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Ahmet Yıldırım  
*Ege University*

Canan Ünlü  
*İstanbul University*

Syed T. Mohyud-Din  
*HITEC University*

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## On the Solution of the Vibration Equation by Means of the Homotopy Perturbation Method

**Ahmet Yıldırım**

Department of Mathematics  
Ege University  
35100 Bornova  
İzmir, Turkey

[ahmet.yildirim@ege.edu.tr](mailto:ahmet.yildirim@ege.edu.tr); [ahmetyildirim80@gmail.com](mailto:ahmetyildirim80@gmail.com)

**Canan Ünlü**

Department of Mathematics  
İstanbul University  
34134 Vezneciler  
İstanbul, Turkey

**Syed Tauseef Mohyud-Din**

HITEC University  
Taxila Cantt, Pakistan

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### Abstract

In this paper, we present a reliable algorithm, the homotopy perturbation method, to solve the well-known vibration equation for very large membrane which is given initial conditions. By using initial value, the explicit solutions of the equation for different cases have been derived, which accelerate the rapid convergence of the series solution. Numerical results show that the homotopy perturbation method is easy to implement and accurate when applied to differential equations. Numerical results for different particular cases of the problem are presented graphically.

**Keywords:** Homotopy perturbation method, Vibration equation for very large membrane, Initial conditions, Maple

**MSC 2010 No:** 35C07, 35C09, 35C10

## 1. Introduction

The homotopy perturbation method (HPM) was first proposed by the Chinese mathematician Ji-Huan He [He (2005, 2006)]. The essential idea of this method is to introduce a homotopy parameter, say  $p$ , which takes values from 0 to 1. When  $p = 0$ , the system of equations usually reduces to a sufficiently simplified form, which normally admits a rather simple solution. As  $p$  is gradually increased to 1, the system goes through a sequence of “deformations”, the solution for each of which is “close” to that at the previous stage of “deformation”. Eventually at  $p = 1$ , the system takes the original form of the equation and the final stage of “deformation” gives the desired solution. One of the most remarkable features of the HPM is that usually just a few perturbation terms are sufficient for obtaining a reasonably accurate solution. This technique has been employed to solve a large variety of linear and nonlinear problems [Yıldırım (2007) and Yıldırım and Öziş (2008)]. The interested reader can see He (2008) for last development of HPM.

In this study, we will use the homotopy perturbation method to obtain the numerical solutions of the vibration equation for very large membrane for different particular cases. The expressions of the displacement for different time and radii of the membrane and also for various wave velocities of free vibration using the initial conditions are deduced and numerical computations are made with the help of Maple (Version 11) and presented through graphs. Recently, Das (2009) used modified decomposition method for solving the vibration equation for very large membrane for different particular cases.

## 2. Solution of the Problem

We will consider the vibration equation of very large membrane is given by the equation

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}, \quad r \geq 0, t \geq 0 \quad (1)$$

with the initial conditions

$$u(r, 0) = f(r), \quad (2)$$

and

$$\frac{\partial}{\partial t} u(r, 0) = c g(r), \quad (3)$$

where  $u(r, t)$  represents the displacement of finding a particle at the point  $r$  in the instant  $t$ ,  $c$  is the wave velocity of free vibration.

To solve (1) by homotopy perturbation method we can construct the following homotopy:

$$\left( \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u_0}{\partial t^2} \right) = p \left( c^2 \frac{\partial^2 u}{\partial r^2} + \frac{c^2}{r} \frac{\partial u}{\partial r} - \frac{\partial^2 u_0}{\partial t^2} \right) \quad (4)$$

and with initial function

$$u_0(r,t) = u(r,0) + t \frac{\partial}{\partial t} u(r,0). \tag{5}$$

Assume the solution of (4) to be in the form

$$u = u_0 + pu_1 + p^2u_2 + p^3u_3 + \dots \tag{6}$$

To find unknown  $u_0, u_1, u_2, \dots$  substituting (6) into (4) and collecting terms of the same of  $p$  give

$$p^0 : \frac{\partial^2 u_0}{\partial t^2} - \frac{\partial^2 u_0}{\partial t^2} = 0, \tag{7}$$

$$p^1 : \frac{\partial^2 u_1}{\partial t^2} = c^2 \frac{\partial^2 u_0}{\partial r^2} + \frac{c^2}{r} \frac{\partial u_0}{\partial r} - \frac{\partial^2 u_0}{\partial t^2}, \tag{8}$$

$$p^2 : \frac{\partial^2 u_2}{\partial t^2} = c^2 \frac{\partial^2 u_1}{\partial r^2} + \frac{c^2}{r} \frac{\partial u_1}{\partial r}, \tag{9}$$

$$p^3 : \frac{\partial^2 u_3}{\partial t^2} = c^2 \frac{\partial^2 u_2}{\partial r^2} + \frac{c^2}{r} \frac{\partial u_2}{\partial r}, \tag{10}$$

### 3. Particular Cases

**Case I:** Taking  $f(r) = r$  and  $g(r) = 1$ .

The given initial value admits the use of

$$u_0 = r + ct.$$

We find the solution

$$u_1 = ct + \frac{c^2 t^2}{2r},$$

$$u_2 = \frac{c^4 t^4}{24r^3},$$

$$u_3 = \frac{c^6 t^6}{80r^5},$$

and so on. Therefore,

$$u(r,t) = r \left[ 1 + c \left( \frac{t}{r} \right) + \frac{c^2}{2} \left( \frac{t}{r} \right)^2 + \frac{c^4}{24} \left( \frac{t}{r} \right)^4 + \frac{c^6}{80} \left( \frac{t}{r} \right)^6 + \dots \right]. \quad (11)$$

The above series will be convergent for the values of  $|t/r| \ll 1$  i.e., for large membrane and small range of time.

**Case II:** Taking  $f(r) = r^2$  and  $g(r) = r$ .

The given initial value admits the use of

$$u_0 = r^2 + crt.$$

We find the solution

$$\begin{aligned} u_1 &= ctr + 2c^2t^2, \\ u_2 &= \frac{c^3t^3}{6r}, \\ u_3 &= \frac{c^5t^5}{120r^3}, \end{aligned}$$

and so on. Thus,

$$u(r,t) = r^2 \left[ 1 + c \left( \frac{t}{r} \right) + 2c^2 \left( \frac{t}{r} \right)^2 + \frac{c^3}{6} \left( \frac{t}{r} \right)^3 + \frac{c^5}{120} \left( \frac{t}{r} \right)^5 + \dots \right]. \quad (12)$$

As of Case I, the above series is also convergent for  $|t/r| \ll 1$ .

**Case III:** Taking  $f(r) = \sqrt{r}$  and  $g(r) = 1/\sqrt{r}$ .

The given initial value admits the use of

$$u_0 = \sqrt{r} + c \frac{t}{\sqrt{r}}.$$

We find the solution

$$\begin{aligned} u_1 &= \frac{ct}{\sqrt{r}} + \frac{c^2t^2}{8r^{3/2}}, \\ u_2 &= \frac{c^3t^3}{24r^{5/2}} + \frac{3c^4t^4}{128r^{7/2}}, \\ u_3 &= \frac{5c^5t^5}{384r^{9/2}} + \frac{49c^6t^6}{5120r^{11/2}} \end{aligned}$$

and so on. Hence,

$$u(r, t) = \sqrt{r} \left[ 1 + c \left( \frac{t}{r} \right) + \frac{c^2}{8} \left( \frac{t}{r} \right)^2 + \frac{c^3}{24} \left( \frac{t}{r} \right)^3 + \frac{3c^4}{128} \left( \frac{t}{r} \right)^4 + \frac{5c^5}{384} \left( \frac{t}{r} \right)^5 + \frac{49c^6}{5120} \left( \frac{t}{r} \right)^6 + \dots \right]. \quad (13)$$

As of Case I, the above series is also convergent for  $|t/r| \ll 1$ .

For Cases I–III, it is kept in mind that for the convergence of the problems the ratio  $t/r$  is to be small. It is observed that for Cases I and III, the displacement decreases with the increase in  $r$  and increases with the increase in  $t$  (Figures 1(a) and 3(a)), but for the Case II, it increases with the increase of both  $r$  and  $t$  (Figure 2(a)) for a fixed value of wave velocity ( $c = 6$ ). It is also seen from Figures 1(b), 2(b) and 3(b) that the displacement increases with the increase in  $t$  and  $c$  both at a fixed value of the radius of the membrane (for  $r = 20$ ). The increase in displacement is faster in Case II than for Cases I and III.

**Case IV:** Taking  $f(r) = r^2$  and  $g(r) = 1$ .

Here, the given initial value admits the use of

$$u_0 = r^2 + ct.$$

We find the solution

$$\begin{aligned} u_1 &= ct + 2c^2t^2 \\ u_n &= 0, \quad n \geq 2. \end{aligned}$$

Therefore,

$$u(r, t) = r^2 + ct + 2c^2t^2. \quad (14)$$

**Case V:** Taking  $f(r) = r^2$  and  $g(r) = r^2$ .

Here, again the given initial value admits the use of

$$u_0 = r^2 + ctr^2.$$

We find the solution

$$\begin{aligned} u_1 &= ctr^2 + 2c^2t^2, \\ u_2 &= \frac{2}{3}c^3t^3, \\ u_n &= 0, \quad n \geq 3. \end{aligned}$$

Finally, we get the following solution:

$$u(r, t) = r^2 + ctr^2 + 2c^2t^2 + \frac{2}{3}c^3t^3 \quad (15)$$

For the Cases IV and V, since the expressions of displacement contain only finite number of terms, so  $u(r, t)$  does not depend on the ratio  $t/r$ . It is seen that in both the cases the displacement increases with the increase in both  $r$  and  $t$  for a fixed value of  $c = 2$  (Figures 4(a) and 5(a)). Figures 4(b) and 5(b) depict that the displacement increases with the increase in  $t$  and  $c$  for fixed value of  $r = 10$ . The increase in displacement in Case V is faster than that in Case IV. All the computations and figures are made using Maple software (Version 11).

#### 4. Conclusion

The homotopy perturbation method is very powerful in finding solutions for various nonlinear problems. Showing its application for vibration of very large membrane, we may conclude that the homotopy perturbation method will be very much useful for solving many physical and engineering problems both analytically and numerically. It is also shown that the advantage of the homotopy perturbation method is its fast convergence of the solution. The numerical results obtained here conform to its high degree of accuracy. Moreover, no linearization or perturbation or discretization is needed.

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Figures

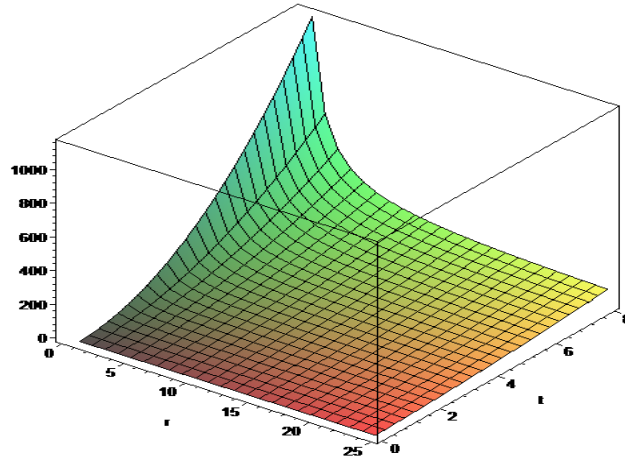


Figure 1(a). Plot of  $u(r,t)$  with respect to  $r$  and  $t$  at  $c = 6$  for Case I.

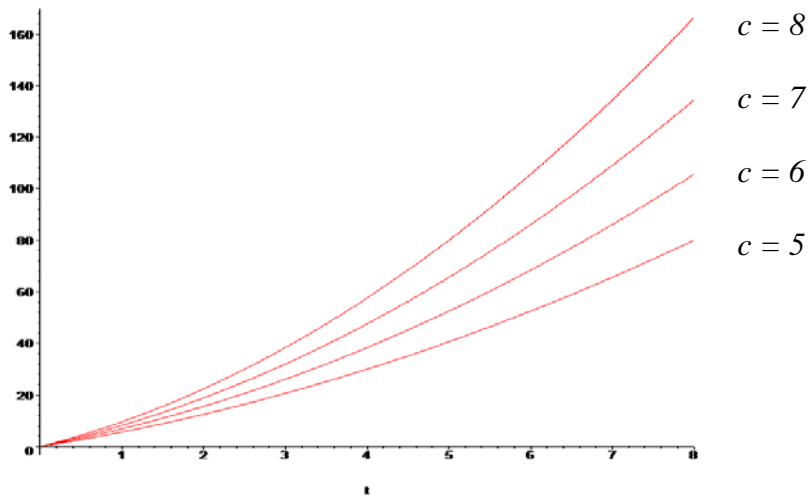


Figure 1(b). Plot of  $u(r,t)$  vs.  $t$  for different values of  $c$  at  $r = 20$  for Case I.



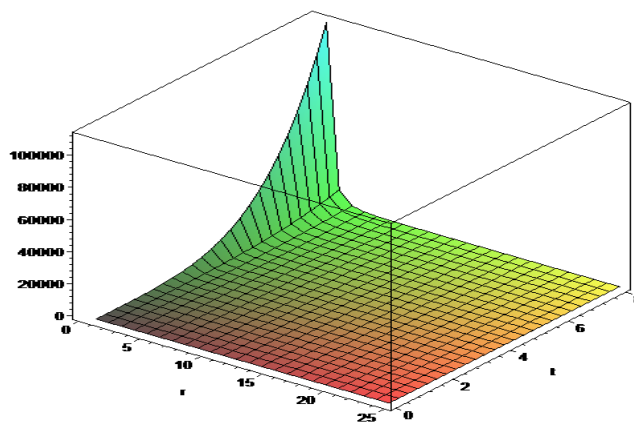


Figure 2(a). Plot of  $u(r,t)$  with respect to  $r$  and  $t$  at  $c = 6$  for Case II.

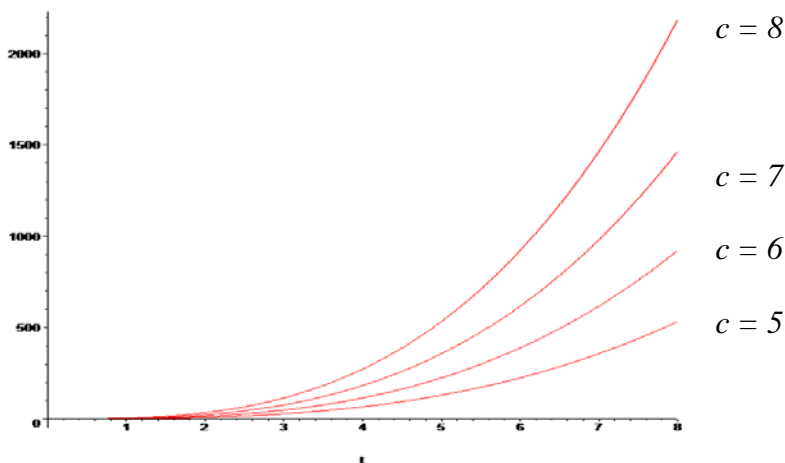


Figure 2(b). Plot of  $u(r,t)$  vs.  $t$  for different values of  $c$  at  $r = 20$  for Case II.

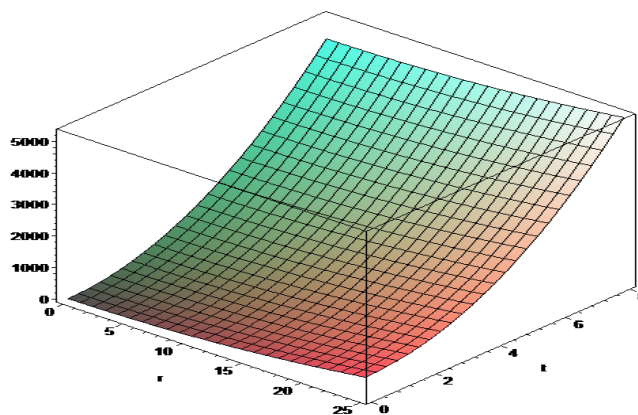
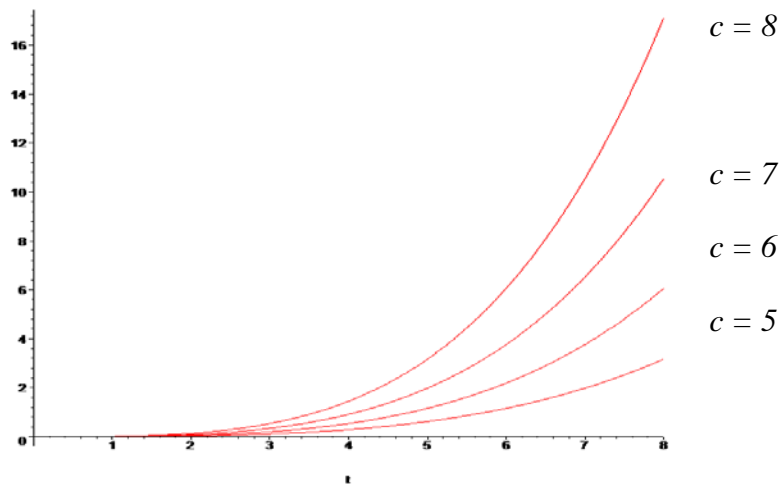
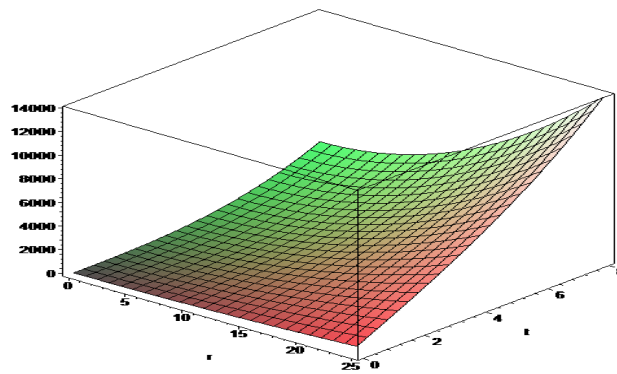


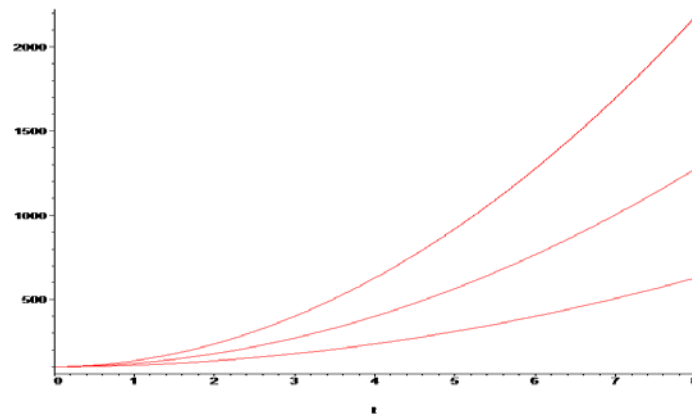
Figure 3(a). Plot of  $u(r,t)$  with respect to  $r$  and  $t$  at  $c = 6$  for Case III.



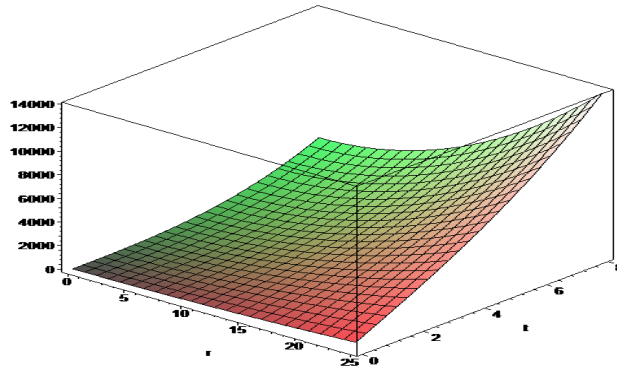
**Figure 3(b).** Plot of  $u(r,t)$  vs.  $t$  for different values of  $c$  at  $r = 20$  for Case III.



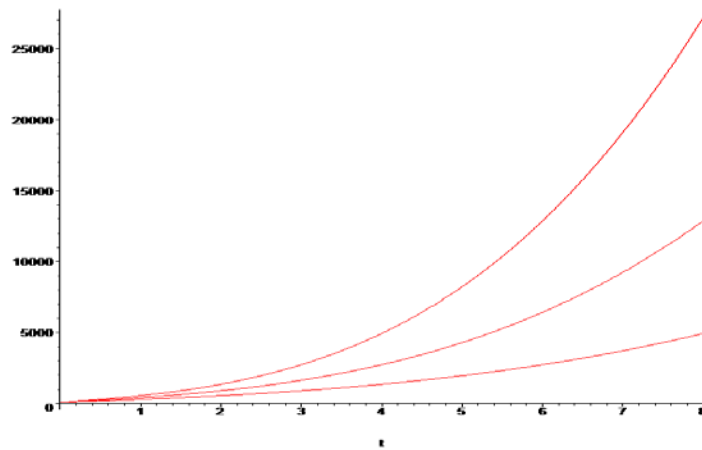
**Figure 4(a).** Plot of  $u(r,t)$  with respect to  $r$  and  $t$  at  $c = 2$  for Case IV



**Figure 4(b).** Plot of  $u(r,t)$  vs.  $t$  for different values of  $c$  ( $c = 4, 3, 2$ ) at  $r = 10$  for Case IV



**Figure 5(a).** Plot of  $u(r,t)$  with respect to  $r$  and  $t$  at  $c = 2$  for Case V.



**Figure 5(b).** Plot of  $u(r,t)$  vs.  $t$  for different values of  $c$  ( $c = 4, 3, 2$ ) at  $r = 10$  for Case V.