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Birkhoff's Ergodic Theorem For Weighted Variable Exponent Amalgam Spaces

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Abstract

In this study, we consider some properties of weighted variable exponent Lebesgue and amalgam spaces. It is known these spaces are considerably used in harmonic and time-frequency analysis including elastic mechanics, electrorheological fluids, image processing, etc. Ergodic theory investigates the long-term averaging properties of measure preserving dynamical systems. This theory has also several applications and problems of statistical physics and mechanics. Moreover, it has influence on many areas of mathematics, especially probability theory and dynamical systems as well as Fourier analysis, functional analysis, and group theory. Therefore, we investigate Ergodic theorem for unweighted variable exponent Lebesgue spaces and also an amalgam space whose local component is weighted one.

Keywords: Weighted Variable Exponent Lebesgue and Amalgam Space; Ergodic Theorem; Probability Measure

MSC 2010 No.: 28D05, 43A15, 46E30

1. Introduction

The variable exponent Lebesgue spaces $L^{p(.)}$ appeared in literature for the first time in 1931 with an article written by Orlicz. Kováčik and Rákosník (1991) presented the variable exponent Lebesgue space $L^{p(.)}(\mathbb{R}^d)$ and the variable exponent Sobolev space $W^{k,p(.)}(\mathbb{R}^d)$. It is known that the variable exponent Lebesgue spaces $L^{p(.)}\left(\mathbb{R}^d\right)$ and the classical Lebesgue spaces $L^p\left(\mathbb{R}^d\right)$ have many similar properties such as Banach space, reflexivity, separability, uniform convexity, Hölder inequalities and embeddings. However, a crucial difference between $L^{p(.)}(\mathbb{R}^d)$ and $L^p(\mathbb{R}^d)$ is that the variable exponent Lebesgue space is not invariant under translation in general (see (Diening (2004), Lemma 2.3) and (Kováčik and Rákosník (1991), Example 2.9)). For a historical journey, we refer Diening et al. (2004), Fan and Zhao (2001), Kováčik and Rákosník, (1991), Musielak (1983) and Samko (2005). Moreover, the Young theorem $\|f * g\|_{p(.)} \leq \|f\|_{p(.)} \|g\|_1$ is not valid for $f \in L^{p(.)}\left(\mathbb{R}^d\right)$ and $g \in L^1\left(\mathbb{R}^d\right)$. The boundedness of the maximal operator was an open problem in $L^{p(.)}(\mathbb{R}^d)$ for a long time. Diening (2004) proved this boundedness over bounded domains where $p(.)$ is locally log-Hölder continuous, that is,

$$|p(x) - p(y)| \leq \frac{C}{-\ln|x-y|}, \; x, y \in \Omega, \; |x-y| \leq \frac{1}{2}.$$

He later extended the boundedness to unbounded domains under some condition with respect to behaviour of the exponent $p(.)$. After this study, many interesting and important papers appeared in non-weighted and weighted variable exponent spaces. Moreover, Aoyama (2009) studied on variable exponent Lebesgue spaces with respect to probability spaces.

Assume that G is a locally compact abelian group with Haar measure μ. For $1 \leq p, q \leq \infty$, an amalgam space $(L^p, \ell^q)(G)$ is a Banach space of measurable functions on G which belong locally to L^p and globally to ℓ^q. Holland (1975) presented the first systematic paper for amalgam spaces. Later, Stewart (1979) extended the Holland's definition to locally compact abelian groups by the Structure Theorem.

In this study, we consider Birkhoff's Ergodic Theorem in the context of weighted variable exponent Lebesgue and amalgam spaces. So, we have more general results in sense to Gorka (2016) in these spaces.

2. Notation and Preliminaries

Definition 2.1.

For a measurable function $p(.) : G \longrightarrow [1, \infty)$ (called the variable exponent on G), we put

$$p^- = \underset{x\in G}{\operatorname{essinf}} p(x), \qquad p^+ = \underset{x\in G}{\operatorname{esssup}} p(x).$$

The variable exponent Lebesgue spaces $L^{p(.)}(G)$ is the set of all measurable functions f on G such that $\varrho_{p(.)}(\lambda f) < \infty$ for some $\lambda > 0$, equipped with the Luxemburg norm

$$\|f\|_{p(.)} = \inf\left\{\lambda > 0 : \varrho_{p(.)}\left(\frac{f}{\lambda}\right) \leq 1\right\},$$

where the modular function $\varrho_{p(.)}$ is defined by

$$\varrho_{p(.)}(f) = \int_G |f(x)|^{p(x)}\, d\mu(x).$$

If $p^+ < \infty$, then $f \in L^{p(.)}(G)$ if and only if $\varrho_{p(.)}(f) < \infty$. The set $L^{p(.)}(G)$ is a Banach space with the norm $\|.\|_{p(.)}$. If $p(x) = p$ is a constant function, then the Luxemburg norm $\|.\|_{p(.)}$ coincides with the usual Lebesgue norm $\|.\|_p$. Note that, if $p^+ < \infty$, then convergence in norm is equivalent to convergence in modular, see Kováčik and Rákosník (1991).

Definition 2.2.

A measurable and locally integrable function $w : G \longrightarrow (0, \infty)$ is called a weight function. The weighted modular is defined by

$$\varrho_{p(.),w}(f) = \int\limits_G |f(x)|^{p(x)} w(x) d\mu(x).$$

The weighted variable exponent Lebesgue space $L_w^{p(.)}(G)$ consists of all measurable functions f on G for which $\|f\|_{L_w^{p(.)}(G)} = \left\| f w^{\frac{1}{p(.)}} \right\|_{p(.)} < \infty$. Also, $L_w^{p(.)}(G)$ is a uniformly convex Banach space and thus reflexive. Moreover, the dual space of $L_w^{p(.)}(G)$ is $L_{w^*}^{q(.)}(G)$, where $\frac{1}{p(.)} + \frac{1}{q(.)} = 1$ and $w^* = w^{1-q(.)}$, see Lahmi et al. (2018). The relationship $\varrho_{p(.),w}(.)$ and $\|.\|_{L_w^{p(.)}(G)}$ are as follows,

$$\min\left\{ \varrho_{p(.),w}(f)^{\frac{1}{p^-}}, \varrho_{p(.),w}(f)^{\frac{1}{p^+}} \right\} \le \|f\|_{L_w^{p(.)}(G)} \le \max\left\{ \varrho_{p(.),w}(f)^{\frac{1}{p^-}}, \varrho_{p(.),w}(f)^{\frac{1}{p^+}} \right\},$$
$$\min\left\{ \|f\|^{p^+}_{L_w^{p(.)}(G)}, \|f\|^{p^-}_{L_w^{p(.)}(G)} \right\} \le \varrho_{p(.),w}(f) \le \max\left\{ \|f\|^{p^+}_{L_w^{p(.)}(G)}, \|f\|^{p^-}_{L_w^{p(.)}(G)} \right\}.$$

Moreover, if $0 < C \le w$, then we have $L_w^{p(.)}(G) \hookrightarrow L^{p(.)}(\Omega)$, since one easily sees that

$$C \int\limits_G |f(x)|^{p(x)} dx \le \int\limits_G |f(x)|^{p(x)} w(x) d\mu(x),$$

and $C \|f\|_{p(.)} \le \|f\|_{L_w^{p(.)}(G)}$ (see Aydın (2012b)). Instead of Haar measure μ, let us take the measure ϑ such that $d\vartheta(x) = w(x)d\mu(x)$.

Definition 2.3.

Let (G, Σ, ϑ) be a measure space. A measurable function $T : G \longrightarrow G$ is called a measure-preserving transformation if

$$\vartheta\left(T^{-1}(A)\right) = \vartheta(A),$$

for all $A \in \Sigma$.

Definition 2.4.

We denote by $L_{loc,w}^{p(.)}(G)$ the space of (equivalence classes of) functions on G such that f restricted to any compact subset K of G belongs to $L_w^{p(.)}(G)$. Note that $L_w^{p(.)}(G) \hookrightarrow L_{loc,w}^{p(.)}(G) \hookrightarrow L_{loc}^1(G)$.

Definition 2.5.

Let G be a locally compact abelian group with Haar measure ϑ. By the Structure Theorem (Hewitt and Ross (1979), Theorem 24.30), $G = \mathbb{R}^a \times G_1$, where a is a nonnegative integer and G_1 is a locally compact abelian group which contains an open compact subgroup H. Let $I = [0,1)^a \times H$ and $J = \mathbb{Z}^a \times T$, where T is a transversal of H in G_1, i.e. $G_1 = \bigcup\limits_{t \in T} (t + H)$ is a coset decomposition of G_1. For $\alpha \in J$ we define $I_\alpha = \alpha + I$, and therefore G is equal to the disjoint union of relatively compact sets I_α.

Let $1 \leq p(.), q < \infty$. The weighted variable exponent amalgam spaces $\left(L_w^{p(.)}, \ell^q\right)$ are defined by

$$\left(L_w^{p(.)}, \ell^q\right) = \left\{f \in L_{loc,w}^{p(.)}(G) : \|f\|_{\left(L_w^{p(.)}, \ell^q\right)} < \infty\right\},$$

where

$$\|f\|_{\left(L_w^{p(.)}, \ell^q\right)} = \left(\sum_{\alpha \in J} \|f\chi_{I_\alpha}\|_{L_w^{p(.)}(G)}^q\right)^{\frac{1}{q}}.$$

It is well known that $\left(L_w^{p(.)}, \ell^q\right)$ is a Banach space and does not depend on the particular choice of I_α. Thus, we have the same spaces $\left(L_w^{p(.)}, \ell^q\right)$. If $w \equiv$ const., then $\left(L_w^{p(.)}, \ell^q\right)$ is denoted by $\left(L^{p(.)}, \ell^q\right)$. Moreover, if $p(.)$ and $w \equiv$ const., then we have the usual amalgam space (L^p, ℓ^q), see Aydın (2017), Holland (1975), Wiener (1926). The dual space of $\left(L_w^{p(.)}, \ell^q\right)$ is isometrically isomorphic to $\left(L_{w^*}^{r(.)}, \ell^t\right)$ where $\frac{1}{p(.)} + \frac{1}{r(.)} = 1$ and $\frac{1}{q} + \frac{1}{t} = 1$. Also, the space $\left(L_w^{p(.)}, \ell^q\right)$ is reflexive. In addition, $\left(L_w^{p(.)}, \ell^q\right)$ is a solid Banach function space by Aydın and Gurkanli (2012) (Proposition 2.2), and $\left(L_w^{p(.)}, \ell^1\right) \hookrightarrow \left(L^{p(.)}, \ell^1\right) \hookrightarrow \left(L^1, \ell^1\right) = L^1$ by Aydın (2012a) (Proposition 3.5).

Definition 2.6 (Aydın (2017) and Squire (1984)).

$L_{c,w}^{p(.)}(G)$ denotes the functions f in $L_w^{p(.)}(G)$ such that supp$f \subset G$ is compact, that is,

$$L_{c,w}^{p(.)}(G) = \left\{f \in L_w^{p(.)}(G) : \text{supp} f \text{ compact}\right\}.$$

Let $K \subset G$ be given. The cardinality of the set

$$S(K) = \{I_\alpha : I_\alpha \cap K \neq \varnothing\},$$

is denoted by $|S(K)|$, where $\{I_\alpha\}_{\alpha \in J}$ is a collection of intervals.

Proposition 2.7 (Aydın (2017)).

If g belongs to $L_{c,w}^{p(.)}(G)$, then

(i) $\|g\|_{\left(L_w^{p(.)}, \ell^q\right)} \leq |S(K)|^{\frac{1}{q}} \|g\|_{L_w^{p(.)}(G)}$ for $1 \leq q < \infty$,
(ii) $\|g\|_{\left(L_w^{p(.)}, \ell^\infty\right)} \leq |S(K)| \, \|g\|_{L_w^{p(.)}(G)}$ for $q = \infty$,
(iii) $L_{c,w}^{p(.)}(G) \subset \left(L_w^{p(.)}, \ell^q\right)$ for $1 \leq q \leq \infty$,

where K is the compact support of g.

Theorem 2.8.

$L_{c,w}^{p(.)}(G)$ is dense subspace of $\left(L_w^{p(.)}, \ell^q\right)$ for $1 \leq p(.), q < \infty$.

Proof:

If we use the similar techniques of Theorem 3.6 in Squire (1984) or Theorem 7 in Holland (1975), then we can prove the theorem easily. ■

Proposition 2.9.

$C_c(G)$, which consists of continuous functions on G whose support is compact, is dense in $\left(L_w^{p(.)}, \ell^q\right)$ for $1 \leq p(.), q < \infty$.

Proof:

It is clear that $C_c(G)$ is included in $\left(L_w^{p(.)}, \ell^q\right)$. Let $f \in \left(L_w^{p(.)}, \ell^q\right)$. By Theorem 2.8, given $\varepsilon > 0$ there exists $g \in L_{c,w}^{p(.)}(G)$ such that

$$\|f-g\|_{\left(L_w^{p(.)}, \ell^q\right)} < \frac{\varepsilon}{2}. \tag{1}$$

If E is the compact support of g, then there exists h in $C_c(E)$ such that

$$\|g-h\|_{L_w^{p(.)}(E)} < \frac{\varepsilon}{2\,|S(E)|^{\frac{1}{q}}},$$

since $C_c(E)$ is dense in $L_w^{p(.)}(E)$, see Aydın (2012a). Hence, by Proposition 2.7, we have

$$\|g-h\|_{\left(L_w^{p(.)}, \ell^q\right)} \leq |S(E)|^{\frac{1}{q}} \|g-h\|_{L_w^{p(.)}(E)} < \frac{\varepsilon}{2}. \tag{2}$$

Using (1) and (2), we obtain

$$\begin{aligned} \|f-h\|_{\left(L_w^{p(.)}, \ell^q\right)} &\leq \|f-g\|_{\left(L_w^{p(.)}, \ell^q\right)} + \|g-h\|_{\left(L_w^{p(.)}, \ell^q\right)} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

This completes the proof. ■

3. Main Results

The following theorem was proved by Gorka (2016) in unweighted variable exponent Lebesgue spaces.

Theorem 3.1.

Let (G, Σ, ϑ) be a probability space and $T : G \longrightarrow G$ a measure preserving transformation with respect to the measure ϑ. Moreover, if $p(.)$ is T-invariant, i.e., $p(T(.)) = p(.)$, then

(i) the limit

$$f_{av}(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} f\left(T^j(x)\right),$$

exists for all $f \in L_w^{p(.)}(G)$ and almost each point $x \in G$, and $f_{av} \in L_w^{p(.)}(G)$.

(ii) for every $f \in L_w^{p(.)}(G)$, we have

$$f_{av}(x) = f_{av}\left(T(x)\right), \tag{3}$$

$$\int_G f_{av}\, d\vartheta = \int_G f\, d\vartheta, \tag{4}$$

$$\lim_{n\to\infty}\left\|f_{av}-\frac{1}{n}\sum_{j=0}^{n-1}f\circ T^{j}\right\|_{L_w^{p(.)}(G)}=0. \tag{5}$$

Proof:

Since $|G|<\infty$ and $L_w^{p(.)}(G)\hookrightarrow L^{p(.)}(G)\hookrightarrow L^1(G)$, the existence of the limit $f_{av}(x)$ for almost every point of G follows from the standard Birkhoof's Theorem. By Fatou Lemma, we have

$$\begin{aligned}\int_G |f_{av}(x)|^{p(x)}\,d\vartheta &= \int_G \left|\lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}f\left(T^j(x)\right)\right|^{p(x)} d\vartheta\\ &\le \int_G \lim_{n\to\infty}\left(\frac{1}{n}\sum_{j=0}^{n-1}\left|f\left(T^j(x)\right)\right|\right)^{p(x)} d\vartheta\\ &\le \liminf_{n\to\infty}\int_G \left(\frac{1}{n}\sum_{j=0}^{n-1}\left|f\left(T^j(x)\right)\right|\right)^{p(x)} d\vartheta\\ &\le \liminf_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\int_G \left|f\left(T^j(x)\right)\right|^{p(x)} d\vartheta.\end{aligned}$$

Here we used convexity and Jensen inequality in the last step. Moreover, since T is a measure preserving map and $p(.)$ is T-invariant, we get

$$\int_G |f(T(x))|^{p(x)}\,d\vartheta=\int_G |f(T(x))|^{p(T(x))}\,d\vartheta=\int_G |f(x)|^{p(x)}\,d\vartheta.$$

This follows that

$$\int_G |f_{av}(x)|^{p(x)}\,d\vartheta\le\int_G |f(x)|^{p(x)}\,d\vartheta<\infty. \tag{6}$$

Thus we get $f_{av}\in L_w^{p(.)}(G)$. This completes *(i)*.

By the Ergodic Theorem in classical Lebesgue spaces, we have (3) and (4) immediately. In order to prove (5) we assume that $f\in C_c(G)$. Thus, $f\in L^{\infty}(G)$ and

$$\lim_{n\to\infty}\left|f_{av}(x)-\frac{1}{n}\sum_{j=0}^{n-1}f\left(T^j(x)\right)\right|^{p(x)}=0,\ \text{a.e.}$$

$$\|f_{av}\|_{L^\infty(G)}\le\|f\|_{L^\infty(G)}.$$

Therefore, we have

$$\begin{aligned}\left|f_{av}(x)-\frac{1}{n}\sum_{j=0}^{n-1}f\left(T^j(x)\right)\right|^{p(x)} w(x) &\le \left|\|f\|_{L^\infty(G)}+\frac{1}{n}\sum_{j=0}^{n-1}\left\|f\left(T^j\right)\right\|_{L^\infty(G)}\right|^{p(x)} w(x)\\ &\le 2^{p^+}\max\left\{\|f\|_{L^\infty(G)}^{p^+},\|f\|_{L^\infty(G)}^{p^-}\right\}w(x)\in L^1(G).\end{aligned}$$

Hence, by Lebesgue dominated convergence theorem we have (5), provided $f \in C_c(G)$. Since $C_c(G)$ is dense in $L_w^{p(.)}(G)$ with respect to the norm $\|.\|_{L_w^{p(.)}(G)}$ (see Aydın (2012a)), then for any $f \in L_w^{p(.)}(G)$ and $\varepsilon > 0$ there is a $g \in C_c(G)$ such that

$$\varrho_{p(.),w}(f-g) < \varepsilon. \tag{7}$$

By the previous step, there is an n_0 such that the inequality

$$\varrho_{p(.),w}\left(g_{av} - \frac{1}{n}\sum_{j=0}^{n-1} g \circ T^j\right) < \varepsilon \tag{8}$$

holds for $n \geq n_0$. Let $q \geq 1$. Then by convexity of the function $y \mapsto y^q$, the inequality $(a+b+c)^q \leq 3^{q-1}(a^q + b^q + c^q)$ holds for any nonnegative a, b, c. Hence, we get

$$\begin{aligned}\varrho_{p(.),w}\left(f_{av} - \frac{1}{n}\sum_{j=0}^{n-1} f \circ T^j\right) &= \int_G \left| f_{av}(x) - \frac{1}{n}\sum_{j=0}^{n-1} f\left(T^j(x)\right)\right|^{p(x)} d\vartheta \\ &\leq 3^{q^+-1}\left(\varrho_{p(.),w}\left(f_{av} - g_{av}\right) + \varrho_{p(.),w}\left(g_{av} - \frac{1}{n}\sum_{j=0}^{n-1} g \circ T^j\right)\right. \\ &\quad \left. + \varrho_{p(.),w}\left(\frac{1}{n}\sum_{j=0}^{n-1}(f-g)\circ T^j\right)\right).\end{aligned}$$

Thus, using (6), (7), (8) and convexity of $\varrho_{p(.),w}$, we have

$$\varrho_{p(.),w}\left(f_{av} - \frac{1}{n}\sum_{j=0}^{n-1} f \circ T^j\right) < \varepsilon.$$

That is the desired result. ■

Now we give the Ergodic Theorem for the spaces $\left(L_w^{p(.)}, \ell^1\right)$.

Theorem 3.2.

Let (G, Σ, ϑ) be a probability space and $T : G \longrightarrow G$ a measure preserving transformation with respect to the measure ϑ. Moreover, if $p(.)$ is T-invariant, i.e., $p(T(.)) = p(.)$, then

(i) the limit

$$f_{av}(x) = \lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} f\left(T^j(x)\right),$$

exists for all $f \in \left(L_w^{p(.)}, \ell^1\right)$ and almost each point $x \in G$, and $f_{av} \in \left(L_w^{p(.)}, \ell^1\right)$.

(ii) for every $f \in \left(L_w^{p(.)}, \ell^1\right)$, we have

$$f_{av}(x) = f_{av}\left(T(x)\right), \tag{9}$$

$$\int_G f_{av} d\vartheta = \int_G f d\vartheta, \tag{10}$$

$$\lim_{n\to\infty} \left\| f_{av} - \frac{1}{n}\sum_{j=0}^{n-1} f\circ T^j \right\|_{\left(L_w^{p(\cdot)},\ell^1\right)} = 0. \tag{11}$$

Proof:

Since $\left(L_w^{p(\cdot)},\ell^1\right) \hookrightarrow \left(L^{p(\cdot)},\ell^1\right) \hookrightarrow \left(L^1,\ell^1\right) = L^1$, then the existence of the limit $f_{av}(x)$ for almost every point of G follows from the standard Birkhoof's Theorem. Thus, by Fatou Lemma we have

$$\|f_{av}\|_{\left(L_w^{p(\cdot)},\ell^1\right)} = \sum_{\alpha\in J} \|f_{av}\|_{L_w^{p(\cdot)}(I_\alpha)} \le \sum_{\alpha\in J} \|f\|_{L_w^{p(\cdot)}(I_\alpha)} = \|f\|_{\left(L_w^{p(\cdot)},\ell^1\right)},$$

and then $f_{av} \in \left(L_w^{p(\cdot)},\ell^1\right)$. This completes *(i)*. By the Ergodic Theorem in classical Lebesgue spaces, we have (9) and (10), immediately. By Proposition 2.9, given $\varepsilon > 0$ there exists $g \in C_c$ such that

$$\|f-g\|_{\left(L_w^{p(\cdot)},\ell^1\right)} < \frac{\varepsilon}{2}. \tag{12}$$

If E is the compact support of g, then we have

$$\left\| g_{av} - \frac{1}{n}\sum_{j=0}^{n-1} g\circ T^j \right\|_{L_w^{p(\cdot)}(E)} < \frac{\varepsilon}{2\,|S(E)|},$$

by Theorem 3.1. If we consider the Proposition 2.7, we get

$$\left\| g_{av} - \frac{1}{n}\sum_{j=0}^{n-1} g\circ T^j \right\|_{\left(L_w^{p(\cdot)},\ell^1\right)} \le |S(E)| \left\| g_{av} - \frac{1}{n}\sum_{j=0}^{n-1} g\circ T^j \right\|_{L_w^{p(\cdot)}(E)} < \frac{\varepsilon}{2}. \tag{13}$$

Hence, by Lebesgue dominated convergence theorem we have (11), provided $g \in C_c$. Finally, using (12) and (13), we have

$$\begin{aligned}
\left\| f_{av} - \frac{1}{n}\sum_{j=0}^{n-1} f\circ T^j \right\|_{\left(L_w^{p(\cdot)},\ell^1\right)} &\le \|f_{av}-g_{av}\|_{\left(L_w^{p(\cdot)},\ell^1\right)} + \left\| g_{av} - \frac{1}{n}\sum_{j=0}^{n-1} g\circ T^j \right\|_{\left(L_w^{p(\cdot)},\ell^1\right)} \\
&\quad + \left\| \frac{1}{n}\sum_{j=0}^{n-1} (f-g)\circ T^j \right\|_{\left(L_w^{p(\cdot)},\ell^1\right)} \\
&\le 2\,\|f-g\|_{\left(L_w^{p(\cdot)},\ell^1\right)} + \left\| g_{av} - \frac{1}{n}\sum_{j=0}^{n-1} g\circ T^j \right\|_{\left(L_w^{p(\cdot)},\ell^1\right)} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

This completes the proof. ∎

4. Conclusion

In this study, we give some historical informations and importance of weighted variable exponent Lebesgue and amalgam spaces. Moreover, in Gorka (2016), the author studied Birkhoof's Ergodic theorem for unweighted variable exponent Lebesgue spaces. Finally, we have considered this work and have generalized Birkhoof's Ergodic theorem to weighted variable exponent Lebesgue and amalgam spaces.

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