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Forced Oscillations of Nonlinear Hyperbolic Equations with Functional Arguments via Riccati Method

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Abstract

By using integral averaging method and a generalized Riccati technique, sufficient conditions are established for the oscillation of solutions of forced nonlinear hyperbolic equations with functional arguments.

Keywords: Forced oscillation; hyperbolic equations; Riccati inequality; interval criteria

MSC (2010) No: 34K11; 35B05; 35R10

1. Introduction

We are concerned with an oscillation criterion for the hyperbolic equations with functional arguments

$$\frac{\partial}{\partial t}\left(r(t)\frac{\partial}{\partial t}u(x,t)\right)+p(t)\frac{\partial}{\partial t}u(x,t)-a(t)\Delta u(x,t)-\sum_{i=1}^{k}b_i(t)\Delta u(x,\tau_i(t))$$
$$+\sum_{i=1}^{m}q_i(x,t)\varphi_i\big(u(x,\sigma_i(t))\big)=f(x,t),\qquad (x,t)\in\Omega\equiv G\times(0,\infty), \tag{E}$$

where Δ is the Laplacian in $\mathbf{R}^n$ and G is a bounded domain of $\mathbf{R}^n$ with piecewise smooth boundary ∂G and we consider the following boundary conditions

$$u=\psi \qquad on \qquad \partial G\times[0,\infty), \tag{B1}$$
$$\frac{\partial u}{\partial \nu}+\mu u=\tilde{\psi} \qquad on \qquad \partial \mathrm{G}\times[0,\infty), \tag{B2}$$

where ν denotes the unit exterior normal vector to ∂G and $\psi,\tilde{\psi}\in C(\partial G\times(0,\infty);\mathrm{R})$, $\mu\in C(\partial G\times(0,\infty),[0,\infty))$.

We assume throughout this paper that:

$$r(t)\in C^1([0,\infty);(0,\infty)),\quad p(t)\in C([0,\infty);\mathbf{R}),$$
$$a(t),b_i(t)\in C([0,\infty);[0,\infty))\,(i=1,2,\cdots,k),$$
$$q_i(x,t)\in C(\overline{\Omega};[0,\infty))\,(i=1,2,\cdots,m),\quad f(x,t)\in C(\overline{\Omega};\mathbf{R}); \tag{H1}$$

$$\tau_i(t)\in C([0,\infty);\mathbf{R}),\ \lim_{t\to\infty}\tau_i(t)=\infty \quad (i=1,2,\cdots,k),$$
$$\sigma_i(t)\in C([0,\infty);\mathbf{R}),\ \lim_{t\to\infty}\sigma_i(t)=\infty \quad (i=1,2,\cdots,m); \tag{H2}$$

$$\varphi_i(s)\in C^1(\mathbf{R};\mathbf{R})\quad (i=1,2,\cdots,m) \text{ are convex on } [0,\infty) \text{ and}$$
$$\varphi_i(-s)=-\varphi_i(s) \text{ for } s\ge 0. \tag{H3}$$

Definition 1. By a *solution* of equation (E) we mean a function

$$u\in C^2(\overline{G}\times[t_{-1},\infty))\cap C(\overline{G}\times[\tilde{t}_{-1},\infty)),$$

which satisfies (E), where

$$t_{-1} = \min\{0, \min_{1 \le i \le k} \{\inf_{t \ge 0} \tau_i(t)\}\}, \quad \tilde{t}_{-1} = \min\{0, \min_{1 \le i \le m} \{\inf_{t \ge 0} \sigma_i(t)\}\}.$$

Definition 2. A solution u of equation (E) is said to be *oscillatory* in Ω if u has a zero in $G \times (t, \infty)$ for any $t > 0$.

Definition 3. We say that functions (H_1, H_2) belong to a function class $\mathbf{H}$, denoted by

$$(H_1, H_2) \in \mathbf{H}, \text{ if } (H_1, H_2) \in C(D; [0, \infty))$$

satisfy

$$H_i(t,t) = 0, \quad H_i(t,s) > 0, (i = 1,2) \quad \text{for} \quad t > s,$$

where $D = \{(t,s) : 0 < s \le t < \infty\}$. Moreover, the partial derivatives $\partial H_1 / \partial t$ and $\partial H_2 / \partial s$ exists on D such that

$$\frac{\partial H_1}{\partial t}(s,t) = h_1(s,t) H_1(s,t) \quad \text{and} \quad \frac{\partial H_2}{\partial s}(t,s) = -h_2(t,s) H_2(t,s),$$

where $h_1, h_2 \in C_{loc}(D; \mathbf{R})$.

The study of oscillation of solutions of partial differential equations is important in physical problems (see Mohyud-Din, Noor and Noor 2009). Some applications involving dynamics with spatial migrations, chemical reactions, control systems, combinatorics can be found in the monograph (Wu 1996).

The oscillation criteria for forced second order nonlinear differential equations have been established by many authors. We refer to the papers (Li and Agarwal 2000, Li and Li 20000, Cakmak and Tiryaki 2005, Nasr 1998, El-Sayed 1993, Sun et al. 2004, Sun and Wong 2007, Wong 1999 and Yang 2003). Recently, there has been an increase in studying the oscillation for hyperbolic equations by using a generalized Riccati transformation (see Cui and Xu 2009, Li 2000, Li and Cui 1998, Wang et al. 2007, 2008, Zhong and Yuan 2007, for example).

The objective of this paper is to present several oscillation criteria for forced nonlinear hyperbolic equations under the two assumptions of forcing term. By using this assumptions and transforming into Riccati inequality, we obtain the oscillation results of equation (E). In subsections 5.1 and 5.2, we are going to give some examples illustrating obtained our results. Furthermore, it divides into the following cases:

$$\int_{t_0}^{\infty} e^{-\tilde{R}(t)}dt = \infty\,, \tag{C1}$$

$$\int_{t_0}^{\infty} e^{-\tilde{R}(t)}dt < \infty\,, \tag{C2}$$

where

$$\tilde{R}(t) = \int_{t_0}^{t}\left(\frac{r'(s)+p(s)}{r(s)}\right)ds\,.$$

2. Reduction to One-Dimensional Problems

In this section we reduce the multi-dimensional oscillation problems for (E) to one-dimensional oscillation problems. It is known that the first eigenvalue λ_1 of the eigenvalue problem

$$\begin{aligned} -\Delta w &= \lambda w \quad &&\text{in} \quad G\,, \\ w &= 0 \quad &&\text{on} \quad \partial G \end{aligned}$$

is positive, and the corresponding eigenfunction $\Phi(x)$ can be chosen so that $\Phi(x) > 0$ in G.

Now we define

$$q_i(t) = \min_{x \in G} q_i(x,t)\,.$$

The following notation will be used:

$$U(t) = K_{\Phi}\int_G u(x,t)\Phi(x)dx\,, \quad \tilde{U}(t) = \frac{1}{|G|}\int_G u(x,t)dx\,,$$

$$F(t) = K_\Phi \int_G f(x,t)\Phi(x)dx, \quad \tilde{F}(t) = \frac{1}{|G|}\int_G f(x,t)dx,$$

$$\Psi(t) = K_\Phi \int_{\partial G} \psi(x,t)\frac{\partial \Phi}{\partial \nu}(x)dS, \quad \tilde{\Psi}(t) = \frac{1}{|G|}\int_{\partial G} \tilde{\psi}(x,t)dS,$$

where

$$K_\Phi = \left(\int_G \Phi(x)dx\right)^{-1} \text{ and } |G| = \int_G dx.$$

Theorem 1. If the functional differential inequalities

$$\left(r(t)y'(t)\right)' + p(t)y'(t) + \sum_{i=1}^{m} q_i(t)\varphi_i(y(\sigma_i(t))) \le \pm G(t) \tag{1}$$

have no eventually positive solution, then every solution $u(x,t)$ of the problem (E), (B1) is oscillatory in Ω, where

$$G(t) = F(t) - a(t)\Psi(t) - \sum_{i=1}^{k} b_i(\tau_i(t))\Psi(\tau_i(t)).$$

Proof:

Suppose to the contrary that there is a non-oscillatory solution u of the problem (E), (B1). Without loss of generality we may assume that $u(x,t) > 0$ in $G \times [t_0, \infty)$ for some $t_0 > 0$, because the case where $u(x,t) < 0$ can be treated similarly. Since (H2) holds, we see that $u(x, \tau_i(t)) > 0 (i = 1,2,\cdots,k)$ and $u(x, \sigma_i(t)) > 0 (i = 1,2,\cdots,m)$ in $G \times [t_1, \infty)$ for some $t_1 \ge t_0$. Multiplying (E) by $K_\Phi \Phi(x)$ and integrating over G, we obtain

$$\left(r(t)U'(t)\right)' + p(t)U'(t) - a(t)K_\Phi \int_G \Delta u(x,t)\Phi(x)dx - \sum_{i=1}^{k} b_i(t)K_\Phi \int_G \Delta u(x,\tau_i(t))\Phi(x)dx$$
$$+ \sum_{i=1}^{m} K_\Phi \int_G q_i(x,t)\varphi_i(u(x,\sigma_i(t)))\Phi(x)dx = F(t), \qquad t \ge t_1. \tag{2}$$

From Green's formula it follows that

$$K_\Phi \int_G \Delta u(x,t)\Phi(x)dx \le -\Psi(t), \qquad t \ge t_1, \tag{3}$$

$$K_\Phi \int_G \Delta u(x,\tau_i(t))\Phi(x)dx \le -\Psi(\tau_i(t)), \qquad t \ge t_1. \tag{4}$$

An application of Jensen's inequality shows that

$$\sum_{i=1}^{m} K_\Phi \int_G q_i(x,t)\varphi_i(u(x,\sigma_i(t)))\Phi(x)dx \ge \sum_{i=1}^{m} q_i(t)\varphi_i(U(\sigma_i(t))), \qquad t \ge t_1\,. \tag{5}$$

Combining (2)-(5) yields

$$(r(t)U'(t))' + p(t)U'(t) + \sum_{i=1}^{m} q_i(t)\varphi_i(U(\sigma_i(t))) \le G(t), \quad t \ge t_1\,.$$

Therefore, $U(t)$ is an eventually positive solution of (1). This contradicts the hypothesis and completes the proof.

Theorem 2. If the functional differential inequalities

$$(r(t)y'(t))' + p(t)y'(t) + \sum_{i=1}^{m} q_i(t)\varphi_i(y(\sigma_i(t))) \le \pm\tilde{G}(t) \tag{6}$$

have no eventually positive solution, then every solution $u(x,t)$ of the problem (E), (B2) is oscillatory in Ω, where

$$\tilde{G}(t) = \tilde{F}(t) + a(t)\tilde{\Psi}(t) + \sum_{i=1}^{k} b_i(\tau_i(t))\tilde{\Psi}(\tau_i(t))\,.$$

Proof:

Suppose to the contrary that there is a non-oscillatory solution u of problem (E), (B2). Without loss of generality we may assume that $u(x,t) > 0$ in $G \times [t_0,\infty)$ for some $t_0 > 0$. Since (H2) holds, we see that $u(x,\tau_i(t)) > 0 (i = 1,2,\cdots,k)$ and $u(x,\tau_i(t)) > 0 (i = 1,2,\cdots,m)$ in $G \times [t_1,\infty)$ for some $t_1 \ge t_0$. Dividing (E) by $|G|$ and integrating over G, we obtain

$$(r(t)\tilde{U}'(t))' + p(t)\tilde{U}'(t) - \frac{a(t)}{|G|}\int_G \Delta u(x,t)dx - \sum_{i=1}^{k} \frac{b_i(t)}{|G|}\int_G \Delta u(x,\tau_i(t))dx$$
$$+ \frac{1}{|G|}\sum_{i=1}^{m}\int_G q_i(x,t)\varphi_i(u(x,\sigma_i(t)))dx = \tilde{F}(t), \qquad t \ge t_1. \tag{7}$$

It follows from Green's formula that

$$\frac{1}{|G|}\int_G \Delta u(x,t)dx \le \tilde{\Psi}(t), \qquad t \ge t_1, \tag{8}$$
$$\frac{1}{|G|}\int_G \Delta u(x,\tau_i(t))dx \le \tilde{\Psi}(\tau_i(t)), \qquad t \ge t_1. \tag{9}$$

Applying Jensen's inequality, we observe that

$$\frac{1}{|G|}\sum_{i=1}^{m}\int_G q_i(x,t)\varphi_i(u(x,\sigma_i(t)))dx \ge \sum_{i=1}^{m} q_i(t)\varphi_i(\tilde{U}(\sigma_i(t))), \qquad t \ge t_1 \tag{10}$$

Combining (7)-(10) yields

$$(r(t)\tilde{U}'(t))' + p(t)\tilde{U}'(t) + \sum_{i=1}^{m} q_i(t)\varphi_i(\tilde{U}(\sigma_i(t))) \le \tilde{G}(t), \qquad t \ge t_1.$$

Hence, $\tilde{U}(t)$ is an eventually positive solution of (6). This contradicts the hypothesis and completes the proof.

3. Second Order Functional Differential Inequalities

We consider the sufficient conditions for every solution $y(t)$ of the functional differential inequality

$$(r(t)y'(t))' + p(t)y'(t) + \sum_{i=1}^{m} q_i(t)\varphi_i(y(\sigma_i(t))) \le f(t) \tag{11}$$

to have no eventually positive solution, where $f(t) \in C([0,\infty);\mathbf{R})$.

3.1. For the case (C1)

We assume the following hypotheses:

For some $j \in \{1,2,\cdots,m\}$, there exists a positive constant σ such that

$$\sigma_j'(t) \geq \sigma \quad \text{and} \quad \sigma_j(t) \leq t,$$

and $\varphi_j'(s) > 0$, $\varphi_j'(s)$ is non-decreasing for $s > 0$; (H4)

there exists $T_0 \leq a < b$ such that

$$f(t) \leq 0, \quad t \in [a,b], \text{ for some } T_0 \geq 0. \tag{H5}$$

Theorem 3. Assume that (C1), (H4) and (H5) hold. If the Riccati inequality

$$z'(t) + \frac{1}{2}\frac{1}{P_{\tilde{K}}(t)} z^2(t) \leq -Q_{\tilde{K}}(t) \tag{12}$$

has no solution on $[T,\infty)$ for all large T and some $\tilde{K} > 0$, then (11) has no eventually positive solution, where $\phi(t) \in C^1((T_0,\infty);(0,\infty))$ for some $T_0 > 0$ and

$$P_{\tilde{K}}(t) = \phi(t) R_{\tilde{K}}(t),$$

$$Q_{\tilde{K}}(t) = \phi(t) q_j(t) - \frac{1}{2}\left(\frac{\phi(t)p(t)}{r(t)} - \phi'(t)\right)^2 \frac{R_{\tilde{K}}(t)}{\phi(t)},$$

where

$$R_{\tilde{K}}(t) = \frac{r(t)e^{\tilde{R}(\sigma_j(t))}}{\tilde{K}\sigma e^{\tilde{R}(t)}}.$$

Proof:

Suppose that $y(t)$ is a positive solution of (11) on $[t_0,\infty)$ for some $t_0 \geq T > 0$. From (11) there exist $j \in \{1,2,\cdots,m\}$ and $a,b \geq t_0$ such that $f(t) \leq 0$ on the interval $I \in [a,b]$, and so,

$$(r(t)y'(t))' + p(t)y'(t) \leq -q_j(t)\varphi_j(y(\sigma_j(t))) \leq 0, \qquad t \in I, \tag{13}$$

which can be rewritten as

$$r(t)y''(t) + \big(r'(t) + p(t)\big)y'(t) \le 0.$$

Hence, we observe that

$$r(t)\{e^{\tilde{R}(t)}y'(t)\}' = e^{\tilde{R}(t)}\big(r(t)y''(t) + \big(r'(t) + p(t)\big)y'(t)\big) \le 0. \tag{14}$$

Using this fact that $e^{\tilde{R}(t)}y'(t)$ is non-increasing, we have

$$y'(\sigma_j(t)) \ge e^{\tilde{R}(t) - \tilde{R}(\sigma_j(t))}y'(t). \tag{15}$$

A standard argument shows that $y'(t) \ge 0$ (cf. Yoshida 2009). Since $y(t) > 0$, $y'(t) \ge 0$ eventually, $y(\sigma_j(t)) \ge k_0$ for some k_0. Hence, we obtain

$$\varphi_j'(y(\sigma_j(t))) \ge \varphi_j'(k_0) \equiv \tilde{K}.$$

Setting

$$w(t) = \frac{r(t)y'(t)}{\varphi_j(y(\sigma_j(t)))},$$

we conclude from (15) that

$$w'(t) = \frac{(r(t)y'(t))'}{\varphi_j(y(\sigma_j(t)))} - r(t)y'(t)\frac{\varphi_j'(y(\sigma_j(t)))\,y'(\sigma_j(t))\sigma_j'(t)}{\varphi_j^2(y(\sigma_j(t)))}$$

$$\le -\frac{p(t)}{r(t)}w(t) - q_j(t) - \frac{1}{R_{\tilde{K}}(t)}w^2(t), \qquad t \in I. \tag{16}$$

Multiplying (16) by $\phi(t)$, we obtain

$$\phi(t)w'(t) + \frac{\phi(t)p(t)}{r(t)}w(t) + \frac{\phi(t)}{R_{\tilde{K}}(t)}w^2(t) \le -\phi(t)q_j(t). \tag{17}$$

On the other hand, we have

$$(\phi(t)w(t))' = \phi'(t)w(t) + \phi(t)w'(t). \tag{18}$$

Substituting (18) into (17) yields

$$(\phi(t)w(t))' + \left(\frac{\phi(t)p(t)}{r(t)} - \phi'(t)\right)w(t) + \frac{\phi(t)}{R_{\tilde{K}}(t)}w^2(t) \le -\phi(t)q_j(t)\,. \tag{19}$$

By Hölder's inequality, we have

$$\left|\left(\frac{\phi(t)p(t)}{r(t)} - \phi'(t)\right)w(t)\right| = \left|\left(\frac{\phi(t)p(t)}{r(t)} - \phi'(t)\right)\left(\frac{\phi(t)}{R_{\tilde{K}}(t)}\right)^{-\frac{1}{2}}\left(\frac{\phi(t)}{R_{\tilde{K}}(t)}\right)^{\frac{1}{2}}w(t)\right|$$

$$\le \frac{1}{2}\left(\left(\frac{\phi(t)p(t)}{r(t)} - \phi'(t)\right)^2\left(\frac{\phi(t)}{R_{\tilde{K}}(t)}\right)^{-1} + \left(\frac{\phi(t)}{R_{\tilde{K}}(t)}\right)^2 w^2(t)\right),$$

which means that

$$\left(\frac{\phi(t)p(t)}{r(t)} - \phi'(t)\right)w(t) \ge -\frac{1}{2}\left(\frac{\phi(t)}{R_{\tilde{K}}(t)}\right)w^2(t) - \frac{1}{2}\left(\frac{\phi(t)p(t)}{r(t)} - \phi'(t)\right)^2\left(\frac{R_{\tilde{K}}(t)}{\phi(t)}\right). \tag{20}$$

Combining (19) with (20), we have

$$(\phi(t)w(t))' + \frac{1}{2}\left(\frac{\phi(t)}{R_{\tilde{K}}(t)}\right)w^2(t) \le -Q_{\tilde{K}}(t), \qquad t \in I\,.$$

We define

$$z(t) = \phi(t)w(t)\,,$$

then

$$z'(t)+\frac{1}{2}\left(\frac{1}{\phi(t)R_{\tilde{K}}(t)}\right)z^2(t)\le -Q_{\tilde{K}}(t), \qquad t\in I .$$

Therefore, $z(t)$ is a positive solution of (12) on I. This contradicts the hypothesis and completes the proof.

Theorem 4. Assume that (C1), (H4) and (H5) hold. If for *each* $T\ge 0$ and some $\tilde{K}>0$, there exist $(H_1,H_2)\in\mathbf{H}$ and some $c\in(a,b)$ such that $T\le a<b$ and

$$\frac{1}{H_1(c,a)}\int_a^c H_1(s,a)\left\{q_j(s)-\frac{1}{4}R_{\tilde{K}}(s)\lambda_1^2(s,a)\right\}\tilde{\phi}(s)ds + \frac{1}{H_2(b,c)}\int_c^b H_2(b,s)\left\{q_j(s)-\frac{1}{4}R_{\tilde{K}}(s)\lambda_2^2(b,s)\right\}\tilde{\phi}(s)ds>0, \tag{21}$$

then (11) has no eventually positive solution, where $\tilde{\phi}(t)\in C^1((T_0,\infty);(0,\infty))$ for some $T_0>0$ and

$$\lambda_1(s,t)=\frac{\tilde{\phi}'(s)}{\tilde{\phi}(s)}-\frac{p(s)}{r(s)}+h_1(s,t), \qquad \lambda_2(t,s)=\frac{\tilde{\phi}'(s)}{\tilde{\phi}(s)}-\frac{p(s)}{r(s)}-h_2(t,s).$$

Proof:

Suppose that $y(t)$ is a positive solution of (11) on $[t_0,\infty)$ for some $t_0>0$. At first, we assume that $y(t)>0$ on (a,b) for $a,b\ge t_0$. Proceeding as the same proof of Theorem 3, from (17), there exists a positive solution $w(s)$ satisfy

$$\tilde{\phi}(s)q_j(s)\le -\tilde{\phi}(s)w'(s)-\frac{\tilde{\phi}(s)p(s)}{r(s)}w(s)-\frac{\tilde{\phi}(s)}{R_{\tilde{K}}(s)}w^2(s). \tag{22}$$

Multiplying (22) by $H_2(t,s)$ and integrating over $[c,t]$ for $t\in[c,b)$, we have

$$\frac{1}{H_2(t,c)}\int_c^t H_2(t,s)\left\{q_j(s)-\frac{1}{4}R_{\tilde{K}}(s)\lambda_2^2(t,s)\right\}\tilde{\phi}(s)ds\le w(c)\tilde{\phi}(c).$$

Letting $t\to b^-$ in the above, we obtain

$$\frac{1}{H_2(b,c)}\int_c^b H_2(b,s)\left\{q_j(s)-\frac{1}{4}R_{\tilde{K}}(s)\lambda_2^2(b,s)\right\}\tilde{\phi}(s)ds \le w(c)\tilde{\phi}(c). \tag{23}$$

On the other hand, multiplying (22) by $H_1(s,t)$ and integrating over $[t,c]$ for $t\in(a,b]$, analogously, we see that

$$\frac{1}{H_1(c,t)}\int_t^c H_1(s,t)\left\{q_j(s)-\frac{1}{4}R_{\tilde{K}}(s)\lambda_1^2(s,t)\right\}\tilde{\phi}(s)ds \le -w(c)\tilde{\phi}(c).$$

Letting $t\to a^+$ in the above, we obtain

$$\frac{1}{H_1(c,a)}\int_a^c H_1(s,a)\left\{q_j(s)-\frac{1}{4}R_{\tilde{K}}(s)\lambda_1^2(s,a)\right\}\tilde{\phi}(s)ds \le -w(c)\tilde{\phi}(c). \tag{24}$$

Adding (23) and (24), we easily obtain the following

$$\frac{1}{H_1(c,a)}\int_a^c H_1(s,a)\left\{q_j(s)-\frac{1}{4}R_{\tilde{K}}(s)\lambda_1^2(s,a)\right\}\tilde{\phi}(s)ds + \frac{1}{H_2(b,c)}\int_c^b H_2(b,s)\left\{q_j(s)-\frac{1}{4}R_{\tilde{K}}(s)\lambda_2^2(b,s)\right\}\tilde{\phi}(s)ds \le 0,$$

which contradicts the condition (21). Pick up a sequence $\{T_i\}\subset[t_0,\infty)$ such that $T_i\to\infty$ as $i\to\infty$. By assumptions, for each $i\in\mathbf{N}$, there exists $a_i,b_i,c_i\in[0,\infty)$ such that $T_i\le a_i<c_i<b_i$, and (21) holds with a,b,c replaced by a_i,b_i,c_i, respectively. From that, every nontrivial solution $y(t)$ of (11) has no zero $t_i\in(a_i,b_i)$.

Noting that $t_i>a_i\ge T_i$, $i\in\mathbf{N}$ we see that $y(t)$ is a eventually positive solution of (11). This contradiction proves that Theorem 4 holds.

Theorem 5. Assume that (C1), (H4) and (H5) hold. For some functions $(H_1,H_2)\in\mathbf{H}$, each $T\ge 0$ and some $\tilde{K}>0$, *if*

$$\limsup_{t\to\infty}\int_T^t H_1(s,T)\left\{q_j(s)-\frac{1}{4}R_{\tilde{K}}(s)\lambda_1^2(s,T)\right\}\tilde{\phi}(s)ds>0 \tag{25}$$

and

$$\limsup_{t\to\infty}\int_T^t H_2(t,s)\left\{q_j(s)-\frac{1}{4}R_{\tilde{K}}(s)\lambda_2^2(t,s)\right\}\tilde{\phi}(s)ds>0\,, \tag{26}$$

where $\tilde{\phi}(t)\in C^1((T_0,\infty);(0,\infty))$ for some $T_0>0$, then (11) has no eventually positive solution.

Proof:

For any $T\ge t_0$, let $a=T$. In (25) we choose $a=T$. Then there exists $c>a$ such that

$$\limsup_{t\to\infty}\int_a^t H_1(s,a)\left\{q_j(s)-\frac{1}{4}R_{\tilde{K}}(s)\lambda_1^2(s,a)\right\}\tilde{\phi}(s)ds>0\,. \tag{27}$$

In (26), we choose $T=c$. Then, there exists $b>c$ such that

$$\limsup_{t\to\infty}\int_c^t H_2(t,s)\left\{q_j(s)-\frac{1}{4}R_{\tilde{K}}(s)\lambda_2^2(t,s)\right\}\tilde{\phi}(s)ds>0\,. \tag{28}$$

Combining (27) and (28), we obtain (21). The conclusion comes from Theorem 4, and the proof is completed.

There exists an oscillatory function $\theta(t)$ such that

$$r(t)e^{-\tilde{R}(t)}\left(e^{\tilde{R}(t)}\theta'(t)\right)' = f(t)\,, \qquad \lim_{t\to\infty}\theta(t)=0\,. \tag{H6}$$

Theorem 6. Assume that (C1), (H4) and (H6) hold. If the Riccati inequality

$$z'(t)+\frac{1}{2}\frac{1}{\tilde{P}_{\tilde{K}}(t)}z^2(t)\le -\tilde{Q}(t) \tag{29}$$

has no solution on $[T,\infty)$ for all large T, *then* (11) has no eventually positive solution, where

$$\tilde{P}_{\tilde{K}}(t) = \frac{e^{\tilde{R}(\sigma_j(t))}}{2d\tilde{K}\sigma}, \qquad \tilde{Q}(t) = q_j(t)\frac{e^{\tilde{R}(t)}}{r(t)}$$

for some $d \in (0,1)$ and some $\tilde{K} > 0$.

Proof:

Suppose that $y(t)$ is a positive solution of (11) on $[t_0, \infty)$ for some $t_0 > 0$. From (11) there exists a $j \in \{1,2,\cdots,m\}$ such that

$$(r(t)y'(t))' + p(t)y'(t) + q_j(t)\varphi_j(y(\sigma_j(t))) \le f(t), \qquad t \ge t_0,$$

which can be rewritten as

$$\left(e^{\tilde{R}(t)}y'(t)\right)' + q_j(t)\frac{e^{\tilde{R}(t)}}{r(t)}\varphi_j(y(\sigma_j(t))) \le \frac{e^{\tilde{R}(t)}}{r(t)} f(t), \qquad t \ge t_0.$$

We consider

$$w(t) = y(t) - \theta(t),$$

Then,

$$\left(e^{\tilde{R}(t)}w'(t)\right)' \le -q_j(t)\frac{e^{\tilde{R}(t)}}{r(t)}\varphi_j(y(\sigma_j(t))) \le 0. \tag{30}$$

Hence, $e^{\tilde{R}(t)}w'(t)$ is non-increasing, and so,

$$w'(\sigma_j(t)) \ge e^{\tilde{R}(t)-\tilde{R}(\sigma_j(t))}w'(t). \tag{31}$$

By standard argument, we see that $w'(t) \ge 0$. On the other hand, we obvious that $w(t)$ is eventually positive, otherwise, $y(t) \le \theta(t)$ and this contradicts the hypothesis (H6). Since $w(t) > 0$, $w'(t) \ge 0$ eventually, $w(\sigma_j(t)) \ge k_0$ for some $k_0 > 0$. Hence, we obtain $\varphi_j'(dw(\sigma_j(t))) \ge \varphi_j'(dk_0) \equiv \tilde{K}$. Define $z(t)$ by

$$z(t) = \frac{e^{\tilde{R}(t)} w'(t)}{\varphi_j(dw(\sigma_j(t)))},$$

we conclude from (30) and (31) that

$$z'(t) = \frac{(e^{\tilde{R}(t)} w'(t))'}{\varphi_j(dw(\sigma_j(t)))} - de^{\tilde{R}(t)} w'(t) \frac{\varphi_j'(dw(\sigma_j(t)))w'(\sigma_j(t))\sigma_j'(t)}{\varphi_j^2(dw(\sigma_j(t)))}$$

$$\le -\tilde{Q}(t)\frac{\varphi_j(y(\sigma_j(t)))}{\varphi_j(dw(\sigma_j(t)))} - \frac{1}{2\tilde{P}(t)} z^2(t), \qquad t \ge t_1 \tag{32}$$

for some $t_1 \ge t_0$. Furthermore, since $\theta(t) \to 0$ as $t \to \infty$, and $w(t) = y(t) - \theta(t)$, there exists a $t_2 \ge t_1$ such that

$$y(\sigma_j(t)) \ge dw(\sigma_j(t)), \quad \text{for} \quad t \ge t_2 \text{ and } d \in (0,1).$$

By the hypothesis (H4), we have

$$\varphi_j(y(\sigma_j(t))) \ge \varphi_j(dw(\sigma_j(t))), \qquad t \ge t_2. \tag{33}$$

Substituting (33) into (32), we obtain the inequality (29), which has a positive solution $z(t)$ on $[t_2, \infty)$. This contradicts the hypothesis and completes the proof.

Theorem 7. Assume that (C1), (H4) and (H6) hold. If for some $d \in (0,1)$ and some $\tilde{K} > 0$, there exist $(H_1, H_2) \in \mathbf{H}$ and $a, b, c \in \mathbf{R}$ such that $T \le a < c < b$ for each $T \ge 0$ and

$$\frac{1}{H_1(c,a)} \int_a^c H_1(s,a)\left\{\tilde{Q}(s) - \frac{1}{2}\tilde{P}_{\tilde{K}}(s)\tilde{\lambda}_1^2(s,a)\right\}\tilde{\phi}(s)ds$$
$$+ \frac{1}{H_2(b,c)} \int_c^b H_2(b,s)\left\{\tilde{Q}(s) - \frac{1}{2}\tilde{P}_{\tilde{K}}(s)\tilde{\lambda}_2^2(b,s)\right\}\tilde{\phi}(s)ds > 0, \tag{34}$$

then (11) has no eventually positive solution, where $\tilde{\phi}(t) \in C^1((T_0, \infty); (0, \infty))$ for some $T_0 > 0$, and

$$\tilde{\lambda}_1(s,t) = \frac{\tilde{\phi}'(s)}{\tilde{\phi}(s)} + h_1(s,t), \qquad \tilde{\lambda}_2(t,s) = \frac{\tilde{\phi}'(s)}{\tilde{\phi}(s)} - h_2(t,s).$$

Proof:

Suppose that $y(t)$ is a positive solution of (11) on $[t_0,\infty)$ for some $t_0>0$. Proceeding as the same proof of Theorem 3, multiplying (29) by $H_2(t,s)$ and $\tilde{\phi}(s)$, and integrating over $[c,t]$ for $t\in[c,b)$, we have

$$\int_c^t H_2(t,s)\tilde{Q}(s)\tilde{\phi}(s)ds \le H_2(t,c)z(c)\tilde{\phi}(c)+\frac{1}{2}\int_c^t H_2(t,s)\tilde{\lambda}_2^2(t,s)\tilde{P}_{\tilde{K}}(s)\tilde{\phi}(s)ds$$

and letting $t\to b^-$ in the above, we obtain

$$\frac{1}{H_2(b,c)}\int_c^b H_2(b,s)\left\{\tilde{Q}(s)-\frac{1}{2}\tilde{P}_{\tilde{K}}(s)\tilde{\lambda}_2^2(b,s)\right\}\tilde{\phi}(s)ds \le z(c)\tilde{\phi}(c).$$

On the other hand, multiplying (29) by $H_1(s,t)$ and $\tilde{\phi}(s)$, integrating over $[t,c]$ for $t\in(a,b]$, and letting $t\to a^+$ in the above, we obtain

$$\frac{1}{H_1(c,a)}\int_a^c H_2(s,a)\left\{\tilde{Q}(s)-\frac{1}{2}\tilde{P}_{\tilde{K}}(s)\tilde{\lambda}_1^2(s,a)\right\}\tilde{\phi}(s)ds \le -z(c)\tilde{\phi}(c).$$

By adding the above two inequalities, we can lead to the contradiction.

Theorem 8. Assume that (C1), (H4) and (H6) hold. For some functions $(H_1,H_2)\in\mathbf{H}$, each $T>0$ and some $\tilde{K}>0$, if

$$\limsup_{t\to\infty}\int_T^t H_1(s,T)\left\{\tilde{Q}(s)-\frac{1}{2}\tilde{P}_{\tilde{K}}(s)\tilde{\lambda}_1^2(s,T)\right\}\tilde{\phi}(s)ds>0 \tag{35}$$

and

$$\limsup_{t\to\infty}\int_T^t H_2(t,s)\left\{\tilde{Q}(s)-\frac{1}{2}\tilde{P}_{\tilde{K}}(s)\tilde{\lambda}_2^2(t,s)\right\}\tilde{\phi}(s)ds>0, \tag{36}$$

where $\tilde{\phi}(t)\in C^1((T_0,\infty);(0,\infty))$ for some $T_0>0$, then (11) has no eventually positive solution.

3.2. For the case (C2)

In the proof of the subsequent theorem we shall use the following lemma which was extended the result of Kusano and Naito (1975).

Lemma 1: Assume that (C2) holds. If $y(t)$ is an eventually positive solution of (11), then $y(t)$ satisfies the inequality

$$e^{\tilde{R}(t)}y'(t)\pi(t)+y(t)\geq 0,$$

where

$$\pi(t)=\int_t^\infty e^{-\tilde{R}(s)}ds \text{ for all sufficiently large } t\geq t_0 .$$

Proof:

Let $y(t)>0$, $t\geq t_0$ for some $t_0>0$. For the same proof of Theorems 3 and 6, we obvious that

$$\left(e^{\tilde{R}(t)}y'(t)\right)'\leq 0, \qquad t\geq t_0,$$

and so

$$e^{\tilde{R}(u)}y'(u)\leq e^{\tilde{R}(s)}y'(s), \qquad t_0<s<u .$$

Dividing the above inequality by $e^{\tilde{R}(u)}$ and integrating over $[s,t]$ yields

$$0<y(t)\leq y(s)+e^{\tilde{R}(s)}y'(s)\int_s^t e^{-\tilde{R}(u)}du, \qquad t_0<s<u . \tag{37}$$

Then, we consider following two cases:

<u>Case 1.</u> $y'(t)\geq 0$. From (37) we have

$$0<y(t)\leq y(s)+e^{\tilde{R}(s)}y'(s)\pi(s) .$$

<u>Case 2.</u> $y'(t)<0$. Since $y(t)>0$, we see that $y(t)$ is bounded from above. Letting

$t \to \infty$ in (37), we obtain the inequality.

Theorem 9. Assume that (C2), (H4) and (H5). If the Riccati inequalities

$$z_i'(t) + \frac{1}{2}\frac{1}{P_i(t)} z_i^2(t) \le -Q_i(t) \quad (i = 1,2) \tag{38}$$

have no solution on $[T,\infty)$ for all large $T > 0$ and some $\tilde{K} > 0$, then (11) has no eventually positive solution, where $\phi(t) \in C^1((T_0,\infty);(0,\infty))$ for some $T_0 > 0$, and

$$P_1(t) = \phi(t) R_\pi(t), \quad P_2(t) = P_{\tilde{K}}(t),$$

$$Q_1(t) = \phi(t) q_j(t) - \frac{1}{2}\left(\frac{\phi(t)p(t)}{r(t)} - \phi'(t)\right)^2 \frac{R_\pi(t)}{\phi(t)}, \qquad Q_2(t) = Q_{\tilde{K}}(t),$$

where

$$R_\pi(t) = \frac{r(t)}{\varphi_j'(c_1\pi(t))}.$$

Proof:

Suppose that $y(t)$ is a positive solution of (11) on $[t_0,\infty)$ for some $t_0 \ge T > 0$. From (11) there exist $j \in \{1,2,\cdots,m\}$ and $a,b \ge t_0$ such that $f(t) \le 0$ on the interval $I \in [a,b]$, and so, the inequality (13) holds. In view of (14) we see that $y'(t) \ge 0$ or $y'(t) < 0$ eventually.

<u>Case 1.</u> $y'(t) < 0$. It follows from Lemma 1 that

$$y(t) \ge c_1\pi(t), \qquad t \ge t_1$$

for all large $t_1 \ge t_0$, where $c_1 = -e^{\tilde{R}(t_1)} y'(t_1) > 0$. We set

$$v(t) = \frac{r(t)y'(t)}{\varphi_j(y(t))},$$

then

$$v'(t) = \frac{(r(t)y'(t))'}{\varphi_j(y(t))} - r(t)\varphi_j'(y(t))\left(\frac{y'(t)}{\varphi_j(y(t))}\right)^2 \le -\frac{p(t)}{r(t)}v(t) - q_j(t) - \frac{1}{R_\pi(t)}v^2(t) .$$

Multiplying the above inequality by $\phi(t)$, and using Hölder's inequality, we have

$$(\phi(t)v(t))' + \frac{1}{2}\left(\frac{\phi(t)}{R_\pi(t)}\right)v^2(t) \le -Q_1(t), \qquad t \in I .$$

We define

$$z_1(t) = \phi(t)v(t) ,$$

then $z_1(t)$ is a negative solution of (38) on I. This is a contradiction.

Case 2. $y'(t) \ge 0$. There exists a constant k_0 such that $y(\sigma_j(t)) \ge k_0$ for some k_0. Hence, we obtain $\varphi_j'(y(\sigma_j(t))) \ge \varphi_j'(k_0) \equiv \tilde{K}$, and so the subsequent proof proceeds as in the corresponding part of the proof of Theorem 3. The proof is complete.

Theorem 10. Assume that (C2), (H4) and (H5). If for each $T \ge 0$ and some $\tilde{K} > 0$, there exist $(H_1, H_2) \in \mathbf{H}$ and some $c \in (a,b)$ such that (21) and

$$\frac{1}{H_1(c,a)}\int_a^c H_1(s,a)\left\{q_j(s) - \frac{1}{4}R_\pi(s)\lambda_1^2(s,a)\right\}\tilde{\phi}(s)ds$$
$$+\frac{1}{H_2(b,c)}\int_c^b H_2(b,s)\left\{q_j(s) - \frac{1}{4}R_\pi(s)\lambda_2^2(b,s)\right\}\tilde{\phi}(s)ds > 0, \qquad (39)$$

where $\tilde{\phi}(t) \in C^1((T_0,\infty);(0,\infty))$ for some $T_0 > 0$, *then* (11) *has no eventually positive* solution.

Theorem 11. Assume that (C2), (H4) and (H5) hold. For some functions $(H_1, H_2) \in \mathbf{H}$, each $T \ge 0$ and some $\tilde{K} > 0$, *if* (25), (26) and

$$\limsup_{t\to\infty}\int_T^t H_1(s,T)\left\{q_j(s) - \frac{1}{4}R_\pi(s)\lambda_1^2(s,T)\right\}\tilde{\phi}(s)ds > 0 \qquad (40)$$

and

$$\limsup_{t\to\infty}\int_T^t H_2(t,s)\left\{q_j(s)-\frac{1}{4}R_\pi(s)\lambda_2^2(t,s)\right\}\tilde{\phi}(s)ds>0\,, \tag{41}$$

where $\tilde{\phi}(t)\in C^1((T_0,\infty);(0,\infty))$ for some $T_0>0$, then (11) has no eventually positive solution.

Now we will be used the following notation:

$$[\delta(t)]_\pm=\max\{0,\pm\delta(t)\}\,.$$

Theorem 12. Assume that (C2), (H4) and (H6). If the Riccati inequalities

$$z_i'(t)+\frac{1}{2}\frac{1}{\tilde{P}_i(t)}z_i^2(t)\le-\tilde{Q}_i(t)\quad (i=1,2) \tag{42}$$

have no solution on $[T,\infty)$ for all large $T>0$ and some $\tilde{K}>0$, $K_1>0$, then (11) has no eventually positive solution, where $\phi(t)\in C^1((T_0,\infty);(0,\infty))$ for some $T_0>0$, and

$$\tilde{P}_1(t)=\frac{e^{\tilde{R}(t)}}{2\varphi'_j(c_1\pi(t))},\quad \tilde{P}_2(t)=\tilde{P}_{\tilde{K}}(t),$$
$$\tilde{Q}_1(t)=\tilde{Q}(t)\frac{\varphi_j([c_1\pi(\sigma_j(t))+\theta(\sigma_j(t))]_+)}{K_1},\quad \tilde{Q}_2(t)=\tilde{Q}(t)\,.$$

Proof:

Suppose that $y(t)$ is a positive solution of (11) on $[t_0,\infty)$ for some $t_0>0$. By the hypothesis (H6) and the definition $w(t)$, we see that $w(t)>0$. In view of (30), we find that $w'(t)\ge 0$ or $w'(t)<0$ eventually.

<u>Case 1.</u> $w'(t)<0$. It is obvious from Lemma 1 that

$$w(t)\ge c_1\pi(t),\qquad t\ge t_1$$

for all large $t_1\ge t_0$, where $c_1=-e^{\tilde{R}(t_1)}w'(t_1)>0$. Now we define

$$z_1(t) = \frac{e^{\tilde{R}(t)} w'(t)}{\varphi_j(w(t))},$$

and so,

$$z_1'(t) = \frac{(e^{\tilde{R}(t)} w'(t))'}{\varphi_j(w(t))} - e^{\tilde{R}(t)} \varphi_j'(w(t)) \left(\frac{w'(t)}{\varphi_j(w(t))} \right)^2$$

$$\le -\tilde{Q}(t) \frac{\varphi_j(y(\sigma_j(t)))}{\varphi_j(w(t))} - \frac{1}{2\tilde{P}_1(t)} z_1^2(t), \qquad t \ge t_1.$$

Since $w(t) > 0$ and $w'(t) < 0$, there is a constant k_1 such that $k_1 \ge w(t)$, $t \ge t_2$ for some $t_2 \ge t_1$, and hence $\varphi_j(w(t)) \le \varphi_j(k_1) \equiv K_1$. On the other hand, we see that

$$y(t) \ge c_1 \pi(t) + \theta(t), \qquad t \ge t_2.$$

Since $y(t)$ is eventually positive, we have

$$\varphi_j(y(\sigma_j(t))) \ge \varphi_j([c_1 \pi(\sigma_j(t)) + \theta(\sigma_j(t))]_+), \qquad t \ge t_3$$

for some $t_3 \ge t_2$. Thus it is easy to see that

$$z_1'(t) \le -\tilde{Q}_1(t) - \frac{1}{2\tilde{P}_1(t)} z_1^2(t), \quad t \ge t_3.$$

Therefore, $z_1(t)$ is a negative solution of (42), which is a contradiction.

Case 2. $w'(t) \ge 0$. There exists a constant k_0 such that $w(\sigma_j(t)) \ge k_0$ for some $k_0 > 0$. Especially, we obtain $\varphi_j'(dw(\sigma_j(t))) \ge \varphi_j'(dk_0) \equiv \tilde{K}$. The rest of proof is similar to the proof of Theorem 6, and hence the proof is complete.

Theorem 13. Assume that (C2), (H4) and (H6). *If* (34) and

$$\frac{1}{H_1(c,a)}\int_a^c H_1(s,a)\left\{\tilde{Q}_1(s)-\frac{1}{2}\tilde{P}_1(s)\tilde{\lambda}_1^2(s,a)\right\}\tilde{\phi}(s)ds$$
$$+\frac{1}{H_2(b,c)}\int_c^b H_2(b,s)\left\{\tilde{Q}_1(s)-\frac{1}{2}\tilde{P}_1(s)\tilde{\lambda}_2^2(b,s)\right\}\tilde{\phi}(s)ds>0, \tag{43}$$

where $\tilde{\phi}(t)\in C^1((T_0,\infty);(0,\infty))$ for some $T_0>0$, then (11) has no eventually positive solution.

Theorem 14. Assume that (C2), (H4) and (H6) hold. For some functions $(H_1,H_2)\in\mathbf{H}$, *each* $T\geq 0$ *and some* $\tilde{K}>0$, *if* (35), (36) *and*

$$\limsup_{t\to\infty}\int_T^t H_1(s,T)\left\{\tilde{Q}_1(s)-\frac{1}{2}\tilde{P}_1(s)\lambda_1^2(s,T)\right\}\tilde{\phi}(s)ds>0 \tag{44}$$

and

$$\limsup_{t\to\infty}\int_T^t H_2(t,s)\left\{\tilde{Q}_1(s)-\frac{1}{2}\tilde{P}_1(s)\lambda_2^2(t,s)\right\}\tilde{\phi}(s)ds>0, \tag{45}$$

where $\tilde{\phi}(t)\in C^1((T_0,\infty);(0,\infty))$ for some $T_0>0$, then (11) has no eventually positive solution.

5. Oscillation criteria for the equation (E)

We will derive the sufficient conditions for oscillation of solutions of the equation (E) combining the results of Sections 2-4. Using Riccati inequality, we derive the sufficient conditions for oscillation of solutions of hyperbolic equation (E). We need the following lemma which was obtained by Usami (1998).

Lemma 2. If there exists a function $\phi(t)\in C^1([T_0,\infty);(0,\infty))$ such that

$$\int_{T_1}^{\infty}\left(\frac{\overline{p}(t)\,|\phi'(t)|^{\beta}}{\phi(t)}\right)^{\frac{1}{\beta-1}}dt<\infty, \qquad \int_{T_1}^{\infty}\frac{1}{\overline{p}(t)(\phi(t))^{\beta-1}}dt=\infty,$$
$$\int_{T_1}^{\infty}\phi(t)\overline{q}(t)dt=\infty$$

for some $T_1 \ge T_0$, then general Riccati inequality

$$x'(t) + \frac{1}{2}\frac{1}{\bar{p}(t)} |x(t)|^{\beta} \le -\bar{q}(t),$$

where $\beta > 1$, $\bar{p}(t) \in C([T_0,\infty);(0,\infty))$ and $\bar{q}(t) \in C([T_0,\infty);\mathbf{R})$ has no solution on $[T,\infty)$ for all large T.

5.1. Oscillation results by Riccati inequality for the case (C1)

Combining Theorems 1-3 and Lemma 2, we obtain following theorem.

Theorem 15. Assume that (C1), (H1)-(H5), and that there exists $T_0 \le a < b < \tilde{a} < \tilde{b}$ such that

$$G(t) \quad [resp. \tilde{G}(t)] = \begin{cases} \le 0, & t \in [a,b], \\ \ge 0, & t \in [\tilde{a},\tilde{b}] \end{cases}$$

for some $T_0 \ge 0$. (H7)

If

$$\int_{T_1}^{\infty} \left(\frac{P_{\tilde{K}}(t)\phi'(t)^2}{\phi(t)} \right) dt < \infty, \qquad \int_{T_1}^{\infty} \frac{1}{P_{\tilde{K}}(t)\phi(t)} dt = \infty,$$

$$\int_{T_1}^{\infty} \phi(t) Q_{\tilde{K}}(t) dt = \infty,$$

where $\phi(t) \in C^1((T_0,\infty);(0,\infty))$ for some $T_0 > 0$ and some $\tilde{K} > 0$, then every solution $u(x,t)$ *of* (E), (B1) [*resp.* (E), (B2)] is oscillatory in Ω.

Combining Theorems 1-2, 6 and Lemma 2, we obtain following theorems.

Theorem 16. Assume that (C1), (H1)-(H5), and that there exists an oscillatory function $\Theta(t)$ such that

$$r(t) e^{-\tilde{R}(t)} \left(e^{\tilde{R}(t)} \Theta'(t) \right)' = G(t) \quad [resp. \tilde{G}(t)], \quad \lim_{t\to\infty} \Theta(t) = 0. \tag{H8}$$

If

$$\int_{T_1}^{\infty}\left(\frac{\tilde{P}_{\tilde{K}}(t)\phi'(t)^2}{\phi(t)}\right)dt<\infty,\quad \int_{T_1}^{\infty}\frac{1}{\tilde{P}_{\tilde{K}}(t)\phi(t)}dt=\infty\text{, and }\int_{T_1}^{\infty}\phi(t)\tilde{Q}(t)dt=\infty,$$

where $\phi(t)\in C^1((T_0,\infty);(0,\infty))$ for some $T_0>0$, some $d\in(0,1)$ and some $\tilde{K}>0$, then every solution $u(x,t)$ *of* (E), (B1) [resp. (E), (B2)] is oscillatory in Ω.

Example 1. We consider the problem

$$\frac{\partial}{\partial t}\left(t^2\frac{\partial}{\partial t}u(x,t)\right)-2t\frac{\partial}{\partial t}u(x,t)-t^2\Delta u(x,t)-t^4\Delta u(x,t-2\pi)-\Delta u\left(x,t-\frac{\pi}{2}\right)$$
$$+t^4u(x,t-\pi)=\sin x\sin t,\quad (x,t)\in(0,\pi)\times[1,\infty), \tag{46}$$

$$u(0,t)=u(\pi,t)=0,\qquad t\ge 1. \tag{47}$$

Here, $n=m=1$, $k=2$, $r(t)=t^2$, $p(t)=-2t$, $q_1(x,t)=t^4$, $\sigma_1(t)=t-\pi$ and $\varphi'_1(\xi)=1=\mu$. Since

$$\tilde{R}(t)=\int_1^t\left(\frac{2s-2s}{s^2}\right)ds=0,\quad R_{\tilde{K}}(t)=t^2,\quad \int_1^t e^{-\tilde{R}(t)}dt=\infty,\quad G(t)=\frac{\pi}{4}\sin t,$$

hold, (H5) and (H8) are satisfied. If we choose $\phi(t)=t^{-2}$, then

$$P_{\tilde{K}}(t)=t^{-2}\times t^2=1,\quad Q_{\tilde{K}}(t)=t^{-2}\times t^4-\frac{1}{2}\left(\frac{t^{-2}(-2t)}{t^2}+2t^{-3}\right)^2\frac{R_{\tilde{K}}(t)}{t^{-2}}=t^2.$$

By straightforward calculations we obtain

$$\int_1^{\infty}\frac{(-2t^{-3})^2}{t^{-2}}dt=4\int_1^{\infty}\left(\frac{1}{t^4}\right)dt<\infty,\quad \int_1^{\infty}\frac{1}{t^{-2}}dt=\infty,\quad \int_1^{\infty}(t^{-2}\times t^2)dt=\infty.$$

Therefore Theorem 15 is applicable, and hence, every solution $u(x,t)$ of the problem

(46), (47) is oscillatory in $(0,\pi)\times[1,\infty)$. Indeed, $u(x,t)=\sin x\cos t$ is such a solution.

Example 2. We consider the problem

$$\frac{\partial}{\partial t}\left(\frac{1}{t^6}\frac{\partial}{\partial t}u(x,t)\right)+\frac{2}{t^7}\frac{\partial}{\partial t}u(x,t)-\left(t^2+\frac{1}{t^6}\right)\Delta u(x,t)-\left(\frac{6}{t^9}+\frac{3}{t^7}\right)\Delta u\left(x,t-\frac{3}{2}\pi\right)$$
$$-\frac{6}{t^8}\Delta u(x,t-2\pi)+t^2u(x,t-\pi)$$
$$=\left(\frac{6}{t^9}\cos t+\frac{6}{t^8}\sin t-\frac{1}{t^7}\cos t\right)\sin x, \qquad (x,t)\in(0,\pi)\times[1,\infty), \tag{48}$$

$$u(0,t)=u(\pi,t)=0, \qquad t\ge 1. \tag{49}$$

Here, $n=1$, $k=2$, $m=1$, $r(t)=t^{-6}$, $p(t)=2t^{-7}$, $q_1(x,t)=t^2$, $\sigma_1(t)=t-\pi$ and $\varphi'_1(\xi)=1=\mu$. Since

$$\tilde{R}(t)=\ln(t^{-4}), \quad \int_1^\infty e^{-\tilde{R}(t)}dt=\infty, \quad \Theta(t)=\frac{1}{t}\cos t,$$

hold, (H5) and (H9) are fulfilled.

If we choose $\phi(t)=t^2$, then

$$\tilde{P}_{\tilde{K}}(t)=\frac{(t-\pi)^{-4}}{2\times\frac{1}{2}}=\frac{1}{(t-\pi)^4}, \quad \tilde{Q}(t)=t^2\times\frac{t^{-4}}{t^{-6}}=t^4.$$

It is easy to see that

$$\int_1^\infty\left(\frac{(t-\pi)^{-4}\times(2t)^2}{t^2}\right)dt=\int_1^\infty\frac{4}{(t-\pi)^4}dt<\infty, \quad \int_1^\infty\frac{1}{((t-\pi)^{-4}\times t^2)}dt=\int_1^\infty\frac{(t-\pi)^4}{t^2}dt=\infty,$$

and

$$\int_1^\infty(t^2\times t^4)dt=\infty.$$

Therefore, all conditions of Theorem 16 hold, and so, every solution $u(x,t)$ of the problem (48), (49) is oscillatory in $(0,\pi)\times[1,\infty)$. For example, $u(x,t)=\sin x\sin t$ is such a solution.

5.2. Interval oscillation results for the case (C1)

Combining Theorems 1-2 and 4-5, we have following two theorems.

Theorem 17. Assume that (C1), (H1)-(H4) *and* (H7). If for each $T \geq 0$ and some $\tilde{K} > 0$, there exist $(H_1, H_2) \in \mathbf{H}$ and some $c \in (a,b)$, $\tilde{c} \in (\tilde{a}, \tilde{b})$ such that $T \leq a < b < \tilde{a} < \tilde{b}$, (21) and

$$\frac{1}{H_1(\tilde{c},\tilde{a})}\int_{\tilde{a}}^{\tilde{c}} H_1(s,\tilde{a})\left\{q_j(s) - \frac{1}{4}R_{\tilde{K}}(s)\lambda_1^2(s,\tilde{a})\right\}\tilde{\phi}(s)ds$$

$$+\frac{1}{H_2(\tilde{b},\tilde{c})}\int_{\tilde{c}}^{\tilde{b}} H_2(\tilde{b},s)\left\{q_j(s) - \frac{1}{4}R_{\tilde{K}}(s)\lambda_2^2(\tilde{b},s)\right\}\tilde{\phi}(s)ds > 0, \qquad (50)$$

where $\tilde{\phi}(t) \in C^1((T_0,\infty);(0,\infty))$ for some $T_0 > 0$, then every solution $u(x,t)$ of (E), (B1) [resp. (E), (B2)] is oscillatory in Ω.

Theorem 18. Assume that (C1), (H1)-(H4) and (H7). For some functions $(H_1, H_2) \in \mathbf{H}$, *each* $T \geq 0$ and some $\tilde{K} > 0$, if (25) and (26) hold, then every solution $u(x,t)$ *of* (E), (B1) [resp. (E), (B2)] is oscillatory in Ω, where $\tilde{\phi}(t) \in C^1((T_0,\infty);(0,\infty))$ for some $T_0 > 0$.

Combining Theorems 1-2 and 7-8, we have following two theorems.

Theorem 19. Assume that (C1), (H1)-(H4) and (H8). If for each $T \geq 0$, some $d \in (0,1)$ and some $\tilde{K} > 0$, there exist $(H_1, H_2) \in \mathbf{H}$ and $a,b,c \in \mathbf{R}$ such that $T \leq a < c < b$ and (34) hold, then every solution $u(x,t)$ *of* (E), (B1) [resp. (E), (B2)] is oscillatory in Ω, where $\tilde{\phi}(t) \in C^1((T_0,\infty);(0,\infty))$ for some $T_0 > 0$.

Theorem 20. Assume that (C1), (H1)-(H4) and (H8). For some functions $(H_1, H_2) \in \mathbf{H}$, each $T \geq 0$, some $d \in (0,1)$ and some $\tilde{K} > 0$, if (35) and (36) hold, then every solution $u(x,t)$ of (E), (B1) [resp. (E), (B2)] is oscillatory in Ω, where $\tilde{\phi}(t) \in C^1((T_0,\infty);(0,\infty))$ for some $T_0 > 0$.

5.3. Oscillation results by Riccati inequality for the case (C2)

Combining Theorems 1-2, 9 and Lemma 2, we obtain following theorem.

Theorem 21. Assume that (C2), (H1)-(H4) and (H7) hold. If for $i = 1,2$

$$\int_{T_1}^{\infty}\left(\frac{P_i(t)\phi'(t)^2}{\phi(t)}\right)dt < \infty, \quad \int_{T_1}^{\infty}\frac{1}{P_i(t)\phi(t)}dt = \infty, \text{ and } \int_{T_1}^{\infty}\phi(t)Q_i(t)dt = \infty,$$

where $\phi(t) \in C^1((T_0,\infty);(0,\infty))$ for some $T_0 > 0$, then every solution $u(x,t)$ *of* (E), (B1) [resp. (E), (B2)] is oscillatory in Ω.

Combining Theorems 1-2, 12 and Lemma 2, we obtain following theorem.

Theorem 22. Assume that (C2), (H1)-(H4) and (H8). If for $i = 1,2$

$$\int_{T_1}^{\infty}\left(\frac{\tilde{P}_i(t)\phi'(t)^2}{\phi(t)}\right)dt < \infty, \quad \int_{T_1}^{\infty}\frac{1}{\tilde{P}_i(t)\phi(t)}dt = \infty, \text{ and } \int_{T_1}^{\infty}\phi(t)\tilde{Q}_i(t)dt = \infty,$$

then every solution $u(x,t)$ *of* (E), (B1) [resp. (E), (B2)] is oscillatory in Ω, where $\phi(t) \in C^1((T_0,\infty);(0,\infty))$ for some $T_0 > 0$ and

$$\tilde{Q}_1(t) = \tilde{Q}(t)\frac{\varphi_j([c_1\pi(\sigma_j(t)) \pm \Theta(\sigma_j(t))]_+)}{K_1}.$$

Example 3. We consider the problem

$$\frac{\partial}{\partial t}\left(e^{\frac{t}{2}}\frac{\partial}{\partial t}u(x,t)\right) + \frac{1}{2}e^{\frac{t}{2}}\frac{\partial}{\partial t}u(x,t) - \Delta u(x,t) - \Delta u(x,t-\pi)$$
$$+ u(x,t-2\pi) + \sqrt{2}e^t u\left(x,t-\frac{5}{4}\pi\right) + \sqrt{2}\left(e^{\frac{t}{2}} + e^t\right)u\left(x,t-\frac{\pi}{4}\right)$$
$$= \cos x \sin t, \qquad (x,t) \in \left(0,\frac{\pi}{2}\right)\times[1,\infty), \tag{51}$$

$$-u_x(0,t) = 0, \qquad u_x\left(\frac{\pi}{2},t\right) = -\sin t, \qquad t \geq 1. \tag{52}$$

Here, $n = k = 1$, $m = 3$, $r(t) = e^{\frac{t}{2}}$, $p(t) = (1/2)e^{\frac{t}{2}}$, $q_1(x,t) = 1$, $\sigma_1(t) = t - 2\pi$,

$q_2(x,t)=\sqrt{2}e^t$, $\sigma_2(t)=t-5\pi/4$, $q_3(x,t)=\sqrt{2}\left(e^{\frac{t}{2}}+e^t\right)$, $\sigma_3(t)=t-\pi/4$ and $f(x,t)=\cos x\sin t$.

It is easy to check that $\tilde{R}(t)=t-1$ and

$$\tilde{G}(t)=\frac{2}{\pi}\sin t\,,\qquad \int_{t_0}^{\infty}e^{-\tilde{R}(t)}dt=\int_{t_0}^{\infty}e^{-t+1}dt<\infty.$$

If we choose $\phi(t)=e^{-\frac{t}{2}}$, $j=2$, then

$$\int^{\infty}\left(\frac{P_1(t)\phi'(t)^2}{\phi(t)}\right)dt=\int^{\infty}\frac{\left(-\frac{1}{2}e^{-\frac{t}{2}}\right)^2}{e^{-\frac{t}{2}}}dt=\frac{1}{4}\int^{\infty}e^{-\frac{t}{2}}dt<\infty,$$

$$\int^{\infty}\left(\frac{P_2(t)\phi'(t)^2}{\phi(t)}\right)dt=\int^{\infty}\frac{e^{-\frac{5}{4}\pi}\left(-\frac{1}{2}e^{-\frac{t}{2}}\right)^2}{e^{-\frac{t}{2}}}dt=\frac{1}{4}\int^{\infty}e^{-\frac{t}{2}-\frac{5}{4}\pi}dt<\infty,$$

$$\int^{\infty}\left(\frac{1}{P_1(t)\phi(t)}\right)dt=\int^{\infty}\frac{1}{e^{-\frac{t}{2}}}dt=\infty,$$

$$\int^{\infty}\left(\frac{1}{P_2(t)\phi(t)}\right)dt=\int^{\infty}\frac{1}{e^{-\frac{5}{4}\pi}\times e^{-\frac{t}{2}}}dt=\infty$$

$$\int^{\infty}\phi(t)Q_1(t)dt=\int^{\infty}e^{-\frac{t}{2}}\left(\sqrt{2}e^{\frac{t}{2}}-\frac{1}{2}\right)dt=\infty,$$

$$\int^{\infty}\phi(t)Q_2(t)dt=\int^{\infty}e^{-\frac{t}{2}}\left(\sqrt{2}e^{\frac{t}{2}}-\frac{1}{2}e^{-\frac{5}{4}\pi}\right)dt=\infty.$$

Thus, all conditions of Theorem 21 are satisfied. Therefore every solution $u(x,t)$ of the problem (51), (52) is oscillatory in $(0,\pi)\times[1,\infty)$. Indeed, $u(x,t)=\cos x\sin t$ is such a solution.

Example 4. We consider the problem

$$\frac{\partial}{\partial t}\left(t\frac{\partial}{\partial t}u(x,t)\right)+\frac{\partial}{\partial t}u(x,t)-\Delta u(x,t)-2\Delta u\left(x,t-\frac{\pi}{2}\right)$$
$$+tu(x,t-2\pi)=\sin x\cos t,\qquad (x,t)\in(0,\pi)\times[1,\infty), \tag{53}$$

$$u(0,t)=u(\pi,t)=0,\qquad t\geq 1. \tag{54}$$

Here, $n=k=m=1$, $r(t)=t$, $p(t)=1$, $q_1(x,t)=t$, $\sigma_1(t)=t-2\pi$ and $f(x,t)=\sin x\cos t$.

An easy computation shows that $\tilde{R}(t)=\ln(t^2)$ and

$$\int_{t_0}^{\infty} e^{-\tilde{R}(t)}dt=\int_{t_0}^{\infty}\frac{1}{t^2}dt<\infty, \quad \Theta(t)=-\frac{1}{t}\cos t.$$

If we choose $\phi(t)=t^{-2}$, $d=\frac{1}{2}$, then

$$\int^{\infty}\left(\frac{\tilde{P}_1(t)\phi'(t)^2}{\phi(t)}\right)dt=\frac{1}{2}\int^{\infty}\frac{t^2\times 4t^{-6}}{t^{-2}}dt=2\int^{\infty}t^{-2}dt<\infty,$$

$$\int^{\infty}\left(\frac{\tilde{P}_2(t)\phi'(t)^2}{\phi(t)}\right)dt=\int^{\infty}\frac{(t-2\pi)^2\times 4t^{-6}}{t^{-2}}dt=\int^{\infty}\frac{4(t-2\pi)^2}{t^4}dt<\infty,$$

$$\int^{\infty}\left(\frac{1}{\tilde{P}_1(t)\phi(t)}\right)dt=\int^{\infty}\frac{1}{t^2\times t^{-2}}dt=\infty,$$

$$\int^{\infty}\left(\frac{1}{\tilde{P}_2(t)\phi(t)}\right)dt=\int^{\infty}\frac{1}{(t-2\pi)^2\times t^{-2}}dt=\infty,$$

$$\int^{\infty}\phi(t)\tilde{Q}_1(t)dt=\int^{\infty}t^{-2}\times t^2\left[\frac{1}{(t-2\pi)}(1\pm\cos t)\right]_+=\infty,$$

$$\int^{\infty}\phi(t)\tilde{Q}_2(t)dt=\int^{\infty}t^{-2}\times t^2=\infty.$$

Hence, Theorem 22 is applicable. Therefore every solution $u(x,t)$ of the problem (53), (54) is oscillatory in $(0,\pi)\times[1,\infty)$. For example, $u(x,t)=\sin x\cos t$ is such a solution.

5.4. Interval oscillation results for the case (C2)

Combining Theorems 1-2 and 10-11, we have following two theorems.

Theorem 23. Assume that (C2), (H1)-(H4) and (H7). If for each $T\geq 0$ and some $\tilde{K}>0$, there exist $(H_1,H_2)\in\mathbf{H}$ and some $c\in(a,b)$, $\tilde{c}\in(\tilde{a},\tilde{b})$ such that $T\leq a<b<\tilde{a}<\tilde{b}$, (21), (39), (50) and

$$\frac{1}{H_1(\tilde{c},\tilde{a})}\int_{\tilde{a}}^{\tilde{c}} H_1(s,\tilde{a})\left\{q_j(s)-\frac{1}{4}R_\pi(s)\lambda_1^2(s,\tilde{a})\right\}\tilde{\phi}(s)ds$$
$$+\frac{1}{H_2(\tilde{b},\tilde{c})}\int_{\tilde{c}}^{\tilde{b}} H_2(\tilde{b},s)\left\{q_j(s)-\frac{1}{4}R_\pi(s)\lambda_2^2(\tilde{b},s)\right\}\tilde{\phi}(s)ds>0,$$

where $\tilde{\phi}(t)\in C^1((T_0,\infty);(0,\infty))$ for some $T_0>0$, then every solution $u(x,t)$ *of* (E), (B1) [*resp.* (E), (B2)] is oscillatory in Ω.

Theorem 24. Assume that (C2), (H1)-(H4) and (H7). For some functions $(H_1,H_2)\in\mathbf{H}$, each $T\ge 0$ and some $\tilde{K}>0$, if (25), (26), (40) and (41) hold, then every solution $u(x,t)$ of (E), (B1) [resp. (E), (B2)] is oscillatory in Ω, where $\tilde{\phi}(t)\in C^1((T_0,\infty);(0,\infty))$ for some $T_0>0$.

Combining Theorems 1-2 and 13-14, we have following two theorems.

Theorem 25. Assume that (C2), (H1)-(H4) and (H8). If for each $T\ge 0$, some $d\in(0,1)$ and some $\tilde{K}>0$, there exist $(H_1,H_2)\in\mathbf{H}$ and $a,b,c\in\mathbf{R}$ such that $T\le a<c<b$ and (34) and (43) hold, then every solution $u(x,t)$ *of* (E), (B1) [resp. (E), (B2)] is oscillatory in Ω, where $\tilde{\phi}(t)\in C^1((T_0,\infty);(0,\infty))$ for some $T_0>0$.

Theorem 26. Assume that (C2), (H1)-(H4), (H9) and (H8). For some functions $(H_1,H_2)\in\mathbf{H}$, each $T\ge 0$ some $d\in(0,1)$ and some $\tilde{K}>0$, *if* (35), (36), (44) and (45) hold, then every solution $u(x,t)$ *of* (E), (B1) [resp. (E), (B2)] is oscillatory in Ω, where $\tilde{\phi}(t)\in C^1((T_0,\infty);(0,\infty))$ for some $T_0>0$.

6. Conclusion

In this paper, we established the sufficient conditions for every solution of (E) to be oscillatory under the conditions (C1) and (C2).

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