Applications and Applied Mathematics: An International Journal (AAM)

Volume 15 | Issue 2 | Article 40

12-2020

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Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466

Applications and Applied Mathematics:
An International Journal
(AAM)

Vol. 15, Issue 2 (December 2020), pp. 1381 - 1395

On the Unsolvability Conditions for Quasilinear Pseudohyperbolic Equations

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Received: August 19, 2020; Accepted: November 3, 2020

Abstract

In this paper, we study the nonexistence of global weak solutions to the Cauchy problem of quasilinear pseudohyperbolic equations with damping term. The sufficient conditions for nonexistence of nontrivial global weak solutions is obtained in terms of exponents, singularities order and other parameters in the problem. The nonlinear capacity method is applied to prove nonexistence theorems. The proofs of our nonexistence theorems are based on deriving apriori estimates for the possible solutions to the problem by an algebraic analysis of the integral form of inequalities with an optimal choice of test functions. The result is extended to the case of coupled system of quasilinear pseudohyperbolic equations.

Keywords: Pseudohyperbolic equation; System of pseudohyperbolic equations; Nonexistence results; Nonlinear capacity method; Test functions

MSC 1010 No.: 35L30, 35L56, 35L82

1. Introduction

Pseudohyperbolic and pseudoparabolic equations form an important and interesting subclass of Sobolev type equations. These are characterized by having mixed time and space partial derivatives appearing in the highest order terms of the equation. Such an equation was studied by

Sobolev (1954). The pseudohyperbolic equation models a variety of nonstationary wave like physical phenonenas, such as reaction–diffusion (Zhou et al. (2006)), heat and mass transfer (Tzou (1997)), nerve conduction (Nagumo et al. (1962)) and so on. The pseudoparabolic equations arise from a variety of important physical processes, such as lightning propagation (Aslan et al. (2008)), the aggregation of populations (Padron (2004)), dispersive long waves (Benjamin et al. (1972), the heat conduction involving two temperatures (Chen and Gurtin (1968)), the flows of fluids through fissured rocks (Barenblatt et al. (1960)), etc.

Due to the fact that various physical phenomenas can be described by Sobolev type equations, there is considerable interest in psedohyperbolic and psedoparabolic problems. Various types of such equations and systems have been studied by many authors from the point of view of existence, uniqueness and blow up aspects.

Pseudoparabolic problems with source term having constant coefficients were studied by (see, for example, Cao et al. (2009); Liu and Yu (2018)). In their papers, the existence and blow-up results were established. In (Khomrutai and Kitisin (2014)), the authors studied the blow - up property for weak solutions to the pseudoparabolic equation with source term involving nonconstant coefficients and obtained the fujita- type critical exponent of the problem. Many literatures showed that results on existence, uniqueness and blow-up of solutions to various class of pseudoparabolic and parabolic equations were obtained; we refer readers to (Nhan and Truong (2017); Lu and Fei (2016); Chen and Tian (2015); Khomrutai (2015); Di and Shang (2014), Galakhov and Salieva (2020, 2018), Galakhov et al. (2018)) and the reference therein.

The problem of existence and uniqueness of solutions to some pseudohyperbolic equations was considered by many authors. Among others, (Zhao and Li (2019)) considered two-dimensional pseudohyperbolic equations with source term independent of unknown function, but depending only independent variables. They proved the existence and uniqueness of approximation solutions using a space-time continuous Galerkin (STCG) method with mesh change. One dimensional initial - boundary value problem of pseudohyperbolic equations associating integral and Neumann boundary condition in bounded domain was considered by (Boutiani and Temsi (2009)). The authors established the existence, uniqueness and continuous dependence upon the data of weak solution. For the case of pseudohyperbolic problems without damping terms and with source term having constant coefficients, nonexistence results were obtained by (Abdelmalek et al. (2016)). Moreover, existence and uniqueness results were also obtained for nonlocal problems of pseudohyperbolic equations as appeared in (Eltayeb et al. (2017); Merad et al. (2015); Pulkina (2012); Korpusov (2010); Biazar et al. (2010); Boutiani and Temsi (2009)). To the best of our knowledge, earlier the unsolvability conditions for pseudohyperbolic equations with a damping terms (parabolic-hyperbolic case) and with source term having variable coefficients and the corresponding system were not considered.

The purpose of this paper is to establish the unsolvability conditions for the Cauchy problem of quasilinear pseudohyperbolic equations with a damping term and source term having variable coefficient using the method developed by Mitidieri and Pohozaev (2001). We also extend the result to the case of coupled system. The investigation of the unsolvability criteria of such equations and systems helps us to determine the range of parameters and initial data that appear in the mathematical models for which the solution does not exist.

2. Problem statement and preliminaries

In the present work, we are interested with a Cauchy problem of quasilinear pseudohyperbolic equation of the form

$$\begin{cases}\eta u_{tt} + \delta u_t - \nabla \cdot (\nabla u_t + \nabla u) = a(x)|u|^q, & (x,t) \in \mathbb{R}^N \times \mathbb{R}^+, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & x \in \mathbb{R}^N,\end{cases} \tag{2.1}$$

and the corresponding coupled system

$$\begin{cases}\eta u_{tt} + \delta u_t - \nabla \cdot (\nabla u_t + \nabla u) = a(x)|v|^q, & (x,t) \in \mathbb{R}^N \times \mathbb{R}^+, \\ \zeta v_{tt} + \gamma v_t - \nabla \cdot (\nabla v_t + \nabla v) = b(x)|u|^p, & (x,t) \in \mathbb{R}^N \times \mathbb{R}^+, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & x \in \mathbb{R}^N, \\ v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x), & x \in \mathbb{R}^N,\end{cases} \tag{2.2}$$

where $\eta,\ \delta,\ \zeta,\ \gamma > 0,\ \ q,\ p > 1$ are constants and $a(x),\ b(x),\ u_0(x),\ u_1(x),\ v_0(x)$ & $v_1(x)$ are known nonnegative functions.

We study the nonexistence of global weak solutions for the problems above via the nonlinear capacity method. An apriori estimates and integral inequalities are the main tools to formulate nonexistence conditions. We also use parametric Young's and Holder's inequalities to establish apriori estimates for the possible solutions to the problems. Our proofs of nonexistence theorems based on obtaining asymptotically optimal apriori estimates by the algebraic analysis of the integral form of the considered equations under a special choice of the test functions.

In order to establish apriori estimates of the solution of the problem in terms of the capacity type integrals of the test function, let us introduce the family of test functions

$$\varphi_R(x,t) = \psi^\lambda\left(\frac{t^\theta + |x|^\mu}{R^\mu}\right), \tag{2.3}$$

with $R > 0,\ \theta > 0$ and $\mu > 0,$ where $\psi \in C_0^\infty(\mathbb{R};\ [0,1])$ is a nonincreasing standard cut - off function such that $0 \le \psi(s) \le 1$ and

$$\psi(s) = \begin{cases}1, & \text{if } 0 \le s \le 1, \\ 0, & \text{if } s \ge 2,\end{cases} \tag{2.4}$$

for sufficiently large parameter $\lambda > 0$ (which can be specified further according to the nature of the problem).

Differentiating (2.3) with respect to t, we get

$$\frac{\partial}{\partial t}\varphi_R(x,t) = \lambda\theta R^{-\mu} t^{\theta-1}\psi^{\lambda-1}\left(\frac{t^\theta + |x|^\mu}{R^\mu}\right)\psi'\left(\frac{t^\theta + |x|^\mu}{R^\mu}\right),$$

from which it follows that

$$\frac{\partial}{\partial t}\varphi_R(x,0) = 0.$$

In accordance with (2.3), we introduce the new independent variables $x \to \xi$ and $t \to \tau$ which are given by $x = R\xi$ and $t = R^{\frac{\mu}{\theta}}\tau$. Then, we have

$$\varphi_R(x,t) = \psi(s), \quad s = \tau^\theta + |\xi|^\mu.$$

Hereafter, c denotes a generic positive constant independent of the variables *x* ,*t* and *u.* By using the Leibniz rule and mathematical induction, we can show that there exist positive constant $c(.,\rho,N)$ such that

$$\left|D_x^\beta \varphi_R\right|^\rho \varphi_R^{1-\rho} \leq c(|\beta|,\rho,N)R^{-|\beta|\rho}, \quad \lambda > \rho > 1, \quad \beta \in (\mathbb{N}\cup\{0\})^N, \tag{2.5}$$

and

$$\left|\frac{\partial^k \varphi_R}{\partial t^k}\right|^\rho \varphi_R^{1-\rho} \leq c(k,\rho,N)R^{-\frac{\mu k}{\theta}\rho}, \quad \lambda > \rho > 1, \ \ k \in \mathbb{N}, \tag{2.6}$$

where $c(.,\rho,N)$ refers a positive constant independent of *x, t* and *R.*

3. Nonexistence result for single equation

In this section, we state and prove the main nonexistence theorem for the problem (2.1). To begin with, we give the definition of weak solutions to this problem. The solutions to this problem are treated in the weak sense (in the sense of distributions) according to the following definition.

Definition 3.1. (Weak solution)

A function $u \in C^1(\mathbb{R}^+; L^q_{loc}(\mathbb{R}^N))$ such that the initial data $u_0, u_1 \in L^1_{loc}(\mathbb{R}^N)$ and the function $a(x)|u|^q \in L^1_{loc}(\mathbb{R}^+; L^1_{loc}(\mathbb{R}^N))$ is called a weak solution to problem (2.1) if the following integral inequality holds

$$\iint_{\mathbb{R}^N\times\mathbb{R}^+} u(\eta\varphi_{tt} + \Delta\varphi_t - \Delta\varphi - \delta\varphi_t)dxdt + \int_{\mathbb{R}^N} u_0(\eta\varphi_t + \Delta\varphi - \delta\varphi)|_{t=0}dx$$
$$= \iint_{\mathbb{R}^N\times\mathbb{R}^+} a|u|^q\varphi\, dxdt + \eta\int_{\mathbb{R}^N} u_1\varphi|_{t=0}dx,$$

for any nonnegative test function $\varphi \in C_{x,t}^{2,2}(\mathbb{R}^N \times \mathbb{R}^+)$ with compact support.

We will consider the class of positive functions $a(x)$ in (2.1) that satisfies the following assumption. Suppose that there exist two constants $c > 0$ and $\alpha \in \mathbb{R}$ such that

$$a(x) \geq c|x|^\alpha, \tag{3.1}$$

for almost every $x \in \mathbb{R}^N$.

Let us formulate our main nonexistence result of this section as follows.

Theorem 3.1.

Suppose that $a(x)$ satisfies inequality (3.1). Let $q > 1$, $\alpha > -2$ and $0 \leq u_0, u_1 \in L_{loc}^1(\mathbb{R}^N)$ such that $\int_{\mathbb{R}^N}(\delta u_0 + \eta u_1)\, dx > 0$. If $1 < q \leq 1 + (2 + \alpha)/N$, then problem (2.1) does not admit a nontrivial global weak solution.

***Proof*:**

The proof is carried out by contradiction. Suppose that u is a nontrivial global weak solution to problem (2.1). Then multiplying both sides of equation (2.1) by any nonnegative admissible test function φ and integrating by parts leads to

$$\int_{\mathbb{R}^N} (\delta u_0 + \eta u_1)\varphi|_{t=0} dx + \iint_{\mathbb{R}^N \times \mathbb{R}^+} a|u|^q \varphi\, dxdt \leq \int_{\mathbb{R}^N} u_0(\eta|\varphi_t| + |\Delta\varphi|)|_{t=0} dx$$
$$+ \iint_{\mathbb{R}^N \times \mathbb{R}^+} u(\eta|\varphi_{tt}| + |\Delta\varphi_t| + |\Delta\varphi| + \delta|\varphi_t|)\, dxdt. \tag{3.2}$$

Applying parametric Young's inequality to each integral on the right hand side of (3.2) with conjugate exponents q and $q' = q/(q-1)$, we obtain

$$\eta \iint_{\mathbb{R}^N \times \mathbb{R}^+} u|\varphi_{tt}|\, dxdt = \eta \iint_{\mathbb{R}^N \times \mathbb{R}^+} u(a\varphi)^{\frac{1}{q}}|\varphi_{tt}|(a\varphi)^{-\frac{1}{q}}\, dxdt$$
$$\leq \eta\varepsilon \iint_{\mathbb{R}^N \times \mathbb{R}^+} a|u|^q \varphi\, dxdt + \eta c_\varepsilon \iint_{\mathbb{R}^N \times \mathbb{R}^+} |\varphi_{tt}|^{q'} (a\varphi)^{1-q'}\, dxdt, \tag{3.3}$$

for some parameter $\varepsilon > 0$ and constant $c_\varepsilon = q^{-1}\varepsilon^{-\frac{q}{p}}$.

Similarly, we have

$$\iint_{\mathbb{R}^N\times\mathbb{R}^+} u|\Delta\varphi_t|\,dxdt \leq \varepsilon \iint_{\mathbb{R}^N\times\mathbb{R}^+} a|u|^q\varphi\,dxdt + c_\varepsilon \iint_{\mathbb{R}^N\times\mathbb{R}^+} |\Delta\varphi_t|^{q'}(a\varphi)^{1-q'}\,dxdt, \tag{3.4}$$

$$\iint_{\mathbb{R}^N\times\mathbb{R}^+} u|\Delta\varphi|\,dxdt \leq \varepsilon \iint_{\mathbb{R}^N\times\mathbb{R}^+} a|u|^q\varphi\,dxdt + c_\varepsilon \iint_{\mathbb{R}^N\times\mathbb{R}^+} |\Delta\varphi|^{q'}(a\varphi)^{1-q'}\,dxdt, \tag{3.5}$$

$$\delta\iint_{\mathbb{R}^N\times\mathbb{R}^+} u|\varphi_t|\,dxdt \leq \delta\varepsilon \iint_{\mathbb{R}^N\times\mathbb{R}^+} a|u|^q\varphi\,dxdt + \delta\, c_\varepsilon \iint_{\mathbb{R}^N\times\mathbb{R}^+} |\varphi_t|^{q'}(a\varphi)^{1-q'}\,dxdt. \tag{3.6}$$

Combining (3.2) – (3.6), for sufficiently small $\varepsilon > 0$, we obtain

$$\int_{\mathbb{R}^N} (\delta u_0 + \eta u_1)\varphi|_{t=0}dx + \iint_{\mathbb{R}^N\times\mathbb{R}^+} a|u|^q\varphi\,dxdt$$
$$\leq c\big[A_q(\varphi) + B_q(\varphi) + C_q(\varphi) + D_q(\varphi)\big] + \int_{\mathbb{R}^N} u_0(\eta|\varphi_t| + |\Delta\varphi|)|_{t=0}dx, \tag{3.7}$$

where

$$A_q(\varphi) \coloneqq \iint_{\mathbb{R}^N\times\mathbb{R}^+} |\varphi_{tt}|^{q'}(a\varphi)^{1-q'}\,dxdt, \tag{3.8}$$

$$B_q(\varphi) \coloneqq \iint_{\mathbb{R}^N\times\mathbb{R}^+} |\Delta\varphi_t|^{q'}(a\varphi)^{1-q'}\,dxdt, \tag{3.9}$$

$$C_q(\varphi) \coloneqq \iint_{\mathbb{R}^N\times\mathbb{R}^+} |\Delta\varphi|^{q'}(a\varphi)^{1-q'}\,dxdt, \tag{3.10}$$

$$D_q(\varphi) \coloneqq \iint_{\mathbb{R}^N\times\mathbb{R}^+} |\varphi_t|^{q'}(a\varphi)^{1-q'}\,dxdt, \tag{3.11}$$

are known as capacity type integrals of the test function.

By taking the admissible test function $\varphi = \varphi_R$ defined as (2.3) and (2.4), we observe that the following inclusion of sets:

$$\text{supp}(\varphi_R) \subseteq \mathcal{H} := \{(x,t) \in \mathbb{R}^N \times \mathbb{R}^+ : 0 \leq t^\theta + |x|^\mu \leq R^\mu\},$$

$$\text{supp}\left(\frac{\partial\varphi_R}{\partial t}\right),\ \text{supp}\left(\frac{\partial^2\varphi_R}{\partial t^2}\right) \subseteq \mathcal{M} := \{(x,t) \in \mathbb{R}^N \times \mathbb{R}^+ : R^\mu \leq t^\theta + |x|^\mu \leq 2R^\mu\},$$

$$\operatorname{supp}(\Delta\varphi_R),\ \ \operatorname{supp}\left(\Delta\frac{\partial\varphi_R}{\partial t}\right)\subseteq\mathcal{M}:=\left\{(x,t)\in\mathbb{R}^N\times\mathbb{R}^+ : R^\mu\le t^\theta+|x|^\mu\le 2R^\mu\right\}.$$

Now, choose $\lambda > q' = q/(q-1)$ so that the integrals $A_q\left(\psi^\lambda(s)\right)$, $B_q\left(\psi^\lambda(s)\right)$, $C_q\left(\psi^\lambda(s)\right)$, and $D_q\left(\psi^\lambda(s)\right)$ are finite over $1\le\tau^\theta+|\xi|^\mu\le 2$.

In view of (2.5) – (2.6), the capacity type integrals of the test function obtained in (3.8) – (3.11) are estimated as

$$\iint_{\mathcal{M}}\left|\frac{\partial^2\varphi_R}{\partial t^2}\right|^{q'}(a\varphi_R)^{1-q'}\,dxdt\le cR^{-\frac{2\mu}{\theta}q'+\frac{\mu}{\theta}+N+\alpha(1-q')},\tag{3.12}$$

$$\iint_{\mathcal{M}}\left|\Delta\frac{\partial\varphi_R}{\partial t}\right|^{q'}(a\varphi_R)^{1-q'}\,dxdt\le cR^{-\left(2+\frac{\mu}{\theta}\right)q'+\frac{\mu}{\theta}+N+\alpha(1-q')},\tag{3.13}$$

$$\iint_{\mathcal{M}}|\Delta\varphi_R|^{q'}(a\varphi_R)^{1-q'}\,dxdt\le cR^{-2q'+\frac{\mu}{\theta}+N+\alpha(1-q')},\tag{3.14}$$

$$\iint_{\mathcal{M}}\left|\frac{\partial\varphi_R}{\partial t}\right|^{q'}(a\varphi_R)^{1-q'}\,dxdt\le cR^{-\frac{\mu}{\theta}q'+\frac{\mu}{\theta}+N+\alpha(1-q')}.\tag{3.15}$$

For positive constants δ and η, since $u_0,u_1\in L^1_{loc}(\mathbb{R}^N)$ such that $\int_{\mathbb{R}^N}(\delta u_0+\eta u_1)\,dx>0$, there exists a test function $0\le\varphi_R\le 1$ satisfying

$$\int_{\mathbb{R}^N}(\delta u_0+\eta u_1)\varphi_R(x,0)dx\ge 0.$$

Then, by the result (3.12) – (3.15), inequality (3.7) becomes

$$\begin{aligned}\iint_{\mathbb{R}^N\times\mathbb{R}^+}a|u|^q\varphi_R\,dxdt&\le c\sum_{j=1}^{4}R^{\gamma_j(q,\alpha)}+c\int_{R^\mu\le|x|^\mu\le 2R^\mu}u_0|\Delta\varphi_R(x,0)|dx\\&\le c\sum_{j=1}^{4}R^{\gamma_j(q,\alpha)}+\frac{c}{R^2}\int_{R^\mu\le|x|^\mu\le 2R^\mu}u_0\,dx,\end{aligned}\tag{3.16}$$

where

$$\gamma_1(q,\alpha)=-\frac{2\mu}{\theta}q'+\frac{\mu}{\theta}+N+\alpha(1-q'),$$

$$\gamma_2(q,\alpha) = -\left(2+\frac{\mu}{\theta}\right)q' + \frac{\mu}{\theta} + N + \alpha(1-q'),$$

$$\gamma_3(q,\alpha) = -2q' + \frac{\mu}{\theta} + N + \alpha(1-q'),$$

$$\gamma_4(q,\alpha) = -\frac{\mu}{\theta}q' + \frac{\mu}{\theta} + N + \alpha(1-q').$$

Choosing $\theta > 0$ and $\mu > 0$, leads to

$$\max_{j\in\{1,2,3,4\}}\{\gamma_j(q,\alpha)\} = \max\{\gamma_3(q,\alpha), \gamma_4(q,\alpha)\} := \sigma(q,\alpha).$$

Consequently, the expression to the right hand side of the inequality (3.16) is minimal if and only if both terms under summation have the same exponent. That is the optimal value of the exponent is $\sigma(q,\alpha) = N - (2+\alpha)/(q-1)$.

Now, we required that $\sigma(q,\alpha) \le 0$, which is equivalent to $q \le 1 + (2+\alpha)/N$. Thus, there are two cases.

First consider the case $1 < q < 1 + (2+\alpha)/N$. Since $\varphi_R \equiv 1$ on $\mathcal{H}$, it follows that

$$0 < \int\limits_{|x|<R} (\delta u_0 + \eta u_1)dx + \iint\limits_{0\le t^\theta + |x|^\mu \le R^\mu} a|u|^q\,dxdt \le cR^{\sigma(q,\alpha)} + \frac{c}{R^2}\int\limits_{\mathbb{R}^N} u_0\,dx, \qquad (3.17)$$

for all $R \gg 1$.

Passing to the limit, the right hand side of inequality (3.17) converges to zero as $R \to \infty$. Hence, the integral

$$0 < \int\limits_{\mathbb{R}^N} (\delta u_0 + \eta u_1)dx + \iint\limits_{\mathbb{R}^N\times\mathbb{R}^+} a|u|^q\,dxdt = 0,$$

which implies $u \equiv 0$ contradicting the nontriviality of the solution u.

Next, consider the case $1 < q = 1 + (2+\alpha)/N$. Then, the right hand side of inequality (3.17) is bounded as $R \to \infty$. Hence,

$$0 < \int\limits_{\mathbb{R}^N} (\delta u_0 + \eta u_1)dx + \iint\limits_{\mathbb{R}^N\times\mathbb{R}^+} a|u|^q\,dxdt \le c < \infty.$$

Therefore, $a|u|^q$ is integrable. By applying Holder's inequality with conjugate exponents q and $q' = q/(q-1)$ from (3.2), we obtain

$$\int_{\mathbb{R}^N} (\delta u_0 + \eta u_1)\varphi_R dx + \iint_{\mathbb{R}^N\times\mathbb{R}^+} a|u|^q \varphi_R \, dxdt$$
$$\leq c\left[\left(A_q(\varphi_R)\right)^{\frac{1}{q'}} + \left(B_q(\varphi_R)\right)^{\frac{1}{q'}} + \left(C_q(\varphi_R)\right)^{\frac{1}{q'}} + \left(D_q(\varphi_R)\right)^{\frac{1}{q'}}\right]\left(\iint_{\mathcal{M}} a|u|^q \varphi_R \, dxdt\right)^{\frac{1}{q}}$$
$$+\frac{c}{R^2}\int_{\mathbb{R}^N} u_0 \, dx.$$

By virtue of (3.8) – (3.11), we have

$$\int_{\mathbb{R}^N} (\delta u_0 + \eta u_1)\varphi_R dx + \iint_{\mathbb{R}^N\times\mathbb{R}^+} a|u|^q \varphi_R \, dxdt$$
$$\leq cR^{\frac{\sigma(q,\alpha)}{q'}}\left(\iint_{R^\mu \leq t^\theta + |x|^\mu \leq 2R^\mu} a|u|^q \, dxdt\right)^{\frac{1}{q}} + \frac{c}{R^2}\int_{\mathbb{R}^N} u_0 \, dx$$
$$\leq c\left(\iint_{R^\mu \leq t^\theta + |x|^\mu \leq 2R^\mu} a|u|^q \, dxdt\right)^{\frac{1}{q}} + \frac{c}{R^2}\int_{\mathbb{R}^N} u_0 \, dx. \qquad (3.18)$$

The absolute integrablity of $a|u|^q$ implies that $\iint_{R^\mu \leq t^\theta + |x|^\mu \leq 2R^\mu} a|u|^q \, dxdt \to 0,$ as $R \to \infty$. Therefore, by letting $R \to \infty$, we obtain from (3.18) that

$$0 < \int_{\mathbb{R}^N} (\delta u_0 + \eta u_1) dx + \iint_{\mathbb{R}^N\times\mathbb{R}^+} a|u|^q \, dxdt = 0,$$

which implies that $u \equiv 0$, and again leads to a contradiction. This completes the proof of the Theorem 3.1. ■

4. Nonexistence result for coupled system

Here, we provide a sufficient condition for the nonexistence of global weak solutions to a coupled system of quasilinear pseudohyperbolic equations. Let us start with the following definition of weak solutions to the problem (2.2).

Definition 4.1. (Weak solution)

A pair of functions $(u, v) \in C^1(\mathbb{R}^+; L^p_{loc}(\mathbb{R}^N) \times L^q_{loc}(\mathbb{R}^N))$ with the initial datum (u_0, v_0), $(u_1, v_1) \in L^1_{loc}(\mathbb{R}^N) \times L^1_{loc}(\mathbb{R}^N)$ is called a weak solution (in the sense of distributions) to the problem (2.2) if $a(x)|v|^q,\ b(x)|u|^p \in L^1_{loc}(\mathbb{R}^+; L^1_{loc}(\mathbb{R}^N))$ and for any nonnegative admissible test function $\varphi \in C_{x,t}^{2,2}(\mathbb{R}^N \times \mathbb{R}^+)$ with compact support the following two integral inequalities hold

$$\iint_{\mathbb{R}^N\times\mathbb{R}^+} u(\eta\varphi_{tt} + \Delta\varphi_t - \Delta\varphi - \delta\varphi_t)dxdt \ + \int_{\mathbb{R}^N} u_0(\eta\varphi_t + \Delta\varphi - \delta\varphi)|_{t=0}dx$$

$$= \iint_{\mathbb{R}^N\times\mathbb{R}^+} a|v|^q\varphi\, dxdt + \eta \int_{\mathbb{R}^N} u_1\varphi|_{t=0}dx,$$

and

$$\iint_{\mathbb{R}^N\times\mathbb{R}^+} v(\zeta\varphi_{tt} + \Delta\varphi_t - \Delta\varphi - \gamma\varphi_t)dxdt \ + \int_{\mathbb{R}^N} v_0(\zeta\varphi_t + \Delta\varphi - \gamma\varphi)|_{t=0}dx$$

$$= \iint_{\mathbb{R}^N\times\mathbb{R}^+} b|u|^p\varphi\, dxdt + \zeta \int_{\mathbb{R}^N} v_1\varphi|_{t=0}dx.$$

Suppose that the functions $a(x),\ b(x) > 0$ in the system (2.2) have an order of power growth at least $\alpha > -2$ and $\beta > -2$ respectively in the sense that there exist positive constants $c > 0$ and $\alpha, \beta \in \mathbb{R}$ such that

$$a(x) \geq c|x|^\alpha, \qquad b(x) \geq c|x|^\beta, \tag{4.1}$$

for almost every $x \in \mathbb{R}^N$.

Theorem 4.1.

Suppose that $a(x)$ and $b(x)$ satisfy condition (4.1). Let $p, q > 1$, α, $\beta > -2$ and (u_0, v_0), $(u_1, v_1) \in L_{loc}^1(\mathbb{R}^{\mathrm{N}}) \times L_{loc}^1(\mathbb{R}^{\mathrm{N}})$ such that

$$\int_{\mathbb{R}^N} (\delta u_0 + \eta u_1)\, dx > 0, \qquad \int_{\mathbb{R}^N} (\gamma v_0 + \zeta v_1)\, dx > 0.$$

If

$$\max\left\{N\left(1 - \frac{1}{pq}\right) - \frac{q(\beta+2) + \alpha + 2}{pq},\ N\left(1 - \frac{1}{pq}\right) - \frac{p(\alpha+2) + \beta + 2}{pq}\right\} \leq 0,$$

then the problem (2.2) admits no nontrivial global weak solution.

***Proof*:**

Suppose that (u, v) is a nontrivial weak solution to the problem (2.2). Then, for any nonnegative admissible test function φ, we have

$$\int_{\mathbb{R}^N} (\delta u_0 + \eta u_1)\varphi|_{t=0} dx + \iint_{\mathbb{R}^N \times \mathbb{R}^+} a|v|^q \varphi \, dxdt \leq \int_{\mathbb{R}^N} u_0(\eta|\varphi_t| + |\Delta\varphi|)|_{t=0} dx$$

$$+ \iint_{\mathbb{R}^N \times \mathbb{R}^+} u(\eta|\varphi_{tt}| + |\Delta\varphi_t| + |\Delta\varphi| + \delta|\varphi_t|) \, dxdt, \tag{4.2}$$

and

$$\int_{\mathbb{R}^N} (\gamma v_0 + \zeta v_1)\varphi|_{t=0} dx + \iint_{\mathbb{R}^N \times \mathbb{R}^+} b|u|^p \varphi \, dxdt \leq \int_{\mathbb{R}^N} v_0(\zeta|\varphi_t| + |\Delta\varphi|)|_{t=0} dx$$

$$+ \iint_{\mathbb{R}^N \times \mathbb{R}^+} v(\zeta|\varphi_{tt}| + |\Delta\varphi_t| + |\Delta\varphi| + \gamma|\varphi_t|) \, dxdt. \tag{4.3}$$

Applying Holder's inequality to the first integrals of the right hand side of (4.2) and (4.3) with conjugate exponents p & $p' = p/(p-1)$ and q & $q' = \frac{q}{q-1}$, respectively, we obtain

$$\iint_{\mathbb{R}^N \times \mathbb{R}^+} a|v|^q \varphi_R \, dxdt + \int_{\mathbb{R}^N} (\delta u_0 + \eta u_1)\varphi_R|_{t=0} dx - \int_{\mathbb{R}^N} u_0 \left(\eta \left|\frac{\partial \varphi_R}{\partial t}\right| + |\Delta\varphi_R|\right)\Big|_{t=0} dx$$
$$\leq c\left[\left(A_p(\varphi_R)\right)^{\frac{1}{p'}} + \left(B_p(\varphi_R)\right)^{\frac{1}{p'}} + \left(C_p(\varphi_R)\right)^{\frac{1}{p'}} + \left(D_p(\varphi_R)\right)^{\frac{1}{p'}}\right]\left(\iint_{\mathbb{R}^N \times \mathbb{R}^+} b|u|^p \varphi_R dxdt\right)^{\frac{1}{p}},$$

and

$$\iint_{\mathbb{R}^N \times \mathbb{R}^+} b|u|^p \varphi_R \, dxdt + \int_{\mathbb{R}^N} (\gamma v_0 + \zeta v_1)\varphi_R|_{t=0} dx - \int_{\mathbb{R}^N} v_0 \left(\zeta \left|\frac{\partial \varphi_R}{\partial t}\right| + |\Delta\varphi_R|\right)\Big|_{t=0} dx$$
$$\leq c\left[\left(A_q(\varphi_R)\right)^{\frac{1}{q'}} + \left(B_q(\varphi_R)\right)^{\frac{1}{q'}} + \left(C_q(\varphi_R)\right)^{\frac{1}{q'}} + \left(D_q(\varphi_R)\right)^{\frac{1}{q'}}\right]\left(\iint_{\mathbb{R}^N \times \mathbb{R}^+} a|v|^q \varphi_R dxdt\right)^{\frac{1}{q}}.$$

By the same argument as in Theorem 3.1, for R large enough, we have

$$\int_{\mathbb{R}^N} (\delta u_0 + \eta u_1)\varphi_R|_{t=0} dx - \int_{\mathbb{R}^N} u_0 \left(\eta \left|\frac{\partial \varphi_R}{\partial t}\right| + |\Delta\varphi_R|\right)\Big|_{t=0} dx \geq 0,$$

and

$$\int_{\mathbb{R}^N} (\gamma v_0 + \zeta v_1)\varphi_R|_{t=0} dx - \int_{\mathbb{R}^N} v_0 \left(\zeta \left|\frac{\partial \varphi_R}{\partial t}\right| + |\Delta \varphi_R|\right)\bigg|_{t=0} dx \geq 0.$$

Then, in view of (3.12) – (3.15), we have

$$\iint_{\mathbb{R}^N\times\mathbb{R}^+} a|v|^q \varphi_R \, dxdt \leq cR^{\frac{\sigma(p,\beta)}{p'}} \left(\iint_{\mathbb{R}^N\times\mathbb{R}^+} b|u|^p \, \varphi_R dxdt \right)^{\frac{1}{p}}, \tag{4.4}$$

and

$$\iint_{\mathbb{R}^N\times\mathbb{R}^+} b|u|^p \varphi_R \, dxdt \leq cR^{\frac{\sigma(q,\alpha)}{q'}} \left(\iint_{\mathbb{R}^N\times\mathbb{R}^+} a|v|^q \, \varphi_R dxdt \right)^{\frac{1}{q}}. \tag{4.5}$$

Combining (4.4) and (4.5) yields

$$\left(\iint_{\mathbb{R}^N\times\mathbb{R}^+} a|v|^q \, \varphi_R dxdt \right)^{\frac{pq-1}{pq}} \leq cR^{\frac{\sigma(p,\beta)}{p'}+\frac{\sigma(q,\alpha)}{pq'}}, \tag{4.6}$$

and

$$\left(\iint_{\mathbb{R}^N\times\mathbb{R}^+} b|u|^p \varphi_R \, dxdt \right)^{\frac{pq-1}{pq}} \leq cR^{\frac{\sigma(q,\alpha)}{q'}+\frac{\sigma(p,\beta)}{p'q}}. \tag{4.7}$$

Now, we require that

$$\frac{\sigma(p,\beta)}{p'} + \frac{\sigma(q,\alpha)}{pq'} \leq 0 \ , \qquad \frac{\sigma(q,\alpha)}{q'} + \frac{\sigma(p,\beta)}{p'q} \leq 0,$$

which is equivalent to

$$\mathcal{S}_1 := N\left(1 - \frac{1}{pq}\right) - \frac{q(\beta+2)+\alpha+2}{pq} \leq 0,$$

and

$$\mathcal{S}_2 := N\left(1 - \frac{1}{pq}\right) - \frac{p(\alpha+2)+\beta+2}{pq} \leq 0.$$

Consequently, if $\max\{\mathcal{S}_1, \mathcal{S}_2\} < 0$ holds, then passing to the limit when $R \to \infty$ in (4.6), we obtain

$$\iint_{\mathbb{R}^N \times \mathbb{R}^+} a|v|^q \varphi_R dxdt = 0 \Rightarrow v \equiv 0,$$

which in turn leads $u \equiv 0$ by (4.5). This is a contradiction to the nontriviality of (u, v). The case, $\max\{\mathcal{S}_1, \mathcal{S}_2\} = 0$ can be treated in the same way as in the proof of Theorem 3.1. ■

5. Conclusion

In this work, the unsolvablitity conditions for the quasilinear pseudohyperbolic equations and systems with damping terms have been obtained. The nonexistence theorems constitute an important part of the theory of partial differential equations, which proposes the necessary conditions for the existence of global weak solutions to the considered differential equations and systems. The results obtained can be applied to establish unsolvability criteria for certain nonstationary wave like phenonenas whose governing equations can be pseudohyperbolic equations and systems. It enabled us to determine the range of parameters and initial datum that appear in the governing equations for which the solution does not exist.

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