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Dual Pole Indicatrix Curve and Surface

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Abstract

In this paper, the vectorial moment vector of the unit Darboux vector, which consists of the motion of the Frenet vectors on any curve, is reexpressed in the form of Frenet vectors. According to the new version of this vector, the parametric equation of the ruled surface corresponding to the unit dual pole indicatrix curve is given. The integral invariants of this surface are rederived and illustrated by presenting with examples.

Keywords: Dual space; Dual angle of pitch; Darboux vector; Ruled surface; Dual pole indicatrix curve; Vectorial moment; The pitch; The angle of pitch

MSC 2010 No.: 14H45, 14H50, 53A04

1. Introduction

In mathematics, the differential geometry of surfaces deals with the differential geometry of smooth surfaces with various additional structures, most often; a Riemannian metric. Surfaces have been extensively studied from various perspectives: extrinsically, relating to their embedding in Euclidean space; and intrinsically, reflecting their properties determined solely by the distance within the surface as measured along curves on the surface. Their characterizations are obtained variable properties by the curves on the surfaces, such as shape operator, principal curvatures, Gaussian curvature and mean curvature.

If p is a point of M, so for each tangent vector X to M at p,

$$S_p(X) = \nabla_X Z, \tag{1}$$

where Z is a normal vector field on a neighborhood of p in M. S_p is called the shape operator of M at p (O'Neill (2006)).

Gauss curvature is an important concept for differential geometry. A surface M in R^3 is flat provided its Gaussian curvature is zero. This curvature are found by O'Neill (2006),

$$K = detS_p. \tag{2}$$

There are many studies on the classical differential geometry of curve and surface theories and are still being studied. Ruled surfaces have an important place in surfaces, because they are used in civil engineering. Since building materials such as wood are straight, they can be thought of as straight lines. The result is that if engineers are planning to construct something with curvature, they can use a ruled surface.

A ruled surface in IR^3 is a surface which contains at least one 1-parameter family of straight lines. Thus a ruled surface has a parametrization in the form

$$\vec{\varphi}(s, v) = \vec{\alpha}(s) + v\vec{x}(s), \tag{3}$$

where we call α the anchor curve and x the generator vector as ruled surface. When the above ruled surface satisfies $\varphi(s + 2\pi, v) = \varphi(s, v)$ it is called closed ruled surface. The properties of the ruled surface obtained according to the condition of the anchor curve or the generator vector are available in the books of differential geometry (Gray et al. (2005); O'Neill (2006); Do Carmo (1976)). Osman Gürsoy's study showed that the dual integral invariant of a closed ruled surface, the dual angle of pitch, corresponds to the dual spherical surface area described by the dual spherical indicatrix of the closed ruled surface. Further, geometric interpretations of the real angle of pitch and the real pitch of a closed ruled surface were given (Gürsoy (1990b)). In Güven et al. (2011), the pitch, the angle of pitch and the dual angle of pitch of closed ruled surface corresponding to a closed curve on dual unit sphere were investigated. In Rashad (2003), he studied a ruled surface as a curve on the dual unit sphere by using Blaschke approach. By investigating one parameter spherical motion in ID^3 with two different kinds of dual indicatrice curves, Yaylı and Saraçoğlu obtained the ruled surfaces that correspond to tangent, principal normal and binormal indicatrices of the dual curve were developable. Further, this study gave a link between the classical surface theory and dual spherical curves on the dual unit spheres (Yaylı and Saraçoğlu (2011)). Applying curves in with tangent, normal, binormal and darboux lines, Yaylı and Saraçoğlu studied their dual spherical indicatrices. In addition, the dual angles and lengths of pitch of the closed ruled surfaces were given. They showed that tangent and binormal indicatrice curves were involutes of Darboux indicatrice curves (Yaylı and Saraçoğlu (2012)).

2. Preliminaries

In E^3, the standard inner product is given by

$$\langle x, x\rangle = x_1^2 + x_2^2 + x_3^2, \tag{4}$$

where $x = (x_1, x_2, x_3) \in E^3$. Let $\alpha : I \to E^3$ be a unit speed curve denote by $\{T, N, B\}$ the moving Frenet frame. $T(s)$ is the tangent vector field, $N(s)$ is the principal normal vector field and $B(s)$ is the binormal vector field of curve $\alpha(s)$, respectively. The Frenet formulas are given by Do Carmo (1976),

$$T'(s) = \kappa(s)N(s),\ N'(s) = -\kappa(s)T(s) + \tau(s)B(s),\ B'(s) = -\tau(s)N(s). \tag{5}$$

Here curvature and torsion of the curve $\alpha(s)$ are defined with (Do Carmo (1976))

$$\kappa(s) = \|\alpha''(s)\|,\ \tau(s) = \frac{\langle \alpha'(s) \wedge \alpha''(s), \alpha'''(s)\rangle}{\|\alpha'(s) \wedge \alpha''(s)\|^2}. \tag{6}$$

Let $f(s)$, $g(s)$ and $h(s)$ be at least C^3- functions. $\alpha(s)$ can be written in the form of

$$\alpha(s) = f(s)T(s) + g(s)N(s) + h(s)B(s), \tag{7}$$

as a linear combination of the Frenet vectors (Tunçer (2017)). By differentiating both sides of (7), the following one obtained (Tunçer (2017))

$$f'(s) - g(s)\kappa(s) = 1,\ h'(s) + g(s)\tau(s) = 0,\ g'(s) + f(s)\kappa(s) - h(s)\tau(s) = 0. \tag{8}$$

For any unit speed curve $\alpha : I \to E^3$, the vector W is called the Darboux vector defined by $W = \tau T + \kappa B$. If we consider the normalization of the Darboux, we have (Fenchel (1951))

$$C = \sin\varphi T + \cos\varphi B, \tag{9}$$

where $\angle(W, B) = \varphi$, $\sin\varphi = \frac{\tau}{\|W\|},\ \cos\varphi = \frac{\kappa}{\|W\|}$.

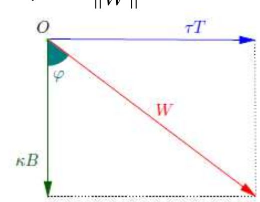

Figure 1. Darboux vector

Let $ID = \{\bar{a} = (a, a^*) : a, a^* \in IR\}$ be the set of the pairs. The element $\varepsilon = (0, 1) \in ID$ satisfies the relationships $\varepsilon \neq 0, \varepsilon^2 = 0$. Then, by the multiplication rule we have that

$$\bar{a} = (a, a^*) = (a, 0) + (0, a^*) = a + \varepsilon a^*.$$

Then $\bar{a} = a + \varepsilon$ is called dual number. The set of dual numbers is given by

$$ID = \{\bar{a} = a + \varepsilon a^* : a, a^* \in IR, \varepsilon^2 = 0\}. \tag{10}$$

The set ID forms a commutative group under addition. The associative laws hold for multiplication. Dual numbers are distributive and form a ring over the real number field (Gürsoy (1990a); Hacısalihoğlu (1972)).

The set

$$ID^3 = \{\widetilde{a} = (\bar{a}_1, \bar{a}_2, \bar{a}_3) : \bar{a}_i \in ID, i = 1, 2, 3\}, \tag{11}$$

is called a dual space. The elements of the set are called dual vectors. Analogue to the dual numbers, a dual vector may be expressed as $\widetilde{a} = \vec{a} + \varepsilon\vec{a}^*$, where $\vec{a}, \vec{a}^*$ are the vectors of IR^3.

A dual vector $\widetilde{a}$ with norm $1 + \varepsilon 0$ is called a dual unit vector. The set of dual unit vectors is given by

$$\widetilde{S}^2 = \{\widetilde{a} = (\bar{a}_1, \bar{a}_2, \bar{a}_3) \in ID^3 : \langle \widetilde{a}, \widetilde{a} \rangle = 1 + \varepsilon 0\}, \tag{12}$$

and called a dual unit sphere (Hacısalihoğlu (1972)).

Theorem 2.1.

There exists a one-to-one correspondence between the lines of line space and the points of dual unit sphere (Blaschke (1945); Hacısalihoğlu (1972)).

If the vectorial moment of the x vector is denoted by x^*, then $x^* = \alpha \wedge x$. If X has the norm $\|X\| = 1$, then it is a dual point on the dual unit sphere. According to the theorem (2.1), there exists a one-to-one transformation between the dual points on the unit dual sphere and the oriented lines in IR^3. A one-parameter set of points (a dual curve) on dual unit sphere corresponds to a one-parameter family of oriented lines in E^3, which defines a ruled surface. This dual curve is called the dual spherical image of the ruled surface (Gürsoy (1990a); Hacısalihoğlu (1972)).

The dual expression of the closed ruled surface in (3) is

$$\vec{\varphi}(s, u) = \vec{x}(s) \wedge \vec{x}^*(s) + u\vec{x}(s), \tag{13}$$

where the $\vec{x}(s) \wedge \vec{x}^*(s)$ is the anchor curve. s is not the arc-parameter of this curve (Gürsoy (1990a); Hacısalihoğlu (1972)).

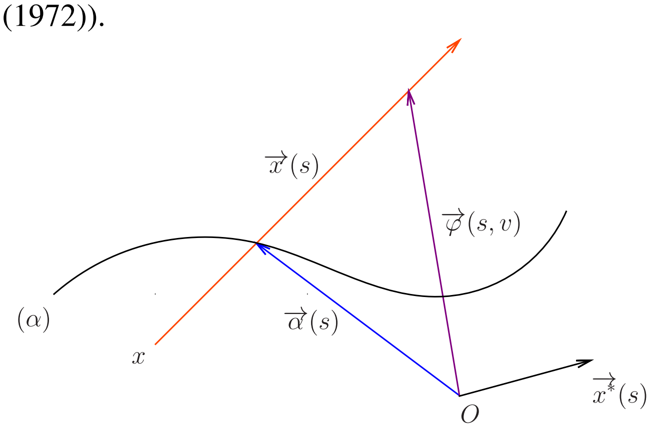

Figure 2. The dual expression of a ruled surface

The dual angle of pitch of surface in (13) is defined by

$$\Lambda_X = -\langle D, X \rangle = \lambda_x - \varepsilon L_x. \tag{14}$$

Here, λ_x and L_x are real integral invariants (Gürsoy (1990a)).

3. Dual Pole Indicatrix Curve and Surface

Since $\widehat{C} = C + \varepsilon C^*$ is the dual vector, the geometric location of this vector draws a curve on the dual sphere. This curve is called a dual pole indicatrix curve. The dual representation of the closed ruled surface corresponding to the dual pole indicatrix curve $(\widehat{C})$ is

$$\psi(s,v) = \beta(s) + vC(s), \beta(s) = C(s) \wedge C^*(s). \tag{15}$$

The vectorial moment of a unit Darboux vector is

$$\begin{aligned} C^* &= \alpha \wedge C = (fT + gN + hB) \wedge (\sin\varphi T + \cos\varphi B) \\ &= g\cos\varphi T + (h\sin\varphi - f\cos\varphi)N - g\sin\varphi B. \end{aligned} \tag{16}$$

Necessary calculations are made for the vector $C(s) \wedge C^*(s)$. We receive

$$\begin{aligned} C \wedge C^* &= (\sin\varphi T + \cos\varphi B) \wedge (g\cos\varphi T + (h\sin\varphi - f\cos\varphi)N - g\sin\varphi B), \\ C \wedge C^* &= -\cos\varphi(h\sin\varphi - f\cos\varphi)T + gN + \sin\varphi(h\sin\varphi - f\cos\varphi)B. \end{aligned}$$

So, the dual representation of this surface is

$$\begin{aligned} \psi(s,v) = &(-\cos\varphi(h\sin\varphi - f\cos\varphi) + v\sin\varphi)T + gN \\ &+(\sin\varphi(h\sin\varphi - f\cos\varphi) + v\cos\varphi)B. \end{aligned}$$

Theorem 3.1.

Distribution parameter of the closed ruled surface $\psi(s,v)$ is zero.

Proof:

It is known that the distribution parameter of the closed ruled surface is calculated by

$$P_C = \frac{det((C \wedge C^*)', C, C')}{\|C'\|^2}. \tag{17}$$

It can be written that

$$C \wedge C^* = -\cos\varphi(h\sin\varphi - f\cos\varphi)T + gN + \sin\varphi(h\sin\varphi - f\cos\varphi)B,$$

where $A = h\sin\varphi - f\cos\varphi$.

When a derivative is taken, the above equation becomes

$$(C \wedge C^*)' = (-(A\cos\varphi)' - g\kappa)T + (-A(\kappa\cos\varphi + \tau\sin\varphi) + g')N + ((A\sin\varphi)' + g\tau)B. \tag{18}$$

Also, with the value of

$$\begin{aligned} -A\|W\| + g' &= -(h\sin\varphi - f\cos\varphi)\|W\| + g' \\ &= -h\sin\varphi\|W\| + f\cos\varphi\|W\| + g' \\ &= -h\tau + h\kappa + g' = 0, \end{aligned}$$

Equation (18) can be concluded that

$$(C \wedge C^*)' = (-(A\cos\varphi)' - g\kappa)T + ((A\sin\varphi)' + g\tau)B. \tag{19}$$

Then, the following result is obtained:

$$P_{\widehat{C}} = \frac{det((C\wedge C^*)', C, C')}{\|C'\|^2} = \frac{\begin{vmatrix} -(A\cos\varphi)' - g\kappa & 0 & (A\sin\varphi)' + g\tau \\ \sin\varphi & 0 & \cos\varphi \\ \varphi'\cos\varphi & 0 & \varphi'\sin\varphi \end{vmatrix}}{\|C'\|^2} = 0 \qquad \blacksquare$$

Theorem 3.2.

The closed ruled surface corresponding to the dual pole indicatrix curve is flat.

Proof:

For the ruled surface $\psi(s,v)$, the partial derivative is taken according to s and v. It is found

$$\begin{aligned} \psi_v(s,v) &= C = \sin\varphi T + \cos\varphi B, \\ \psi_s(s,v) &= (C\wedge C^*)' + vC' = \big(-(A\cos\varphi)' - g\kappa + v\varphi'\cos\varphi\big)T \\ &\quad + \big((A\sin\varphi)' + g\tau - v\varphi'\sin\varphi\big)B. \end{aligned}$$

Taking into account that inner product, we have

$$\begin{aligned} \langle\psi_s(s,v), \psi_v(s,v)\rangle &= \big(-(A\cos\varphi)' - g\kappa + v\varphi'\cos\varphi\big)\sin\varphi \\ &\quad + \big((A\sin\varphi)' + g\tau - v\varphi'\sin\varphi\big)\cos\varphi \\ &= -(A\cos\varphi)'\sin\varphi + (A\sin\varphi)'\cos\varphi + g(\tau\cos\varphi - \kappa\sin\varphi) \\ &= A\varphi' \\ &\neq 0. \end{aligned}$$

Using the Gram-Schmidt process, the vectors of orthonormal base for $\psi_v(s,v) = x_1$ and $\psi_s(s,v) = x_2$ are

$$\begin{aligned} y_1 &= x_1, \\ y_2 &= -\frac{\langle y_1, x_2\rangle}{\langle y_1, y_1\rangle} y_1 + x_2 \\ &= \big(-(A\cos\varphi)' - g\kappa - A\varphi'\sin\varphi + v\varphi'\cos\varphi\big)T + \big((A\sin\varphi)' + g\tau \\ &\quad -A\varphi'\cos\varphi - v\varphi'\sin\varphi\big)B \\ &= \big(-A'\cos\varphi + v\varphi'\cos\varphi - g\|W\|\cos\varphi)T + (A'\sin\varphi - v\varphi'\sin\varphi \\ &\quad +g\|W\|\sin\varphi\big)B, \end{aligned}$$

$$\begin{aligned} E_1 &= \frac{y_1}{\|y_1\|} = C, \\ E_2 &= \frac{y_2}{\|y_2\|} = \frac{(-A'\cos\varphi + v\varphi'\cos\varphi - g\kappa)T + (A'\sin\varphi - v\varphi'\sin\varphi + g\tau)B}{(A' - v\varphi') + g\|W\|}. \end{aligned}$$

For a ruled surface with parametrization $\psi(s,v)$, the normal vector is given by

$$\begin{aligned} N_{\widehat{C}} &= E_1 \wedge E_2 \\ &= -\frac{(A' - v\varphi') + g(\tau\cos\varphi + \kappa\sin\varphi)}{(A' - v\varphi') + g\|W\|} N \\ &= -N. \end{aligned}$$

On the other hand, we compute

$$S(E_2) = D_{E_2} N_{\widehat{C}} = \kappa T - \tau B \Rightarrow \langle S(E_2), E_1\rangle = \langle \kappa T - \tau B, \sin\varphi T + \cos\varphi B\rangle = 0. \tag{20}$$

Since the shape operator is self-adjoint, we can write $\langle S(E_2), E_1\rangle = \langle S(E_1), E_2\rangle$. If the main direction of the surface is the asymptotic direction, the shape operator is $\langle S(E_1), E_1\rangle = 0$ (Gray et al. (2005)). From Equation (2), the Gauss curvature is

$$K(P) = det(S_P) = \begin{bmatrix} \langle S(E_1), E_1\rangle & \langle S(E_1), E_2\rangle \\ \langle S(E_2), E_1\rangle & \langle S(E_2), E_2\rangle \end{bmatrix} = 0. \tag{21}$$

Then, the closed ruled surface corresponding to the dual pole indicatrix curve is flat. ■

Theorem 3.3.

The instantaneous pfaffian vector and the dual Steiner vector generetaed by the motion of the dual Frenet vectors $\{\widehat{T}, \widehat{N}, \widehat{B}\}$ are given by

$$\widehat{W} = \tau T + \kappa B + \varepsilon(f'T + g'N + h'B), \tag{22}$$

and

$$D = d + \varepsilon d^* = T\oint \tau + B\oint \kappa + \varepsilon(T\oint f' + N\oint g' + B\oint h'), \tag{23}$$

respectively.

Proof:

Let us assume that the instantaneous pfaffian vector generated by the motion of the dual Frenet vectors $\{\widehat{T}, \widehat{N}, \widehat{B}\}$ is

$$\widehat{W} = \widehat{\tau}\widehat{T} + \widehat{\kappa}\widehat{B}. \tag{24}$$

The dual Frenet vectors $\{\widehat{T}, \widehat{N}, \widehat{B}\}$ are given by linear combination of Frenet vectors $\{T, N, B\}$ as below:

$$\widehat{T} = T + \varepsilon T^*, T^* = \alpha \wedge T \Rightarrow \widehat{T} = T + \varepsilon(hN - gB), \tag{25}$$

$$\widehat{N} = N + \varepsilon N^*, N^* = \alpha \wedge N \Rightarrow \widehat{N} = N + \varepsilon(-hT + fB), \tag{26}$$

$$\widehat{B} = B + \varepsilon B^*, B^* = \alpha \wedge B \Rightarrow \widehat{B} = B + \varepsilon(gT - fN). \tag{27}$$

Because of the definition of dual curvature, we can write

$$\begin{aligned} \widehat{\kappa} &= \sqrt{\langle \widehat{T}', \widehat{T}'\rangle} \\ &= \sqrt{\langle \kappa N + \varepsilon\kappa(-hT + fB), \kappa N + \varepsilon\kappa(-hT + fB)\rangle} \\ &= \kappa. \end{aligned} \tag{28}$$

Then, $\kappa^* = 0$ is found. We know that

$$\widehat{\tau} = \frac{\langle \widehat{T} \wedge \widehat{T}', \widehat{T}'' \rangle}{\langle \widehat{T} \wedge \widehat{T}', \widehat{T} \wedge \widehat{T}' \rangle}. \tag{29}$$

By setting

$$\begin{aligned} \widehat{T}'' &= \kappa' N + \kappa(-\kappa T + \tau B) + \varepsilon\big(\kappa'(-hT + fB) \\ &\quad + \kappa(-h'T - h\kappa N + f'B - f\tau N)\big) \\ &= (-\kappa^2 - \varepsilon(h\kappa)')T + (\kappa' - \varepsilon\kappa(h\kappa + f\tau))N + (\kappa\tau + \varepsilon(f\kappa)')B, \end{aligned}$$

Equation (29) becomes

$$\begin{aligned} \widehat{\tau} &= \frac{\langle \widehat{T} \wedge \widehat{T}', \widehat{T}'' \rangle}{\langle \widehat{T} \wedge \widehat{T}', \widehat{T} \wedge \widehat{T}' \rangle} = \begin{vmatrix} 1 & \varepsilon h & -\varepsilon g \\ -\varepsilon h\kappa & \kappa & \varepsilon f\kappa \\ -\kappa^2 - \varepsilon(h\kappa)' & \kappa' - \varepsilon\kappa(h\kappa + f\tau) & \kappa\tau + \varepsilon(f\kappa)' \end{vmatrix} \\ &= \tau + \varepsilon. \end{aligned} \tag{30}$$

Using Equations (25), (27), (28) and (29) in (24), it follows that

$$\begin{aligned} \widehat{W} &= \widehat{\tau}\widehat{T} + \widehat{\kappa}\widehat{B} \\ &= \widehat{\tau}(T + \varepsilon(hN - gB)) + \widehat{\kappa}(B + \varepsilon(gT - fN)) \\ &= (\tau + \varepsilon)(T + \varepsilon(hN - gB)) + \kappa(B + \varepsilon(gT - fN)) \\ &= \tau T + \kappa B + \varepsilon((1 + g\kappa)T + (hT - f\kappa)N - g\tau B) \\ &= \tau T + \kappa B + \varepsilon(f'T + g'N + h'B). \end{aligned}$$

Also, by taking definition of dual Steiner vector (Gürsoy (1990a)), we obtain

$$D = d + \varepsilon d^* = T \oint \tau + B \oint \kappa + \varepsilon(T \oint f' + N \oint g' + B \oint h'). \qquad \blacksquare$$

Theorem 3.4.

The dual angle of pitch of the closed ruled surface corresponding to the dual pole indicatrix curve is

$$\Lambda_{\widehat{C}} = \sin\varphi\lambda_T + \cos\varphi\lambda_B - \varepsilon(-g\cos\varphi\lambda_T + g\sin\varphi\lambda_B + \sin\varphi \oint f' + \cos\varphi \oint h'). \tag{31}$$

Here λ_T and λ_B are the angle of pitches of closed ruled surface corresponding to the T and B, respectively.

Proof:

We will use Equations (14) and (23). If we take into account these equations, the dual angle of pitch of the closed ruled surface corresponding to the dual pole indicatrix curve is

$$\Lambda_{\widehat{C}} = -\Big\langle D, \widehat{C} \Big\rangle = -\Big\langle d + \varepsilon d^*, C + \varepsilon C^* \Big\rangle,$$

$$\begin{aligned}\Lambda_{\widehat{C}} &= -\Big\langle T \oint \tau + B \oint \kappa, \sin\varphi T + \cos\varphi B \Big\rangle - \varepsilon\Big[\Big\langle T \oint \tau + B \oint \kappa, g\cos\varphi T \\ &\quad + (h\sin\varphi - f\cos\varphi)N - g\sin\varphi B \Big\rangle + \Big\langle T \oint f' + N \oint g' + B \oint h', \sin\varphi T + \cos\varphi B \Big\rangle\Big] \\ &= -\sin\varphi \oint \tau - \cos\varphi \oint \kappa - \varepsilon\Big(g\cos\varphi \oint \tau - g\sin\varphi \oint \kappa + \sin\varphi \oint f' + \cos\varphi \oint h'\Big) \\ &= \sin\varphi\lambda_T + \cos\varphi\lambda_B - \varepsilon\Big(- g\cos\varphi\lambda_T + g\sin\varphi\lambda_B + \sin\varphi \oint f' + \cos\varphi \oint h'\Big).\end{aligned}$$

■

Example 3.1.

Let $\alpha(s) = \frac{1}{\sqrt{2}}(-\cos s, -\sin s, s)$ be a circular helix curve. Then, it is easy to show that

$$\begin{aligned} T(s) &= \frac{1}{\sqrt{2}}\big(\sin s, -\cos s, 1\big), & \kappa(s) &= \frac{1}{\sqrt{2}}, \\ N(s) &= \big(\cos s, \sin s, 0\big), & \tau(s) &= \frac{1}{\sqrt{2}}, \\ B(s) &= \frac{1}{\sqrt{2}}\big(- \sin s, \cos s, 1\big), & C(s) &= \big(0, 0, 1\big). \end{aligned}$$

We can write vectorial moment

$$C^* = \alpha \wedge C = \frac{1}{\sqrt{2}}\big(- \sin s, \cos s, 0\big).$$

If Equation (13) is taken into account, then we obtain the closed ruled surface corresponding to the unit dual pole indicatrix curve $(\widehat{C})$ as

$$\begin{aligned}\psi_1(s, v) &= C \wedge C^* + vC \\ &= (0, 0, 1) \wedge \frac{1}{\sqrt{2}}\big(- \sin s, \cos s, 0\big) + v(0, 0, 1) \\ &= \Big(- \frac{1}{\sqrt{2}}\sin s, -\frac{1}{\sqrt{2}}\cos s, v\Big). \end{aligned} \tag{32}$$

This ruled surface is shown in Figure 3.

Let us find the functions $f(s), g(s), h(s)$. From Equation (7), we can write

$$\begin{aligned} -\cos s &= f(s)\sin s + g(s)\sqrt{2}\cos s - h(s)\sin s, \\ -\sin s &= -f(s)\cos s + g(s)\sqrt{2}\sin s + h(s)\cos s, \\ s &= f(s) + h(s). \end{aligned}$$

The solutions of $f(s), g(s), h(s)$ are given by $f(s) = \dfrac{s}{2}, g(s) = -\dfrac{1}{\sqrt{2}}$ and $h(s) = \dfrac{s}{2}$. By taking into account values of the $f, g, h, \kappa, \tau, \sin\varphi$ and $\cos\varphi$, the dual angle of pitch of closed ruled surface in Equation (32) is

$$\begin{aligned}\Lambda_{\widehat{C}} &= \sin\varphi\lambda_T + \cos\varphi\lambda_B - \varepsilon\Big(- g\cos\varphi\lambda_T + g\sin\varphi\lambda_B + \sin\varphi\oint f' + \cos\varphi\oint h'\Big)\\ &= \frac{1}{\sqrt{2}}\lambda_T + \frac{1}{\sqrt{2}}\lambda_B - \varepsilon\Big(\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\lambda_T - \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\lambda_B + \frac{1}{\sqrt{2}}\oint\frac{1}{2}ds + \cos\varphi\oint\frac{1}{2}ds\Big)\\ &= \frac{1}{\sqrt{2}}(\lambda_T + \lambda_B) - \varepsilon\Big(\frac{1}{2}(\lambda_T - \lambda_B) + \frac{1}{\sqrt{2}}\underbrace{\oint ds}_{L_T}\Big)\\ &= \frac{1}{\sqrt{2}}(\lambda_T + \lambda_B) - \varepsilon\Big(\frac{1}{2}(\lambda_T - \lambda_B) + \frac{1}{\sqrt{2}}L_T\Big).\end{aligned}$$

Here L_T is the pitch of closed ruled surface corresponding to the T.

Herein, we remark that the dual angle of pitch of this surface depends on the real integral invariants of the closed ruled surface which corresponds to T and B.

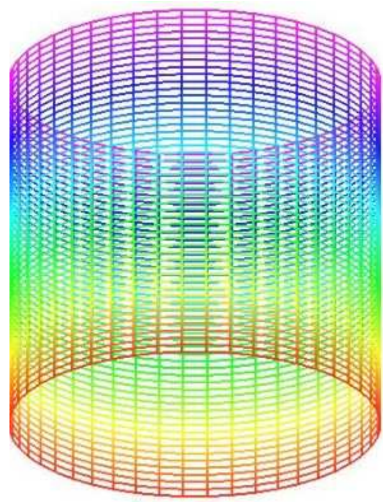

Figure 3. $\psi_1(s,v)$-ruled surface

Example 3.2.

Let $\alpha(s) = \left(\frac{4}{5}\cos s, 1-\sin s, -\frac{3}{5}\cos s\right)$ be a curve. Then, it is easy to show that

$$\begin{aligned} T(s) &= \Big(-\frac{4}{5}\sin s, -\cos s, \frac{3}{5}\sin s\Big), \quad \kappa(s) = 1,\\ N(s) &= \Big(-\frac{4}{5}\cos s, \sin s, \frac{3}{5}\cos s\Big), \qquad \tau(s) = 0,\\ B(s) &= \Big(-\frac{3}{5}, 0, -\frac{4}{5}\Big), \quad C(s) = \Big(-\frac{3}{5}, 0, -\frac{4}{5}\Big).\end{aligned}$$

We can write vectorial moment

$$C^* = \alpha \wedge C = \Big(-\frac{4}{5}(1-\sin s), \cos s, \frac{3}{5}(1-\sin s)\Big).$$

If Equation (13) is taken into account, then we obtain the closed ruled surface corresponding to the unit dual pole indicatrix curve $(\widehat{C})$ as

$$\begin{aligned}\psi_2(s,v) &= C \wedge C^* + vC\\ &= \Big(-\frac{3}{5},0,-\frac{4}{5}\Big) \wedge \Big(-\frac{4}{5}(1-\sin s),\cos s,\frac{3}{5}(1-\sin s)\Big) + v\Big(-\frac{3}{5},0,-\frac{4}{5}\Big)\\ &= \Big(\frac{4}{5}\cos s - v\frac{3}{5}, 1-\sin s, -\frac{3}{5}\cos s - v\frac{4}{5}\Big). \end{aligned} \tag{33}$$

This ruled surface is shown in Figure 4.

Let us find the functions $f(s), g(s), h(s)$. From Equation (7), we have

$$\begin{aligned}\Big(\frac{4}{5}\cos s, 1-\sin s, -\frac{3}{5}\cos s\Big) = f(s)&\Big(-\frac{4}{5}\sin s, -\cos s, \frac{3}{5}\sin s\Big)\\ +g(s)&\Big(-\frac{4}{5}\cos s, \sin s, \frac{3}{5}\cos s\Big)\\ +h(s)&\Big(-\frac{3}{5},0,-\frac{4}{5}\Big), \end{aligned} \tag{34}$$

or we can write

$$\begin{aligned} 4\cos s &= -4f(s)\sin s - 4g(s)\cos s - 3h(s),\\ 1-\sin s &= -f(s)\cos s + g(s)\sin s,\\ -3\cos s &= 3f(s)\sin s + 3g(s)\cos s - 4h(s). \end{aligned}$$

The solutions of $f(s), g(s), h(s)$ are given by $f(s) = -\cos s, g(s) = \sin s - 1$ and $h(s) = 0$. By taking into account values of the $f, g, h, \kappa, \tau, \sin\varphi$ and $\cos\varphi$ in Equation(33), the dual angle of pitch of closed ruled surface in Equation (33) is

$$\begin{aligned}\Lambda_{\widehat{C}} &= \sin\varphi\lambda_T + \cos\varphi\lambda_B - \varepsilon\Big(-g\cos\varphi\lambda_T + g\sin\varphi\lambda_B + \sin\varphi\oint f' + \cos\varphi\oint h'\Big)\\ &= \lambda_B - \varepsilon(1-\sin s)\lambda_T, \end{aligned}$$

where c is an arbitrary constant known as the integration constant.

Herein, we stress that the dual angle of pitch of this surface $\psi_2(s,v)$ depends on the angles of pitch of closed ruled surface corresponding to the B and T, respectively.

4. Conclusion

In this study, the C^* vectorial moment of the C unit Darboux vector formed by the Frenet vectors of any curve is rewritten using the data in Equation (7). The dual represenstation of the ruled surface which corresponds to the dual pole indicatrix curve drawn by the dual vector $\widehat{C} = C + \epsilon C^*$ on the dual sphere is expressed in terms of Frenet vectors of the anchor curve. Then, it is shown that a dual curvature is only real. The dual torsion is determined as $\widehat{\tau} = \tau + \epsilon$. The dral, the Gaussian curvature, Pfaff vector, dual Steiner vector and dual angle of pitch of this surface, which are obtained from these new data, are recalculated.

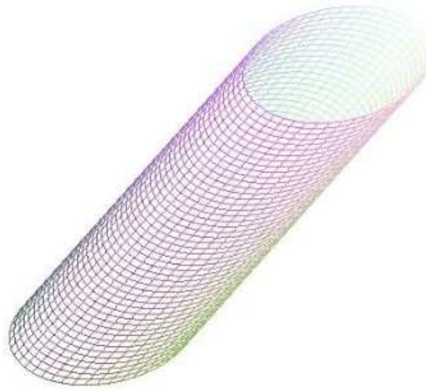

Figure 4. $\psi_2(s,v)$-ruled surface

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