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Impatient Customers in an Markovian Queue with Bernoulli Schedule Working Vacation Interruption and Setup Time

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Abstract

In this paper, using probability generating function method, Impatient customers in an Markovian queue with Bernoulli schedule working vacation interruption and setup time is discussed. Customers impatience is due to the servers vacation. During the working vacation period, if there are customers in the queue, the vacation can be interrupted at a service completion instant and the server begins a regular service period with probability $(1 - b)$ or continues the vacation with probability b. We obtain the probability generating functions of the stationary state probabilities, performance measures, sojourn time of a customer and stochastic decomposition of the queue length, waiting time and numerical results.

Keywords: $M/M/1$ queue; Impatient customers; Working Vacation Interruption; Setup time

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1. Introduction

Queueing systems with server's vacations have been extensively studied by many authors including Tian and Zhang (2006). In 2002, Servi and Finn (2002) first introduced a new vacation policy and studied an $M/M/1$ queue. In telecommunication systems, this period of absence may represent the server's working period on some secondary job. In manufacturing systems, these unavalible periods may represent performing maintenance activities, or equipment breakdowns. These systems are received considerable attention in the literature by the survey of Doshi (1986). While making decision for the number of servers needed in the service system to meet time varying demand, the balking and reneging probabilities can be used to estimate the amount of lost business in more practical consideration for the managers as given in Liao (2007). Haghighi and Dimitar (2016), discussed busy period of a single server Poisson queueing system with spliting and batch delayed-feedback. Vikas and Deepali (2012), studied the state dependent bulk service queue with balking, reneging, and server vacation. Recently, Vijaya Laxmi et al. (2013) analyzed an M/M/1/N working vacations queue with balking and reneging and Vijaya Laxmi et al. (2019) dealt with analysis of a Markovian queueing system with single working vacation and impatience of customers. Abou-El-Ata (1991) discussed the finite buffer single server queueing system with balking and reneging. Analytical solutions of the single server Markovian overflow queue with balking, reneging and an additional server for longer queue were discussed in Abou-El-Ata and Shawky (1992). A computational algorithm and parameter optimization for a multi server queue with unreliable server and impatient customers have been discussed by Chia and Jau-Chaun (2010).

Queueing models with impatience and vacation seem to go back to Van Der Duyn Schouten (1978). The author considers the model $M/G/1$ with finite capacity for the workload and server multiple vacations. Takine and Hasegawa (1990) considered an $M/G/1$ queue with balking customers and deterministic deadline customers. Impatience has been dealt with in the queueing literature mainly in the context of customers abandoning the queue due to either a long wait already experienced or a long wait anticipated upon arrival. Many authors treated the impatience phenomenon under various assumptions including Shakir Majid and Manoharan (2018) and Manoharan and Jeeva (2019). Queueing systems with vacation interruption have been investigated by Baba Baba (2010), Chen et al. (2009), Li and Tian (2007), Zhang and Hou (2010), and Zhang and Shi (2009). Queueing systems with Bernoulli schedule working vacations and vacation interruption have been investigated by Vijayashree and Janani (2018), Sivaraman and Bharti (2017), and Manoharan and Ashok (2018). Bouchentouf and Guendouzi (2020) presented $M^X/M/c$ Bernoulli feedback queue with variant multiple working vacations and impatient customers.

In the present scenario, everybody is using internet to collect any unknown information through browsing and so we are in need of mobile data. A mobile station receives data from a base station. Arriving massages are stored in the base station and the mobile station downloads these messages from the base station. Upon the completion of a download, if there are no messages in the base station, the mobile station is turned off (the duration of setup time starts) in order to save energy(power saving mode). If a new messages arrives the base station sends (at once the duration of setup time ends) a signal in order to wake up the mobile station. So the mobile station needs some random setup time to be active so as to receive waiting messages. This is a process underwent during browsing time. In our models we refer the power saving mode as setup time.

So, queueing models with setup time are inevitable to meet out the present needs. In this paper, we analyze impatient customers in an $M/M/1$ queueing system with Bernoulli schedule single working vacation interruption and setup time. The server works at a slow service rate during the working vacation period, customers also impatient due to this working vacation period. At the service completion epoch the vacation is interrupted and the server move to normal busy period if there are customers in the queue with probability $(1-b)$ or remains in the vacation with probability b. We derive the probability generating functions of the number of customers in the system when the server is in a normal busy period, Bernoulli schedule vacation interruption, setup period respectively. Some performance measures, mean sojourn time, stochastic decomposition and numerical results are obtained.

The paper is organized as follows. In Section 2, we describe the model. In Section 3, the stationary analysis of the system is carried out and deduced the probability generating functions of the number of customers in the system when the server is in a normal service period, Bernoulli schedule vacation interruption and in a setup period respectively. In Section 4, performance measures of the model are presented. In Section 5, the sojourn time of the system is analyzed. In Section 6, stochastic decomposition results are presented. In Section 7, the numerical analysis is carried out.

2. Model Description

We consider impatient customers in an $M/M/1$ queueing system with Bernoulli vacation interruption and setup time. Customers arrive according to a Poisson process with mean arrival rate λ. The service times during a normal service period, a working vacation period and the vacation times are exponentially distributed with rates μ_b, μ_v and γ respectively. The customers are assumed to be impatient during the single working vacations. Whenever a customer arrives at the system is on working vacation, the customer activates an impatient timer T, which is exponentially distributed with rate α. If the server finishes the working vacation before the impatient timer expires, the customer remains in the system till his service completion. However, if the impatient timer expires when the server is still on working vacation, the customer abandons the system and never returns. During the working vacation period, a customer is served at a lower rate and at the instants of the service completion, the vacation is interrupted and the server resumes a regular service period with probability $\bar{b} = (1 - b)$ or remains in the vacation with probability b. At the end of vacation the server begins a closed-down period. During a closed-down period, an arriving customer cannot be served immediately and the server experiences a period of setup time. Setup time duration follows an exponential distribution with parameter β and a regular busy period starts after a setup period. We assume that inter arrival times, service times, Impatient times, working vacation times and setup times are mutually independent. In addition, the service order is First In First Out (FIFO).

3. Stationary Probabilities

At time t, let $Q(t)$ denote the total number of customers in the system and $J(t)$ denotes the state of the server with

$$J(t)= \begin{cases} 0, & \text{at time t the server is busy in working vacation period,} \\ 1, & \text{at time t the server is free during in setup period,} \\ 2, & \text{at time t the server is busy in normal service period.} \end{cases}$$

Then, the pair $\{(Q(t), J(t)); t \geq 0\}$ defines a two dimensional continuous time Markov chain with state space

$$\Omega = \{\{(0,0), (0,1)\} \cup \{(i,j); i = 1, 2, ..., j = 0, 1, 2\}\},$$

Let

$$p_{i,j} = \lim_{t\to\infty} Pr\left\{(Q(t) = i, J(t) = j); t \geq 0\right\}.$$

denote the stationary probabilities of the Markov process $\{(Q(t), J(t))\ ; t \geq 0\}$. Therefore, using the theory of Markov process, under the stability condition $\rho = (\frac{\lambda}{\mu_b}) < 1$, we obtain the following set of balance equations:

$$(\lambda + \gamma)p_{0,0} = (\alpha + \mu_v)p_{1,0} + \mu_b p_{1,2}, \tag{1}$$

$$(\lambda + n\alpha + \mu_v + \gamma)p_{n,0} = \lambda p_{n-1,0} + (b\mu_v + (n+1)\alpha)p_{n+1,0}, n \geq 1, \tag{2}$$

$$\lambda p_{0,1} = \gamma p_{0,0}, \tag{3}$$

$$(\lambda + \beta)p_{n,1} = \lambda p_{n-1,1}, n \geq 1, \tag{4}$$

$$(\lambda + \mu_b)p_{1,2} = \gamma p_{1,0} + \mu_b p_{2,2} + \bar{b}\mu_v p_{2,0} + \beta p_{1,1}, \tag{5}$$

$$(\lambda + \mu_b)p_{n,2} = \lambda p_{n-1,2} + \beta p_{n,1} + \gamma p_{n,0} + \mu_b p_{n+1,2} + \bar{b}\mu_v p_{n+1,0}, n \geq 2. \tag{6}$$

Define the Probability generating functions

$$P_0(z) = \sum_{n=0}^{\infty} p_{n,0}z^n; P_1(z) = \sum_{n=0}^{\infty} p_{n,1}z^n; P_2(z) = \sum_{n=1}^{\infty} p_{n,2}z^n,$$

$$\text{with } P_0(1) + P_1(1) + P_2(1) = 1 \text{ and } P_0^{'}(z) = \sum_{n=1}^{\infty} nz^{n-1}p_{n,0}.$$

Multiplying the equation (2) with z^n and summing over all possible values of n and adding with (1), we get

$$\alpha z(1-z)P_0'(z) + A(z)P_0(z) + A_1 z - A_2(1-z) = 0, \tag{7}$$

where

$$A(z) = (\lambda z^2 - (\lambda + \mu_v + \gamma)z + b\mu_v), \tag{8}$$

$$A_1 = \mu_b p_{1,2} + \bar{b}\mu_v p_{1,0} + \bar{b}\mu_v p_{0,0}, \tag{9}$$

$$A_2 = b\mu_v p_{0,0}. \tag{10}$$

Multiplying Equation (4) with z^n and summing over all possible values of n and adding with (3), we get

$$(\lambda + \beta - \lambda z)P_1(z) = (\lambda + \beta)p_{0,1}. \tag{11}$$

Multiplying Equation (6) with z^n and summing over all possible values of n and adding with (5), we get

$$(1-z)(\lambda z-\mu_b)P_2(z)=(\gamma z+\bar{b}\mu_v)P_0(z)+\beta zP_1(z)$$
$$-(\mu_b p_{1,2}+\bar{b}\mu_v p_{1,0}+\bar{b}\mu_v p_{0,0}+\beta p_{0,1})z-\bar{b}\mu_v(1-z)p_{0,0}. \quad (12)$$

For solving the differential equation (7), it can be written as

$$P_0'(z)+\left[\left(-\frac{\lambda}{\alpha}\right)+\left(\frac{b\mu_v}{\alpha z}\right)-\frac{(\gamma+\bar{b}\mu_v)}{\alpha(1-z)}\right]P_0(z)=-\left(\frac{1}{\alpha}\right)\left[\frac{A_1}{(1-z)}-\frac{A_2}{z}\right]. \quad (13)$$

Solving (13), we get

$$P_0(z)=\frac{-A_1B_1(z)+A_2B_2(z)}{\alpha e^{(-\frac{\lambda}{\alpha})z}z^{(\frac{b\mu_v}{\alpha})}(1-z)^{\frac{(\gamma+\bar{b}\mu_v)}{\alpha}}},$$

where

$$B_1(z)=\int_0^z e^{-(\frac{\lambda}{\alpha})y}y^{\frac{b\mu_v}{\alpha}}(1-y)^{\frac{(\gamma+\bar{b}\mu_v)}{\alpha}-1}dy,$$
$$B_2(z)=\int_0^z e^{-(\frac{\lambda}{\alpha})y}y^{\frac{b\mu_v}{\alpha}-1}(1-y)^{(\frac{\gamma+\bar{b}\mu_v}{\alpha})}dy.$$

Setting $\alpha=0$ and $\bar{b}=1$ in (7), we get

$$P_0(z)=\frac{\mu_v(1-z)p_{0,0}-\mu_b z p_{1,2}}{(\lambda z^2-(\lambda+\mu_v+\gamma)z+\mu_v)}. \quad (14)$$

For $z=1$, Equation (12) becomes

$$(\gamma+\bar{b}\mu_v)P_0(1)+\beta P_1(1)=\mu_b p_{1,2}+\bar{b}\mu_v p_{1,0}+\bar{b}\mu_v p_{0,0}+\beta p_{0,1}. \quad (15)$$

Using (15) in (12), we get

$$P_2(z)=\frac{(\gamma z+\bar{b}\mu_v)P_0(z)+\beta zP_1(z)-z\big[(\gamma+\bar{b}\mu_v)P_0(1)+\beta P_1(1)\big]}{(\lambda z-\mu_b)(1-z)}-\frac{\bar{b}\mu_v p_{0,0}}{(\lambda z-\mu_b)}. \quad (16)$$

Using L'Hospital's rule and applying $z=1$, we get

$$P_2(z)\Big|_{z=1}=\left[\frac{\begin{array}{l}\Big[(\gamma z+\bar{b}\mu_v)P_0'(z)+\gamma P_0(z)+\beta zP_1'(z)\\ \quad +\beta P_1(z)-[(\gamma+\bar{b}\mu_v)P_0(1)+\beta P_1(1)]\Big]\end{array}}{\lambda(1-z)-(\lambda z-\mu_b)}+\frac{\bar{b}\mu_v p_{0,0}}{(\mu_b-\lambda)}\right]\Bigg|_{z=1} \quad (17)$$

$$\Rightarrow P_2(1)=\left[\frac{(\gamma+\bar{b}\mu_v)P_0'(1)+\beta P_1'(1)}{(\mu_b-\lambda)}\right]+\frac{\bar{b}\mu_v}{(\mu_b-\lambda)}(p_{0,0}-P_0(1)). \quad (18)$$

4. Performance Measures

Let $E[L_0]$ denote the mean length of the vacation period, $E[L_1]$ denote the mean length of the setup period and $E[L_2]$ denote the mean length of the normal busy period. Then, $E[L_0] = P_0'(1)$, $E[L_1] = P_1'(1)$, $E[L_2] = P_2'(1)$ and $P_2(1) = 1 - P_0(1) - P_1(1)$. Substituting $P_0'(1), P_1'(1)$ and $P_2'(1)$ values in (18), we get

$$E[L_0] = \frac{(\mu_b - \lambda)}{(\gamma + \bar{b}\mu_v)}\big[1 - P_0(1) - P_1(1)\big] - \frac{\beta(\mu_b - \lambda)}{(\gamma + \bar{b}\mu_v)} - \frac{\bar{b}\mu_v}{(\gamma + \bar{b}\mu_v)}\big[p_{0,0} - P_0(1)\big]. \quad (19)$$

Adding (2), (4), (6) and rearranging the terms, we get

$$\begin{aligned}\lambda p_{n,0} + \lambda p_{n,1} + \lambda p_{n,2} - [(\mu_v + (n+1)\alpha)p_{n+1,0} + \mu_b p_{n+1,2}] &= \lambda p_{n-1,0} + \lambda p_{n-1,1} \\ &+ \lambda p_{n-1,2} - [(\mu_v + n\alpha)p_{n,0} + \mu_b p_{n,2}], n \geq 1. \quad (20)\end{aligned}$$

Using (1) and (3) in (20), we get

$$\begin{aligned}\lambda p_{n,0} + \lambda p_{n,1} + \lambda p_{n,2} - [(\mu_v + (n+1)\alpha)p_{n+1,0} + \mu_b p_{n+1,2}] &= \lambda p_{0,0} + \lambda p_{0,1} \\ &- \big[(\mu_v + \alpha)p_{1,0} + \mu_b p_{1,2}\big],\end{aligned}$$

$$\lambda P_0(1) + \lambda P_1(1) + \lambda P_2(1) = \mu_b P_2(1) + \mu_v(P_0(1) - p_{0,0}) + \alpha \sum_{n=0}^{\infty}(n+1)p_{n+1,0}, n \geq 0. \quad (21)$$

Using $E[L_0] = \sum_{n=0}^{\infty}(n+1)p_{n+1,0}$ and $P_2(1) = 1 - P_0(1) - P_1(1)$ in (21), we get

$$\begin{aligned}(\mu_b - \lambda)(\gamma + \bar{b}\mu_v) + \alpha(\mu_b - \lambda) &= [\alpha(\mu_b - \lambda) + (\mu_b - \mu_v)(\gamma + \bar{b}\mu_v) - \alpha\bar{b}\mu_v]p_{0,0} \\ &+ \alpha(\mu_b - \lambda)[P_1(1) + \beta E[L_1]]. \quad (22)\end{aligned}$$

Taking $\lim_{z\to 1}$ to (14) and then using (9) and (10), we get

$$\begin{aligned}P_0(1) = \frac{e^{\frac{\lambda}{\alpha}}}{\alpha}&\Big[-\big\{\big[(\gamma + \bar{b}\mu_v)P_0(1) + \beta P_1(1)\big] + \beta P_{0,1}\big\} B_1(1) + b\mu_v P_{0,0} B_2(1)\Big] \\ &\times \lim_{z\to 1}(1-z)^{\frac{-(\gamma + \bar{b}\mu_v)}{\alpha}}. \quad (23)\end{aligned}$$

Since

$$0 \leq P_0(1) = \sum_{n=0}^{\infty} p_{n,0} \leq 1 \text{and } \lim_{z\to 1}(1-z)^{\frac{-(\gamma+\bar{b}\mu_v)}{\alpha}} \to \infty,$$

we must have

$$-\big[(\gamma + \bar{b}\mu_v)P_0(1) + \beta P_1(1)\big] B_1(1) + \beta p_{0,1} B_1(1) + b\mu_v p_{0,0} B_2(1) = 0 \quad (24)$$

$$\Rightarrow p_{0,0} = \frac{\big[(\gamma + \bar{b}\mu_v)P_0(1) + \beta P_1(1) - \beta p_{0,1}\big] B_1(1)}{b\mu_v B_2(1)}. \quad (25)$$

Using (25) in (22), we get

$$\begin{aligned}P_0(1) = \big[&(\alpha+\gamma+\bar{b}\mu_v)(\mu_b-\lambda)bB_2(1) - \alpha(\mu_b-\lambda)\beta E[L_1]bB_2(1)\\ &+(\alpha\bar{b}+\gamma+\bar{b}\mu_v)\beta P_0(1)B_1(1)\\ &-((\alpha\bar{b}+\gamma+\bar{b}\mu_v)\beta B_1(1)+\alpha(\mu_b)bB_2(1))P_1(1)(\alpha(\mu_b-\lambda)\\ &+(\mu_b-\mu_v)(\gamma+\bar{b}\mu_v)bB_2(1)+(\alpha\bar{b}+\gamma+\bar{b}\mu_v)(\gamma+\bar{b}\mu_v)B_1(1))\big]^{-1}.\end{aligned} \tag{26}$$

Using (1), (3) and (19), the unknown $p_{0,1},\ p_{1,0}, p_{1,2},$ we obtain the following:

$$p_{1,0} = \frac{\begin{bmatrix}(\lambda+\gamma+\bar{b}\mu_v)((\gamma+\bar{b}\mu_v)P_0(1)+\beta P_1(1)-\beta p_{01})B_1(1)\\ -b\mu_v B_2(1)(\beta P_1(1)+(\gamma+\bar{b}\mu_v)P_0(1))\end{bmatrix}}{b\mu_v B_2(1)(\alpha+b\mu_v)}. \tag{27}$$

From Equations (3) and (25), we get

$$p_{0,1} = \frac{\gamma((\gamma+\bar{b}\mu_v)P_0(1)+\beta P_1(1))B_1(1)}{\lambda b\mu_v B_2(1)+\beta\gamma B_1(1)}. \tag{28}$$

From Equations (1) and (27), we get

$$p_{1,2} = \frac{\begin{bmatrix}((\alpha+\mu_v)b\mu_v B_2(1)(P_0(1)(\gamma+\bar{b}\mu_v)+\beta P_1(1))\\ -B_1(1)(\lambda\mu_v\bar{b}+(\alpha+\mu_v)(\gamma+\bar{b}\mu_v)-\gamma(\alpha+b\mu_v))\\ (P_0(1)(\gamma+\bar{b}\mu_v)+\beta P_1(1)-\beta p_{0,1}))\end{bmatrix}}{b\mu_v\mu_b(\alpha+n\mu_v)B_2(1)}. \tag{29}$$

Differentiating $P_1(z)$ with respect to z, we get

$$E[L_1] = \frac{\lambda(\lambda+\beta)p_{0,1}}{\beta^2}.$$

Now $E(L_2)$, the expected number of customers in the system when the server is busy, is obtained from (16) using L'Hospital's rule as follows:

$$\begin{aligned}E(L_2) &= P_2'(1),\\ &= \frac{\begin{bmatrix}(\mu_b-\lambda)(2\gamma P_0'(1)+2\beta P_1'(1)+(\gamma+\bar{b}\mu_v)P_0''(1)+\beta P_1''(1))\\ +2\lambda((\gamma+\bar{b}\mu_v)P_0'(1)+\beta P_1'(1)+\bar{b}\mu_v(p_{0,0}-P_0(1)))\end{bmatrix}}{2(\mu_b-\lambda)^2}.\end{aligned} \tag{30}$$

Differentiating (7) twice, respectively, we get

$$\alpha(1-z)P_0'(z)+\alpha z(1-z)P_0''(z)-\alpha zP_0'(z)+A'(z)P_0(z)+A(z)P_0'(z)+A_1+A_2=0,$$

$$\begin{aligned}\big[&\alpha(1-z)P_0''(z)-\alpha P_0'(z)+\alpha z(1-z)P_0'''(z)-\alpha zP_0''(z)+\alpha(1-z)P_0''(z)\\ &-\alpha P_0'(z)-\alpha zP_0''(z)+A''(z)P_0(z)+2A'(z)P_0'(z)+A(z)P_0''(z)\big]=0.\end{aligned}$$

At $z=1$, the above becomes

$$-2\alpha P_0'(1)-2\alpha P_0''(1)+2A'(z)P_0'(1)+A''(z)P_0(1)+A(1)P_0''(1)=0,$$

$$A''(1)P_0(1)+2(A'(1)-\alpha)P_0'(1)+(A(1)-2\alpha)P_0''(1)=0. \tag{31}$$

But $A(1)=b\mu_v-(\mu_v+\gamma)=-(\gamma+\bar{b}\mu_v),\quad A'(1)=(\lambda-\mu_v-\gamma)$ and $\quad A''(1)=2\lambda.$

Using all these values in (31), we get

$$P_0''(1) = 2\left\{ \left(\frac{\lambda}{\gamma + \bar{b}\mu_v + 2\alpha}\right) P_0(1) - \left(\frac{\alpha + \mu_v + \gamma - \lambda}{\gamma + \bar{b}\mu_v + 2\alpha}\right) E(L_0) \right\}, \tag{32}$$

$$E(L_1) = P_1'(1) = \frac{\lambda(\lambda + \beta)}{\beta^2} p_{0,1}, \tag{33}$$

$$P_1''(1) = \frac{2\lambda^2(\lambda + \beta)}{\beta^3}. \tag{34}$$

Using (32), (33), (34) in (30), we get $E[L_2]$. The expected number of customers in the system can be computed as

$$E[L] = E[L_0] + E[L_1] + E[L_2].$$

5. Sojourn Time

Let W be the total sojourn time of a customer in the system, measured from the instant of arrival till departure, with the departure either due to completion of service or as a consequence of abandonment. By Little's rule

$$E[W] = \frac{1}{\lambda}[E[L_0] + E[L_1] + E[L_2]],$$

$$E(W_{n,2}) = \frac{(n+1)}{\mu_b}, n = 1, 2, 3, \ldots. \tag{35}$$

For $n \geq 1$, we get

$$\begin{aligned} E(W_{n,0}) = p_{(n+1,0)(n+1,1)} &\left[\frac{1}{(\lambda + \gamma + \mu_v + (n+1)\alpha)} + E[W_{n,1}]\right] \\ + p_{(n+1,0)(n+2,0)} &\left[\frac{1}{(\lambda + \gamma + \mu_v + (n+1)\alpha)} + E[W_{n,0}]\right] \\ + p_{(n+1,0)(n,0)} &\left[\frac{1}{(\lambda + \gamma + \mu_v + (n+1)\alpha)} + E[W_{n-1,0}]\right] \\ + p_{(n+1,0)(n,1)} &\left[\frac{1}{(\lambda + \gamma + \mu_v + (n+1)\alpha)} + E[W_{n-1,1}]\right]. \end{aligned} \tag{36}$$

For $n = 0, 1, 2, ...$, we have

$$\begin{aligned} p_{(n+1,0)(n+1,1)} &= \frac{\gamma}{(\lambda + \gamma + \mu_v + (n+1)\alpha_{n+1})}, \\ p_{(n+1,0)(n,1)} &= \frac{\bar{b}\mu_v}{(\lambda + \gamma + \mu_v + (n+1)\alpha_{n+1})}, \\ p_{(n+1,0)(n+2,0)} &= \frac{\lambda}{(\lambda + \gamma + \mu_v + (n+1)\alpha_{n+1})}, \\ p_{(n+1,0)(n,0)} &= \frac{(n\alpha + b\mu_v)}{(\lambda + \gamma + \mu_v + (n+1)\alpha_{n+1})}. \end{aligned} \tag{37}$$

Thus, $E[W_{n,0}]$ is given as

$$\begin{aligned} E[W_{n,0}] =& \frac{\gamma}{[\lambda+\gamma+\mu_v+(n+1)\alpha]}\left[\frac{1}{[\lambda+\gamma+\mu_v+(n+1)\alpha]}+E[W_{n,1}]\right] \\ &+ \frac{\lambda}{[\lambda+\gamma+\mu_v+(n+1)\alpha]}\left[\frac{1}{[\lambda+\gamma+\mu_v+(n+1)\alpha]}+E[W_{n,1}]\right] \\ &+ \frac{\bar{b}\mu_v}{[\lambda+\gamma+\mu_v+(n+1)\alpha]}\left[\frac{1}{[\lambda+\gamma+\mu_v+(n+1)\alpha]}+E[W_{n,1}]\right] \\ &+ \frac{n\alpha+b\mu_v}{[\lambda+\gamma+\mu_v+(n+1)\alpha]}\left[\frac{1}{[\lambda+\gamma+\mu_v+(n+1)\alpha]}+E[W_{n-1,0}]\right]. \end{aligned} \tag{38}$$

Putting $n=0$ in (38), we get

$$\begin{aligned} E[W_{0,0}] &= \frac{\gamma}{(\lambda+\gamma+\mu_v+\alpha)}\left[\frac{1}{(\lambda+\gamma+\mu_v+\alpha)}+E[W_{1,2}]\right] \\ &\quad+\frac{\lambda}{(\lambda+\gamma+\mu_v+\alpha)}\left[\frac{1}{(\lambda+\gamma+\mu_v+\alpha)}+E[W_{0,0}]\right] \\ &\quad+\frac{\mu_v}{(\lambda+\gamma+\mu_v+\alpha)^2}, \\ \frac{(\gamma+\mu_v+\alpha)}{(\lambda+\gamma+\mu_v+\alpha)}E[W_{0,0}] &= \frac{1}{(\lambda+\gamma+\mu_v+\alpha)}\left[\frac{\gamma}{(\lambda+\gamma+\mu_v+\alpha)}+\frac{\gamma}{\mu_b}\right. \\ &\quad\left.+\frac{\lambda}{(\lambda+\gamma+\mu_v+\alpha)}+\frac{\mu_v}{(\lambda+\gamma+\mu_v+\alpha)}\right], \\ E[W_{0,0}] &= \frac{1}{(\gamma+\mu_v+\alpha)}\left[\frac{(\lambda+\gamma+\mu_v)}{(\lambda+\gamma+\mu_v+\alpha)}+\frac{\gamma}{\mu_b}\right]. \end{aligned} \tag{39}$$

Recursively iterating (38) for $n \geq 0$ and using (39), we get

$$\begin{aligned} E[W_{n,0}] &= \frac{1}{(\gamma+\mu_v+(n+1)\alpha)}\left[\frac{(\lambda+\gamma+\mu_v+n\alpha)}{(\lambda+\gamma+\mu_v+(n+1)\alpha)}+\frac{((n+1)\gamma+n\bar{b}\mu_v)}{\mu_b}\right] \\ &\quad+\sum_{k=1}^{n}\frac{1}{(\gamma+\mu_v+k\alpha)}\left[\frac{(\lambda+\gamma+\mu_v+(k-1)\alpha)}{(\lambda+\gamma+\mu_v+k\alpha)}+\frac{(k\gamma+(k-1)\bar{b}\mu_v)}{\mu_b}\right] \\ &\quad\times\prod_{i=k}^{n}\frac{(b\mu_v+i\alpha)}{(\gamma+\mu_v+(i+1)\alpha)}. \end{aligned}$$

$$\therefore E[W_{n,1}] = \frac{1}{\beta}\left[1+\frac{\beta(n+1)}{\mu_b}\right].$$

Let W_s denote the mean waiting time of customers served. Then,

$$\begin{aligned} E[W_s] &= \sum_{n=0}^{\infty}p_{n,0}E[W_{n,0}]+\sum_{n=0}^{\infty}p_{n,1}E[W_{n,1}]+\sum_{n=1}^{\infty}p_{n,2}E[W_{n,2}], \\ &= \sum_{n=0}^{\infty}p_{n,0}E[W_{n,0}]+\sum_{n=0}^{\infty}p_{n,1}E[W_{n,1}]+\frac{1}{\mu_b}\big[E(L_2)+P_2(1)\big]. \end{aligned} \tag{40}$$

The probability that the system is in normal busy period (P_b), the probability that the system is in working vacation period (P_{wv}) and the probability that the system is in setup period (P_{st}) are, given by

$$P_b = \sum_{n=1}^{\infty} p_{n,2} = P_2(1)\ , \quad P_{wv} = \sum_{n=0}^{\infty} p_{n,0} = P_0(1)\ , \quad P_{st} = \sum_{n=0}^{\infty} p_{n,1} = P_1(1). \tag{41}$$

6. Stochastic Decomposition

Theorem 6.1.

If $\rho < 1$, the stationary queue length L can be decomposed into the sum of two independent random variables $L = L_0 + L_d$, where L_0 is the stationary queue length of a $M/M/1$ queue and L_d is the additional stationary queue length due to the Bernoulli schedule vacation interruption and setup time has a distribution with pgf

$$L_d(z) = \frac{1}{(1-\rho)}\Bigg\{P_0(z)\left[1-\rho z - \frac{(\gamma z+\bar{b}\mu_v)}{\mu_b(1-z)}\right] + P_1(z)\left[1-\rho z - \frac{\beta z}{\mu_b(1-z)}\right] + \frac{\bar{b}\mu_v p_{0,0}}{\mu_b} + \frac{z((\gamma+\bar{b}\mu_v)P_0(1)+\beta P_1(1))}{\mu_b(1-z)}\Bigg\}. \tag{42}$$

Proof:

Consider

$$\begin{aligned}
L(z) &= P_0(z) + P_1(z) + P_2(z),\\
&= P_0(z)\left[1+\frac{(\gamma z+\bar{b}\mu_v)}{(\lambda z-\mu_b)(1-z)}\right] + P_1(z)\left[1+\frac{\beta z}{(\lambda z-\mu_b)(1-z)}\right]\\
&\quad -\frac{z((\gamma+\bar{b}\mu_v)P_0(1)+\beta P_1(1))}{(\lambda z-\mu_b)(1-z)} - \frac{\bar{b}\mu_v p_{0,0}}{(\lambda z-\mu_b)},\\
&= \frac{(\mu_b-\lambda)}{(\mu_b-\lambda z)}\Bigg\{P_0(z)\left[\frac{\mu_b(1-\rho z)}{\mu_b-\lambda} - \frac{\gamma z+\bar{b}\mu_v}{(\mu_b-\lambda)(1-z)}\right]\\
&\quad + P_1(z)\left[\frac{\mu_b(1-\rho z)}{\mu_b-\lambda} - \frac{\beta z}{(\mu_b-\lambda)(1-z)}\right]\\
&\quad + \frac{z((\gamma+\bar{b}\mu_v)P_0(1)+\beta P_1(1))}{(\lambda z-\mu_b)(1-z)} + \frac{\bar{b}\mu_v p_{0,0}}{(\lambda z-\mu_b)}\Bigg\},\\
&= \frac{(1-\rho)}{(1-\rho z)}L_d(z).
\end{aligned}$$

$L_d(z)$ can be expressed in series expansion as

$$L_d(z) = \frac{1}{(1-\rho)}\Bigg[\sum_{n=0}^{\infty} p_{n,0}z^n - \rho\sum_{n=0}^{\infty} p_{n,0}z^{n+1} + \frac{\gamma}{\mu_b}\sum_{n=1}^{\infty}\sum_{k=0}^{\infty} p_{n+k,0}z^n + \frac{\bar{b}\mu_v}{\mu_b}\sum_{n=1}^{\infty}\sum_{k=0}^{\infty} p_{n+k+1,0}z^n + \sum_{n=0}^{\infty} p_{n,1}z^n - \rho\sum_{n=0}^{\infty} p_{n,1}z^{n+1} + \frac{\beta}{\mu_b}\sum_{n=1}^{\infty}\sum_{k=0}^{\infty} p_{n+k,1}z^n\Bigg].$$

$L_d(z) = \sum_{n=0}^{\infty} C_n z^n$ where

$$C_0 = \frac{p_{0,0}}{(1-\rho)} \text{ and}$$

$$C_n = \frac{1}{(1-\rho)}\left\{p_{n,0} - \rho p_{n-1,0} + \frac{\gamma}{\mu_b}\sum_{k=0}^{\infty} p_{n+k,0} + \left(\frac{\bar{b}\mu_v}{\mu_b}\right)\sum_{k=0}^{\infty} p_{n+k+1,0} + p_{n,1} - \rho p_{n-1,1} + \left(\frac{\beta}{\mu_b}\right)\sum_{k=0}^{\infty} p_{n+k,1}\right\}, n \geq 0.$$

Consider

$$\begin{aligned}\sum_{n=0}^{\infty} C_n &= \frac{1}{(1-\rho)}\Bigg\{(1-\rho)\sum_{n=0}^{\infty} p_{n,0} + \frac{\gamma}{\mu_b}\sum_{n=1}^{\infty} np_{n,0} + \frac{\bar{b}\mu_v}{\mu_b}\sum_{n=2}^{\infty}(n-1)p_{n,0} \\ &\quad +(1-\rho)\sum_{n=0}^{\infty} p_{n,1} + \frac{\beta}{\mu_b}\sum_{n=1}^{\infty} np_{n,1}\Bigg\}, \\ &= \frac{1}{(1-\rho)}\Bigg\{(1-\rho)\sum_{n=0}^{\infty} p_{n,0} + \frac{(\gamma + \bar{b}\mu_v)}{\mu_b}\sum_{n=1}^{\infty} np_{n,0} - \frac{\bar{b}\mu_v}{\mu_b}\sum_{n=2}^{\infty}(n-1)p_{n,0} \\ &\quad +(1-\rho)\sum_{n=0}^{\infty} p_{n,1} + \frac{\beta}{\mu_b}\sum_{n=1}^{\infty} np_{n,1}\Bigg\}.\end{aligned}$$

Using (19) in the above, we get

$$\begin{aligned}\sum_{n=0}^{\infty} C_n &= \sum_{n=0}^{\infty} p_{n,0} + \sum_{n=0}^{\infty} p_{n,1} + \frac{1}{(1-\rho)}\Bigg\{(1-\rho)(1 - P_0(1) - P_1(1) - \beta E[L_1]) \\ &\quad -\frac{\bar{b}\mu_v}{\mu_b}\sum_{n=2}^{\infty} p_{n,0} - \frac{\bar{b}\mu_v}{\mu_b}(p_{0,0} - P_0(1)) + \frac{\beta}{\mu_b}\sum_{n=1}^{\infty} np_{n,1}\Bigg\}, \\ &= \Bigg\{\sum_{n=0}^{\infty} p_{n,0} + \sum_{n=0}^{\infty} p_{n1} + 1 - P_0(1) - P_0(1) - \beta E[L_1] + \frac{\bar{b}\mu_v}{\mu_b(1-\rho)}P_0(1) \\ &\quad -\frac{\bar{b}\mu_v}{\mu_b(1-\rho)}\sum_{n=0}^{\infty} p_{n,0} + \frac{\beta}{\mu_b(1-\rho)}\Bigg\} = 1,\end{aligned}$$

$$\Rightarrow \sum_{n=0}^{\infty} C_n = 1.$$

Hence, $L_d(z)$ is a PGFs of the additional queue length due to the Bernoulli schedule vacation interruption. ■

Theorem 6.2.

If $\rho < 1$, the stationary waiting time can be decomposed into the sum of two independent random variables $W = W_0 + W_d$, where W_0 is the stationary waiting time of a $M/M/1$ queue without vacation which has an exponential distribution with the parameter $\mu_b(1-\rho)$ and W_d is the additional stationary waiting time due to Bernoulli Schedule vacation interruption and setup time has a distribution with its Laplace Stieltjes transform (LST)

$$\begin{aligned} W_d^*(s) = \frac{1}{(\mu_b-\lambda)s}\Bigg\{ & \left[(\mu_b-\lambda+s)s-\gamma(\lambda-s)-\lambda\bar{b}\mu_v\right] P_0\left[1-\left(\frac{s}{\lambda}\right)\right] \\ & +\Big[(\mu_b-\lambda+s)s-\beta(\lambda-s)\Big] P_1\left[1-\left(\frac{s}{\lambda}\right)\right] \\ & +\bar{b}\mu_v s p_{0,0}+(\lambda-s)\big[(\gamma+\bar{b}\mu_v)P_0(1)+\beta P_1(1)\big]\Bigg\}. \end{aligned} \tag{43}$$

Proof:

The relationship between the Probability generating function and LST of waiting time is given by

$Q(z) = W^*\big[\lambda(1-z)\big]$. Assume that $s = \lambda(1-z);$ so $z = \left[1-\left(\frac{s}{\lambda}\right)\right]$ and $(1-z) = \left(\frac{s}{\lambda}\right)$.

Applying the above relations in (43), we obtain the desired result. Taking $\lim_{s\to 0} W_d^*(s)$, we get

$$\begin{aligned} W_d = \frac{1}{(\mu_b-\lambda)}\Big[& P_0(1)(\mu_b-\lambda+\gamma)+(\gamma+\bar{b}\mu_v)P_0'(1) \\ & +P_1(1)(\mu_b-\lambda+\beta)+\beta P_1'(1)+\bar{b}\mu_v p_{0,0}\Big]. \end{aligned}$$

Substituting the value of $P_0'(1)$, $P_1'(1)$, we get $\lim_{s\to 0} W_d^*(s)$=1, where W_d is a random variable of the additional waiting time. ■

7. Numerical results

By fixing the values of $\gamma = 0.8$, $\beta = 4.2$, $b = 0.3$, $\alpha = 0.4$ and $\mu_v = 1.8$ subject to the stability condition and varying λ from 1.0 to 2.0 in steps of 0.1 the values of E(L) is calculated for the values of $\mu_b = 5.7$ and $\mu_b = 6.5$. The corresponding line graphs are drawn in Figure 1. We observe that as λ increases E(L) is also increases. Again, by fixing the values of $\gamma = 0.9$, $\beta = 3.8$, $b = 0.5$, $\alpha = 0.8$ and $\mu_v = 2.3$ subject to the stability condition and varying λ from 1.0 to 2.0 in steps of 0.1 the values of E(W) is calculated for $\mu_b = 6.8$ and $\mu_b = 7.4$ respectively. The corresponding line graphs are drawn in Figure 2. We observe that as λ increases E(W) also increases.

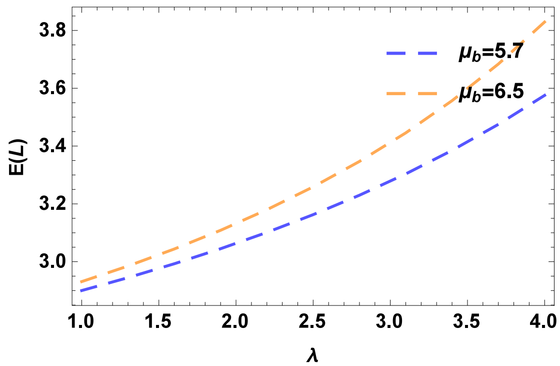

Figure 1: The changing curve of E(L)

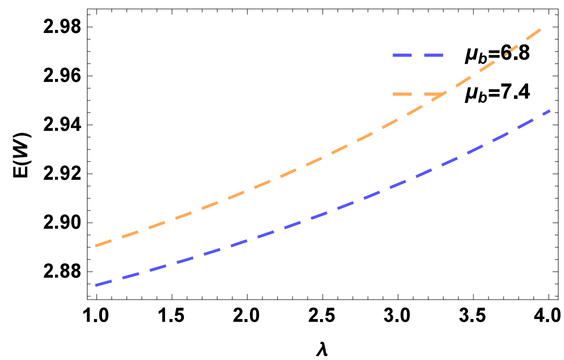

Figure 2: The changing curve of E(W)

8. Conclusion

Models with impatient customers in queues have been studied by various authors in the past, where the source of impatience has always been taken to be either a long wait already been experienced upon arrival at a queue, or a long wait anticipated by a customer upon arrival. In this paper we have studied impatient customers in an Markovian queue with Bernoulli schedule working vacation interruption and setup time where the customers' impatience is due to a slow service rate in a working vacation. We have derived probability generating functions of the number of customers in the system when the server is in a normal service period, Bernoulli schedule vacation interruption and in a setup period respectively. Various performance measures such as the mean system size, and the mean sojourn time of a customer served are derived. Meanwhile, the stochastic decomposition structures of the queue length and waiting time are also found. Finally, some numerical results to show the impact of model parameters on performance measures of the system are presented. The effects of some parameters on the performance measures of the system have been investigated numerically.

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