index function, maximum principle is also sufficient condition for the optimal control function. In order to introduce the maximum principle, it is required to define the adjoint system corresponding to state system given by Equations (1)-(3). Let us define adjoint system as follows,

$$v_{xx} - \frac{13\ell^2}{12} v_{xxxx} + \frac{\ell^4}{72} v_{xxxxxx} - \frac{1}{c^2} v_{tt} + \frac{1}{c^2}\frac{\ell^2}{3}\frac{\rho'}{\rho} v_{ttxx} = 0, \tag{13}$$

with boundary conditions

$$v(x,t) = v_{xx}(x,t) = v_{xxxx}(x,t) = 0 \quad \text{at} \quad x = 0 \quad \text{and} \quad x = \ell, \tag{14}$$

and terminal conditions

$$\frac{1}{c^2}\frac{\ell^2}{3}\frac{\rho'}{\rho} v_{txx}(x,t) - \frac{1}{c^2} v_t(x,t) = 2\mu_1 w(x,t) \quad \text{at} \quad t = t_f, \tag{15a}$$

$$v(x,t) = -2\mu_2 w_t(x,t) \quad \text{at} \quad t = t_f. \tag{15b}$$

3.2. Maximum principle

For the optimal control function $C_v^\circ(t) \in \mathcal{C}_{ad}$, the corresponding optimal state function $w^\circ(x,t) = w(x,t;C_v^\circ)$ satisfies Equations (1)-(3) and the adjoint variable $v^\circ(x,y,t) = v(x,y,t;V_e^\circ)$ satisfies Equations (13)-(15), respectively.

Theorem 3.1.

If

$$\mathcal{H}[x_1,x_2,t;v^\circ,C_v^\circ] = \max_{C_v \in \mathcal{C}_{ad}} \mathcal{H}[x_1,x_2,t;v,C_v], \tag{16}$$

in which the Hamiltonian is defined by the equation

$$\mathcal{H}[x_1,x_2,y_1,y_2,t;v,V_e] = [v_x(x_2,t) - v_x(x_1,t)]C_v(t) - \mu_3 C_v^2(t), \tag{17}$$

then,

$$\mathcal{J}[C_v^\circ] = \min_{C_v \in \mathcal{C}_{ad}} \mathcal{J}[C_v], \quad C_v \in \mathcal{C}_{ad}. \tag{18}$$

Proof:

Let w, w° be two displacement functions corresponding to control function $C_v(t)$ and $C_v^\circ(t)$, respectively. Also, define the difference between them as follows,

$$\Delta w = w - w^\circ, \quad \Delta C_v(t) = C_v(t) - C_v^\circ(t). \tag{19}$$

Let us define the operator Φ

$$\Phi(\Delta w) = \Delta w_{xx} - \frac{13\ell^2}{12}\Delta w_{xxxx} + \frac{\ell^4}{72}\Delta w_{xxxxxx} - \frac{1}{c^2}\Delta w_{tt} + \frac{1}{c^2}\frac{\ell^2}{3}\frac{\rho'}{\rho}\Delta w_{ttxx}, \tag{20}$$

and its adjoint operator Φ^* as follows,

$$\Phi^*(v) = v_{xx} - \frac{13\ell^2}{12}v_{xxxx} + \frac{\ell^4}{72}v_{xxxxxx} - \frac{1}{c^2}v_{tt} + \frac{1}{c^2}\frac{\ell^2}{3}\frac{\rho'}{\rho}v_{ttxx}. \quad (21)$$

Operator Φ is subject to the following boundary conditions,

$$\Delta w(x,t) = \Delta w_{xx}(x,t) = \Delta w_{xxxx}(x,t) = 0 \quad \text{at} \quad x = 0 \quad \text{and} \quad x = \ell, \quad (22)$$

and initial conditions,

$$\Delta w(x,t) = 0, \quad \Delta w_t(x,t) = 0 \quad \text{at} \quad t = 0. \quad (23)$$

Also, operator Φ^* is subject to the Equations (14)-(15). Then, by using the definitions of Φ and Φ^* next relation can be obtained,

$$\iint_\Omega \Big\{ v\Phi(\Delta w) - \Delta w \Phi^*(v) \Big\} d\Omega = \iint_\Omega \Big\{ v\Delta C_v(t)[H''(x-x_1) - H''(x-x_2)] \Big\} d\Omega. \quad (24)$$

After applying six times integration by parts to left-side of Equation (24) and using boundary conditions Equation (22) and Equation (14) and terminal conditions Equation (15), one obtains

$$\iint_\Omega \Big\{ v\Phi(\Delta w) - \Delta w \Phi^*(v) \Big\} d\Omega = -2\int_0^\ell \Big\{ \mu_1 w(x,t_f \Delta w(x,t_f) + \mu_2 w_t(x,t_f)\Delta w_t(x,t_f) \Big\} dx. (25)$$

For the right hand-side of Equation (24), remember the dirac-delta function, which has the following properties,

$$H''(x-\theta) = \delta'(x-\theta), \quad \int_0^\ell \delta'(x-\theta)\varphi(x)dx = -\varphi'(\theta), \quad \theta \in (0,\ell). \quad (26)$$

By means of Equation (26), the right-hand side of Equation (24) is found as follows,

$$\iint_\Omega v\Delta C_v(t)[\delta'(x-x_1) - \delta'(x-x_2)]d\Omega = \int_0^{t_f} \Delta C_v(t)[v_x(x_2,t) - v_x(x_1,t)]dt. \quad (27)$$

Here, give attention to the deflection of the performance index functional, which is given by

$$\begin{aligned}\Delta\mathcal{J}[C_v] &= \mathcal{J}[C_v] - \mathcal{J}[C_v^\circ] \\ &= \int_0^\ell \Big\{ \mu_1[w^2(x,t_f) - w^{\circ^2}(x,t_f)] + \mu_2[w_t^2(x,t_f) - w_t^{\circ^2}(x,t_f)] \Big\} dx + \int_0^{t_f} \mu_3[C_v^2 - C_v^{\circ^2}]dt.\end{aligned} \quad (28)$$

Expanding $w^2(x,t_f)$ and $w_t^2(x,t_f)$ into Taylor series about $w^{\circ^2}(x,t_f)$ and $w_t^{\circ^2}(x,t_f)$, respectively, leads to the following relation,

$$w^2(x,t_f) - w^{\circ^2}(x,t_f) = 2w^\circ(x,t_f)\Delta w^\circ(x,t_f) + \gamma_1, \quad (29a)$$

$$w_t^2(x,t_f) - w_t^{\circ^2}(x,t_f) = 2w_t^\circ(x,t_f)\Delta w_t^\circ(x,t_f) + \gamma_2, \quad (29b)$$

where

$$\gamma_1 = 2(\Delta w)^2 + \quad \text{higher order terms} > 0,$$
$$\gamma_2 = 2(\Delta w_t)^2 + \quad \text{higher order terms} > 0.$$

Substituting Equation (29) into Equation (28) yields

$$\Delta \mathcal{J}[C_v] = \int_0^{\ell} \Big\{ \mu_1[2w^\circ(x,t_f)\Delta w(x,t_f) + \gamma_1] + \mu_2[2w_t^\circ(x,t_f)\Delta w_t(x,t_f) + \gamma_2] \Big\} dS + \int_0^{t_f} \mu_3[C_v^2 - C_v^{\circ^2}]dt.$$

From Equation (25) and because of $\mu_1\gamma_1 + \mu_2\gamma_2 > 0$, one observes

$$\Delta \mathcal{J}[C_v] \geq \int_0^{t_f} \Big\{ \Delta C_v(t)[v_x(x_1,t) - v_x(x_2,t)] + \mu_3(C_v^2(t) - C_v^{\circ^2}(t)) \Big\} dt \geq 0.$$

Then,

$$C_v(t)[v_x(x_1,t) - v_x(x_2,t)] + \mu_3 C_v^2(t) \geq C_v^\circ(t)[v_x^\circ(x_1,t) - v_x^\circ(x_2,t)] + \mu_3 C_v^{\circ^2}(t).$$

Namely,

$$\mathcal{H}[x_1,x_2,t;v^\circ,C_v^\circ] = \max_{C_v \in \mathcal{C}_{ad}} \mathcal{H}[x_1,x_2,t;v,C_v],$$

and

$$\mathcal{J}[C_v^\circ] = \min_{C_v \in \mathcal{C}_{ad}} \mathcal{J}[C_v], \quad C_v \in \mathcal{C}_{ad}. \tag{30}$$

By taking the first variation of $\mathcal{H}$, control function $C_v(t)$ is obtained optimally as follows,

$$C_v(t) = \frac{v_x(x_2,t) - v_x(x_1,t)}{2\mu_3}. \tag{31}$$

■

4. Numerical results

In this section, theoretical results obtained in previous chapters are presented in table and graphical forms. The solution of the system of partial differential equation including Equations (1)-(3) and Equations (13)-(15) is found by means of MATLAB. In the simulations, terminal time is considered as $t_f = 5$. Also, weighting factors are considered as $\mu_1 = \mu_2 = 1$ and $\mu_3 = 10^{-3}$ for the controlled case. The length of the beam is $\ell = 1m$ and the values of the displacement and velocity of the beam are calculated at the midpoint of the beam $x = 0.5m$. Young's modulus of the beam E is taken into account as 2×10^7 and line density of the beam ρ is considered as 6×10^4. The response of the beam is examined in two cases. In case A, the beam is subject to the following initial conditions and external excitation,

$$w(x,0) = \sqrt{2}\sin(\pi x), \quad w_t(x,0) = \sqrt{2}\sin(\pi x), \quad \mathcal{F}(x,t) = (1-x)\exp(t/2).$$

Displacement of the beam for the case A is plotted in the Figure 1 and it is easily concluded that vibrations are effectively damped out due to applied control algorithm by using minimum level of control voltage. Also, controlled and uncontrolled velocities of the beam corresponding to case A are simulated in Figure 2 and same observation is valid. In case B, the beam is assumed that it is induced by following initial conditions and external excitation,

$$w(x,0)=0, \quad w_t(x,0)=0, \quad \mathcal{F}(x,t)=(1-x)\exp(-t).$$

Midpoint displacement and velocity of the beam are presented in Figures 3 and 4 and the differences between the un/controlled displacement and un/controlled velocity show the robustness of the control. Now, focus on the bandwith of the Figures 1 and 2 and Figures 3 and 4. As it is shown, the bandwidth of the displacement and velocity corresponding to case A and B is not same. The reason of this is that in case A, the beam is induced by means of bigger initial conditions and external excitation than case B. Let us introduce two functionals defining the dynamic response of the beam

$$\mathcal{J}(w)=\int_0^1 w^2(x,t_f)dx+\int_0^1 w_t^2(x,t_f)dx, \tag{32}$$

and used control accumulates over $(0,t_f)$,

$$\mathcal{J}(C_v)=\int_0^{t_f} C_v^2(t)dt. \tag{33}$$

Note that Equation (32) is corresponding to the Equation (11) in case of $\mu_1=\mu_2=1$ and $\mu_3=0$. In Table 1, the dynamic response of the beam and spent control amount are presented for both case A and case B by using different values of the weighting factor μ_3. For these cases, it is concluded from table1 that as the penalty μ_3 on the control function decreases, the dynamic response of the beam decreases corresponding to an increase in the control voltage applied to the piezoelectric patch actuators. Also, the total amount of the applied control for the case A and B is different than each other since in case B, the beam is subjected to smaller conditions than case A. Observing both figures and table, it is concluded that undesirable vibrations in the beam are suppressed effectively by introduced control algorithm.

Table 1. The values of $\mathcal{J}(w)$ and $\mathcal{J}(\mathcal{C}_v)$ for different values of μ_3 in case A and B.

μ_3	$\mathcal{J}_a(w)$	$\mathcal{J}_a(\mathcal{C}_v)$
10^3	415	8 e-3
10^2	304	6 e-1
10^1	52	10
10^0	1.3	35
10^{-1}	2.2 e-2	83
10^{-2}	2.5 e-4	98
10^{-3}	2.5 e-6	100

μ_3	$\mathcal{J}_b(w)$	$\mathcal{J}_b(\mathcal{C}_v)$
10^3	2.8 e-5	5.0 e-10
10^2	2.0 e-5	4.0 e-8
10^1	3.0 e-6	7.0 e-7
10^0	9.0 e-8	2.1 e-5
10^{-1}	1.0 e-9	5.0 e-5
10^{-2}	1.5 e-11	5.7 e-5
10^{-3}	1.5 e-13	6.0 e-5

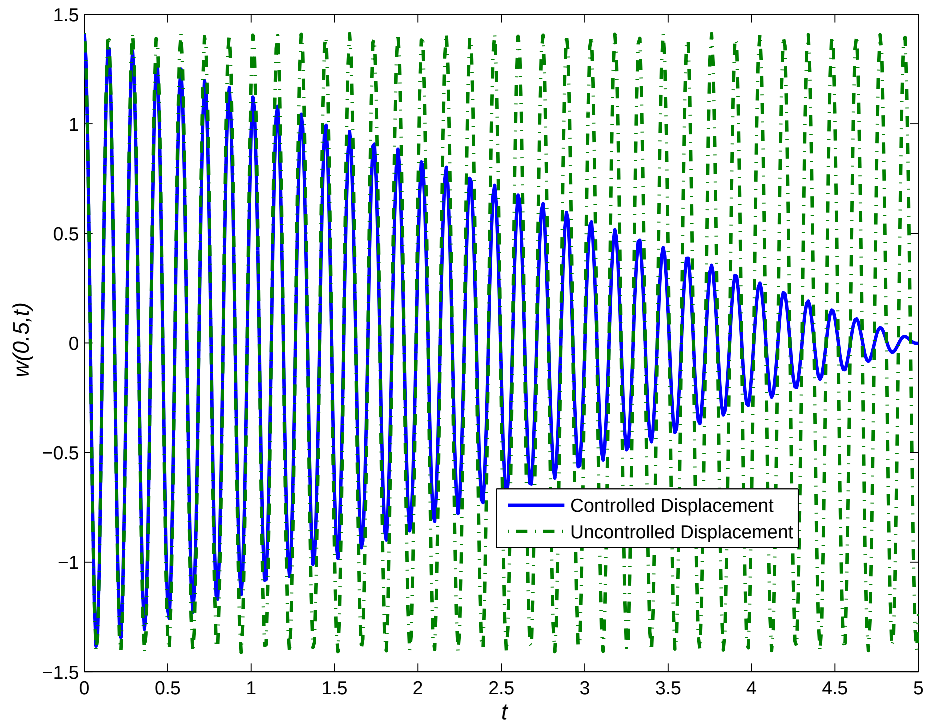

Figure 1. Controlled and uncontrolled displacements at $x = 0.5m$ for case A.

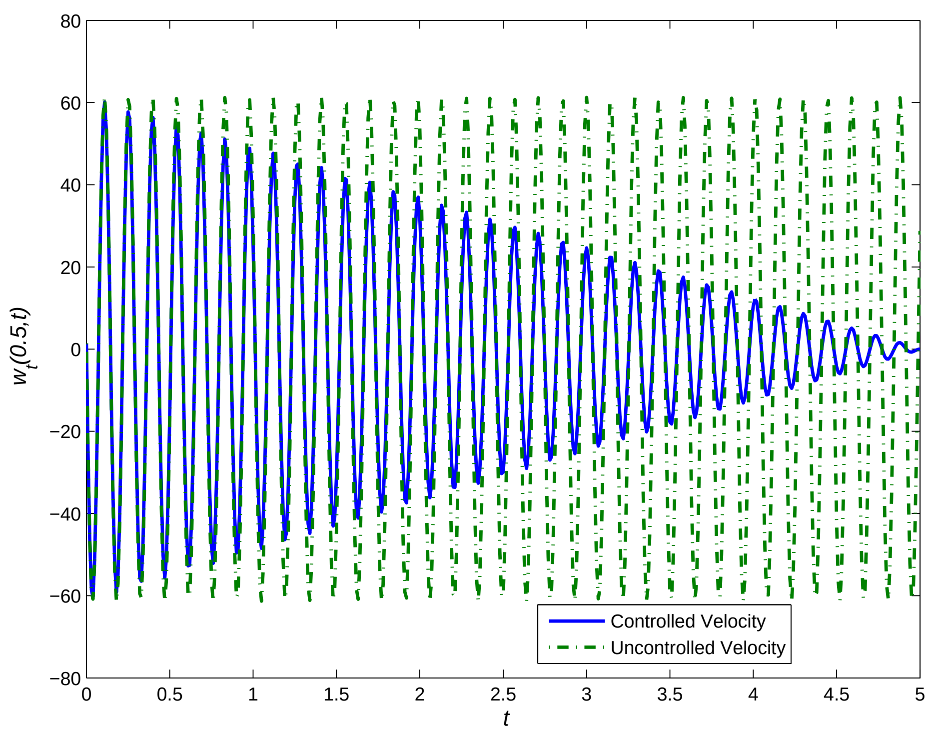

Figure 2. Controlled and uncontrolled velocities at $x = 0.5m$ for case A.

5. Conclusion

In this paper, optimal control of a beam is studied by means of Maximum principle. The performance index functional is chosen a sum of the quadratic functional of the displacement and

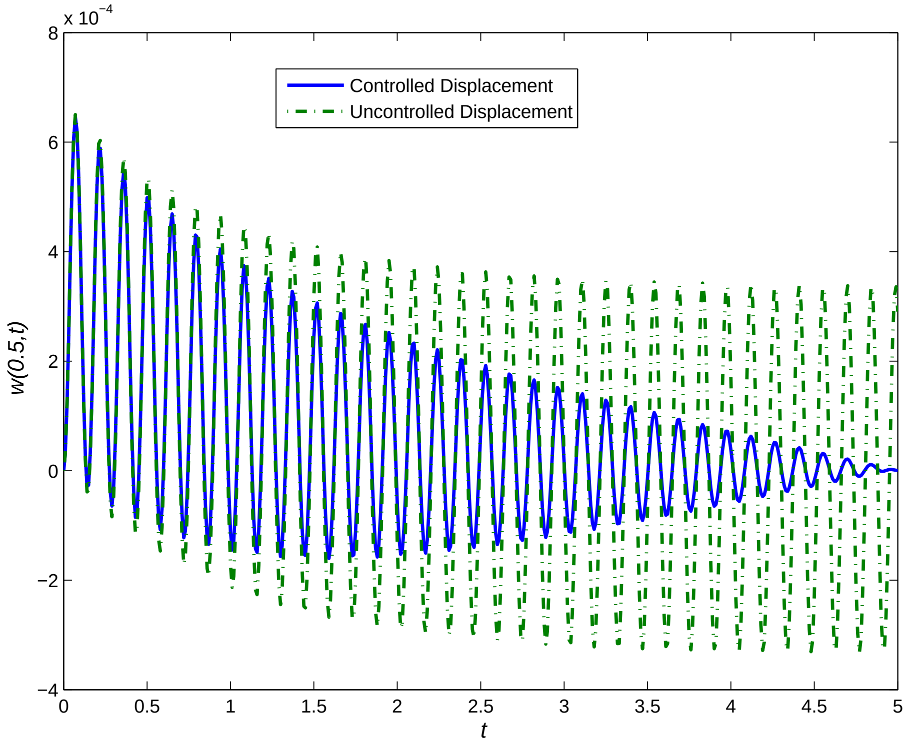

Figure 3. Controlled and uncontrolled displacements at $x = 0.5m$ for case B.

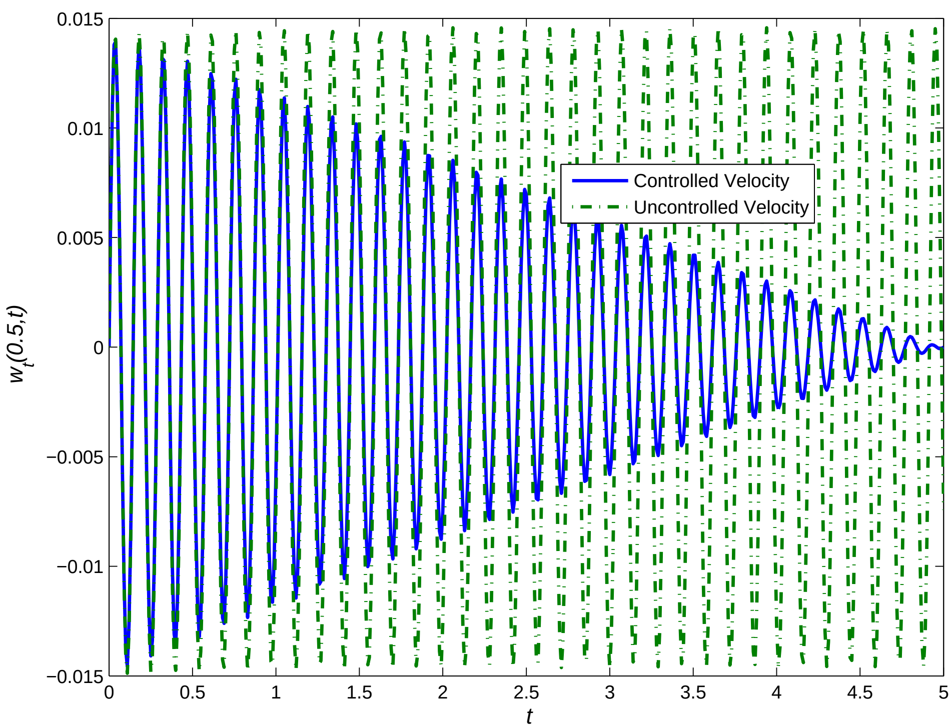

Figure 4. Controlled and uncontrolled velocities at $x = 0.5m$ for case B.

velocity of the beam and performance index functional also includes a quadratic functional of the control as a penalty term. In order to obtain control function analytically, an adjoint variable satisfying the adjoint equation corresponding to state equation is introduced. Introducing the maximum principle, optimal control problem is transformed to the solving a system of partial differential equations including state and adjoint variables, which are linked by terminal-boundary and initial conditions. The solution of this system is obtained by `MATLAB`. In order to show the effectiveness of the control, two cases are examined and results are presented in tables and graphical forms. By observing results, it is concluded that control for suppressing the undesirable vibrations in the beam is very effective and it can be extended to other beam models in different conditions.

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