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On Nearly Kähler Finsler Spaces

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Abstract

Ichijyō introduced (a, b, J)-manifolds as a special class of generalized Randers manifolds. We introduce generalized (a, b, J)-manifolds. A partial negative answer to Ichijyō's open problem on nearly Kähler Finsler manifolds is given. The condition under which generalized (a, b, J)-manifolds are Berwaldian is obtained. Finally, we prove that under a mild assumption a nearly Kähler Finsler manifold is Landsbergian.

Keywords: Rizza manifold,(a, b, J)-metric; Nearly Kählerian Finsler manifold; Projective equivalent; Berwald metric; Landsberg metric

MSC 2010 No.: 53B40, 53B35

1. Introduction

Riemannian geometry has many important subcategories, such as Hermitian geometry, including Kähler geometry. The first discussions and analyses of Riemannian and semi-Riemannian nearly Kähler manifolds emerged during the 1970s with Alfred Gray's works. Gray and Hervella classified almost Hermitian manifolds in 16 classes in 1976 (Gray and Hervella (1980)), one of these classes, known as nearly Kähler manifolds, attracted attention. In his later papers (Gray (1970); Gray (1972); Gray (1976)), Gray studied these manifolds in great detail and surveyed some of the topological characteristics of these manifolds. In (Gray (1972)) he explored the relationship between the nearly Kähler manifolds and the 3-symmetric spaces and proposed a basic classification theorem and provided a decomposition for the nearly Kähler manifolds to the Kähler and strictly nearly Kähler manifolds, which had nice characteristics, such as Einsteinism and the vanishing of the first Chern class (Gray (1976)) (cf. Nagy's papers (Nagy)). It was Grunewald who proved that in dimension 6, nearly Kähler manifolds are related to the existence of a Killing spinor (Grunewald). Relation and position of nearly Kähler manifolds in string theory were revealed by Friedrich and Ivanove's papers (Friedrich and Ivanov (2002)). In 2002, they expressed that nearly Kähler manifolds admit a Hermitian connection with totally skew-symmetric torsion.

The study of homogeneous nearly Kähler manifolds began with Buteruille (Butruille (2010)). He classified six-dimensional complete homogeneous nearly Kähler manifolds and responded positively to the Gray-Wolf conjecture that every homogeneous nearly Kähler manifold is a Riemannian 3-symmetric space. According to the Butruille, only the 6-dimensional, complete, homogeneous and simply connected strictly nearly Kähler manifolds examples are $S^3 \times S^3$, the complex projective space $\mathbb{C}P^3$, the flag manifold $\mathbb{F}^3$, and the sphere S^6, all of which are 3-symmetric.

Finsler geometry, a natural generalization of Riemannian geometry, has been considered noticeable in recent years particularly for its applications in physics, biology, etc. Randers metrics $F = \alpha + \beta$, where α is a Riemannian metric and β is a 1-form, are the first and easiest examples that come to the mind from the Finslerian manifolds, which are a special case of (α, β)- metrics, i.e., $F = \alpha\varphi(\frac{\beta}{\alpha})$. Most of the theorems in Finsler geometry were first investigated for Randers metrics, and possibly later they were probed for a more general case. Recently, researchers have shown an increased interest in Finsler spaces equipped with complex structures. Several attempts have been made to illuminate nature of these spaces (Fukui; Heil; Prakashi). In complex Finsler spaces quite a few efforts had been done to introduce Hermitian manifolds and extend this concept to Finslerian space, but after encountering a few failures, it was understood that this is not a light task. The correct generalization of Hermitian manifolds in Finslerian setting was done by Rizza (Rizza (1962); Rizza (1963)) and we call an almost Hermitian Finsler manifold, a Rizza manifold.

Considering Finsler metric's role in physics and biology and on the other hand the role of nearly Kähler Riemannian manifolds in theoretical physics, it seems that study of nearly Kähler manifolds in Finslerian case may lead to a new approach and understanding of both mathematics and physics phenomenon. The systematic study of Kählerian Finsler manifold was done by Ichijyō. He introduced a special class of Rizza manifolds and named them (a, b, J)- manifolds, which are special generalized Randers metrics, and by contributing of these manifolds, he demonstrated examples of Kählerian Finsler manifold (Ichijyō (1988)-Ichijyō and Hashiguchi (1995b)). Preliminary work on nearly Kähler Finsler manifolds was undertaken by Ichijyō. By presenting concepts of normal and nearly normal manifolds, Ichijy$\bar{o}$ in (Ichijyō and Hashiguchi (1995b)) made an example of nearly Kähler Finsler manifolds. H-S Park and H-T Lee studied nearly Kähler Finsler structures as well and extended some theorems which was obtained by Ichijyō for Kählerian Finsler manifolds to nearly Kähler Finsler manifolds under some additional conditions (Park (1993); Park and Lee (1993)).

In the purpose of achieving more instances of nearly Kähler Finsler spaces, Ichijy$\bar{o}$ questioned that with having a nearly Kähler Riemannian manifold (M, α, J), can we find a non-zero 1-form b on M such that the corresponding (a, b, J)-manifold becomes a nearly Kählerian Finsler manifold? This paper comes out with a negative answer, in nearly normal case, to Ichijy$\bar{o}$'s question (see Proposition 3.3).

Two Finsler metrics are said to be projectively equivalent if they have the same geodesics as point sets. Hilbert's fourth problem deals with projectively flat metrics. Here, a situation under which an (a, b, J)-metric and generalized (a, b, J)-metric are projectively equivalent to their Riemannian parts is attained at Proposition 4.11 and Proposition 4.12, respectively.

Non-Riemannian curvatures such as Berwald and Landsberg curvatures have a major position in Finsler geometry. It is known that Berwlad metrics are nearest Finsler metrics to Riemannian ones. We characterize Berwaldian generalized (a, b, J)-manifolds (see Theorem 4.8). In (Ichijyō (1994)), Ichijy$\bar{o}$ showed that a Kählerian Finsler manifold is Landsbergian. We also acquire the condition under which the same thing holds in nearly Kähler case (see Theorem 4.14).

2. Preliminaries

Let M be an n-dimensional C^∞ manifold. Denote by T_xM the tangent space at $x \in M$, and by $TM = \cup_{x\in M} T_xM$ the tangent bundle of M. Let $\widetilde{TM}$ be $TM - \{0\}$, i.e., the slit tangent bundle of M and let $\rho : \widetilde{TM} \to M$ be the natural projection. A *Finsler metric* on M is a function $F : TM \to [0, \infty)$ which has the following properties: (i) F is C^∞ on slit tangent bundle, (ii) F is positively 1-homogeneous on the fibers of tangent bundle TM, and (iii) for each $y \in T_xM$, the following quadratic form g_y on T_xM is positive definite,

$$g_y(u, v) := \frac{1}{2}\frac{\partial^2}{\partial s \partial t}\left[F^2(y + su + tv)\right]|_{s,t=0}, \ \ u, v \in T_xM.$$

The geodesic curves of a Finsler metric F on a smooth manifold M are determined by the system of 2ed order differential equations $\ddot{c}^i + 2G^i(\dot{c}) = 0$, where the local functions $G^i = G^i(x, y)$ are called the spray coefficients and defined by

$$G^i = \frac{1}{4}g^{il}\left\{\frac{\partial^2 F^2}{\partial x^k \partial y^l} - \frac{\partial F^2}{\partial x^l}\right\}, \tag{1}$$

where $g_{ij} := \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}$ and $(g^{ij}) = (g_{ij})^{-1}$. F is called a Berwald metric, if G^i are quadratic in $y \in T_xM$ for any $x \in M$, and called a Douglas metric if $G^i = \frac{1}{2}\Gamma^i_{jk}(x)y^j y^k + P(x, y)y^i$(Bácsó and Matsumoto (1997); Najafi et al. (2007)). Let $L_{jkl} := -\frac{1}{2}F\frac{\partial F}{\partial y^i}B^i_{jkl}$, where $B^i_{jkl} := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}$ is the Berwald curvature of F. The Landsberg curvature $\mathbf{L}_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ is defined by $\mathbf{L}_y(u,v,w) := L_{ijk}(y)u^iv^jw^k$. A Finsler metric is called a Landsberg metric if $\mathbf{L} = 0$. It is easy to see that every Berwald metric is a Landsberg metric, but the converse is still an open problem in Finsler geometry. On a Berwald manifold (M, F), all tangent spaces T_xM with the induced Minkowski norm F_x are linearly isometric. Moreover, F is affinely equivalent to a Riemannian metric g, namely, F and g have the same spray. Using this fact, S. Zabó determined the local structure of Berwald metrics (Chern and Shen (2004)). Let $x \in M$ and $F_x := F|_{T_xM}$. Putting $G^i_j = \frac{\partial G^i}{\partial y^j}$, the Cartan connection of the Finsler metric F is denoted by $CF = (F^i_{jk}, G^i_j, C^i_{jk})$ and defined by the following,

$$F^i_{jk} = \frac{1}{2}g^{ir}(\delta_k g_{jr} + \delta_j g_{rk} - \delta_r g_{kj}),\ C^i_{jk} = \frac{1}{2}g^{ir}\left(\frac{\partial g_{jr}}{\partial y^k} + \frac{\partial g_{rk}}{\partial y^j} - \frac{\partial g_{kj}}{\partial y^r}\right), \tag{2}$$

where $\delta_k = \frac{\partial}{\partial x^k} - G^i_k\frac{\partial}{\partial y^i}$. Indeed, $C^i_{jk} = g^{ir}C_{rjk}$, where $C_{ijk} = \frac{1}{2}\frac{\partial g_{ij}}{\partial y^k}$ is the Cartan tensor of F, which measures the non-Euclidean feature of F. It is well known that $\mathbf{C} = 0$ if and only if F is Riemannian. For $y \in T_xM_0$, define the mean Cartan torsion $\mathbf{I}_y$ by $\mathbf{I}_y(u) := I_i(y)u^i$, where $I_i := g^{jk}C_{ijk}$. By Deicke Theorem, F is Riemannian if and only if $\mathbf{I} = 0$.

For any Finsler tensor $S^i_j(x,y)$, the h-covariant and v-covariant derivatives with respect to $C\Gamma$ are defined as follows, respectively

$$S^i_{j|k} = \frac{\delta S^i_j}{\delta x^k} + S^m_j\Gamma^i_{mk} - S^i_m\Gamma^m_{jk}, \tag{3}$$

$$S^i_{j}\Big|_k = \frac{\partial S^i_j}{\partial y^k} + S^m_jC^i_{mk} - S^i_mC^m_{jk}. \tag{4}$$

Putting $G^i_{jk} = \frac{\partial G^i_j}{\partial y^k}$, it is well known that F^i_{jk} and G^i_{jk} are positively homogeneous functions of degree 0 with respect to y and the relation $G^i_j = F^i_{jk}y^k$ holds. Moreover, we have this important identity $F^i_{jk} = G^i_{jk} - L^i_{jk}$, where $L^i_{jk} = g^{it}L_{tjk}$. Unlike Riemannian geometry, there are more than one Finsler connection associated to any Finsler metric. The Berwald connection of the Finsler metric F is defined by $B\Gamma = (G^i_{jk}, G^i_j, 0)$. For any Finsler tensor $S^i_j(x,y)$, the h-covariant derivative with respect to $B\Gamma$ is denoted and defined as follows (Matsumoto (1986))

$$S^i_{j;k} = \frac{\delta S^i_j}{\delta x^k} + S^m_jG^i_{mk} - S^i_mG^m_{jk}. \tag{5}$$

Definition 2.1.

Two Finsler metrics F and $\bar{F}$ on a manifold M are said to be projectively equivalent if they have the same geodesics as point sets. More precisely, for any geodesic $\bar{\sigma}(t)$ of $\bar{F}$, after an appropriate oriented reparametrization, $\bar{t} = \bar{t}(t)$, the new map $\sigma(t) := \bar{\sigma}(\bar{t}(t))$ is a geodesic of F, and vice versa.

Definition 2.2.

A Finsler metric $F = F(x,y)$ on an open subset $\mathcal{U} \in \mathbb{R}^n$ is said to be projectively flat if all geodesics are straight in $\mathcal{U}$, i.e., $a(t) = f(t)a + b$ for some constant vectors $a, b \in \mathbb{R}^n$. A Finsler metric F on a manifold M is said to be locally projectively flat if at any point, there is a local coordinate system (x^i) in which F is projectively flat.

Let us recall Rapcsak's characterization of two projectively related Finsler metrics.

Theorem 2.3. (Rapcsak's Theorem (Chern and Shen (2004)))

Let F and $\bar{F}$ be Finsler metrics on a manifold M. F is projectively equivalent to $\bar{F}$ if and only if F satisfies the following system,

$$F_{;k.l}y^k - F_{;l} = 0, \tag{6}$$

in which case, their spray coefficients are related by $G^i = \bar{G}^i + Py^i$ where $P = \frac{F_{;k}}{y}^k 2F$. Here $F_{;k}$ denote the horizontal covariant of F with respect to the Berwald connection of $\bar{F}$ and $F_{;k.l} = (F_{;k})_{y^l}$.

Let M be a $2n$-dimensional manifold. A $(1,1)$-tensor field $J = J^i{}_j dx^j \otimes \frac{\partial}{\partial x^i}$ on M is called an almost complex structure on M if $J \circ J = -I_{TM}$, where I_{TM} is the identity map of TM. A Riemannian metric h is called compatible with the almost complex structure J, if at each point $x \in M$ the endomorphism J_x preserves the inner product h_x induced from h on T_xM. It means that for all tangent vectors $u, v \in T_xM$ $h_x(J_x(u), J_x(v)) = h_x(u, v)$. In this case, the triple (M, J, h) is called an almost Hermitian manifold. The Nijenhuis tensor corresponding to J is defined as follows:

$$N_J(X,Y) = [JX, JY] - J\big([X, JY] + [JX, Y]\big) - [X, Y], \tag{7}$$

where X and Y are any two arbitrary vector fields on M. The almost complex structure is called integrable if N_J vanishes. In this case, manifold is complex and the almost Hermitian manifold (M, J, h) is Hermitian manifold.

Finsler geometry is a natural extension of Riemannian geometry. Hence, it is natural to extend the compatibility of almost complex structure from Riemannian setting to Finslerian one.

Let (M, F) be a Finsler manifold. At each point $x \in M$, the Finsler metric F induces a Minkowski norm on T_xM, instead of inner product. It is known that F makes the pull-back bundle p^*TM to a Riemmanian bundle over $\widetilde{TM}$. The fiber of p^*TM at a point $(x, y) \in \widetilde{TM}$ is just $\{(x, y)\} \times T_xM$. Hence, every vector field X on M is naturally lifted to a section of the pull-back bundle p^*TM by setting $\tilde{X}(x, y) = (x, y, X(x))$. Similarly, any almost complex structure J on M is lifted to an endomorphism of p^*TM as follows $\tilde{J}_{(x,y)}(x, y, u) := (x, y, J_x(u))$. Noticing these facts, N. Prakash tried to extend the compatibility of an almost complex structure J on a manifold M with a Finsler metric F on M as follows,

$$g(\tilde{J}\tilde{X}, \tilde{J}\tilde{Y}) = g(\tilde{X}, \tilde{Y}), \tag{8}$$

where g is the Riemannian metric on the pull-back bundle p^*TM which is constructed from F, and X, Y are any two vector fields on M. Later on, E. Heil and Y. Ichijy$\bar{o}$ showed that (8) implies that F is Riemannian (Fukui (1989)). In the next step, generalized Finsler metrics on a manifold M were considered. Every Riemannian structure on the pull-back bundle p^*TM is called a generalized

Finsler metric on M. It is obvious that every Finsler metric F induces a generalized Finsler metric, but the converse is not true. Let $J = J^i{}_j dx^j \otimes \frac{\partial}{\partial x^i}$ and F be an almost complex structure and Finsler metric on a manifold M, respectively. Suppose that $g_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}$ is the fundamental tensor associated to F. Then, the Moor metric associated to J and F is a generalized Finsler metric on M and defined by the following,

$$\tilde{G}_{ij} = \frac{1}{2}(g_{ij} + g_{pq}J_i^p J_j^q). \tag{9}$$

It is seen that $\tilde{G}_{ij} = \tilde{G}_{pq}J_i^p J_j^q$. This means that the Moor metric associated to J and F is compatible with the complex structure J, while F is not necessary a Riemannian metric.

Furthermore, another stream of thoughts appeared in Rizza's mind. Once more, let $J = J^i{}_j dx^j \otimes \frac{\partial}{\partial x^i}$ and F be an almost complex structure and Finsler metric on a manifold M, respectively. He asked himself: "*How one can make any Minkowski space (T_xM, F_x) to a complex Banach space?*" Then he proposed the following compatibility between J and F,

$$F(x, y\cos\theta + J_x(y)\sin\theta) = F(x, y), \qquad \forall\theta \in \mathbb{R}, \forall y \in T_xM. \tag{10}$$

Ichijy$\bar{o}$ baptized such Finsler manifolds as Rizza manifolds (Ichijyō (1988)). If F is Riemannian, then (M, F, J) is a Rizza manifold if and only if it is an almost Hermitian manifold. Thus, Rizza manifolds are natural extension of almost Hermitian manifolds. Some equivalent conditions to (10) were obtained as follows

(1) $g_{ij}J_k^i y^k y^j = 0$,
(2) $g_{im}J_j^m + g_{jm}J_i^m + 2C_{ijm}J_r^m y^r = 0$.

Y. Ichijy$\bar{o}$ and M. Hashiguchi gave an important class of non-Riemannian Rizza manifolds, namely (a, b, J)-manifolds (Ichijyō and Hashiguchi (1995a)). Let (M, α, J) be a $2n$-dimensional almost Hermitian manifold. For a non-vanishing 1-form $b_i(x)$ on M, we have a symmetric quadratic form

$$\beta(x, y) = (b_{ij}(x)y^i y^j)^{\frac{1}{2}}, \tag{11}$$

where $b_{ij} = b_i b_j + J_i J_j$, $J_i = b_r J_i^r$. Indeed, J_i are the local component of the 1-form $b \circ J$. Now, it is easy to see that the Finsler metric $F = \alpha + \beta$ is a typical example of Rizza manifolds (Ichijyō (1988)). In this case, (M, F, J) is called an (a, b, J)-manifold (Ichijyō and Hashiguchi (1995a)). An (a, b, J)-manifold is called normal if it satisfies

$$\nabla_k b_i = 0, \nabla_k J_j^i = 0. \tag{12}$$

Two 1-forms b_i and J_i given on a Riemannian manifold (M, α) are called cross-recurrent if there exist a 1-form λ_k satisfying

$$\nabla_k b_i = \lambda_k J_i, \ \nabla_k J_i = -\lambda_k b_i, \tag{13}$$

where ∇ is the Levi-Civita connection of α (for more details see (Ichijyō and Hashiguchi (1995a))). An (a, b, J)-manifold is called nearly normal if b_i and J_i are cross-recurrent and the following holds

$$\nabla_k J_j^i + \nabla_j J_k^i = 0. \tag{14}$$

We define the kernel of the symmetric quadratic form β at $x \in M$ as follows,

$$Ker(\beta) = \{y \in T_xM|\ b_{ij}(x)y^iu^j = 0\ \forall u \in T_xM\}.$$

Note that an (a, b, J)-manifold $F = \alpha + \beta$ is singular at every tangent vector y which is in the kernel of β. In Lee ((2003)), N. Lee has determined the maximal domain $\mathcal{D}$ of $\widetilde{TM}$, on which an (a, b, J)-manifold is regular. Then, N. Lee computed the fundamental tensor and Cartan tensor of $F = \alpha + \beta$,

$$g_{ij} = \frac{F}{\alpha}a_{ij} + \frac{F}{\beta}b_ib_j + \frac{F}{\beta}J_iJ_j + F_iF_j - \frac{F}{\alpha}\alpha_i\alpha_j - \frac{L}{\beta}\beta_i\beta_j,$$

where $\alpha_i = \frac{\partial\alpha}{\partial y^i}, \beta_i = \frac{\partial\beta}{\partial y^i}$ and $F_i = \alpha_i + \beta_i$,

$$C_{ijk} := \frac{1}{4}(F^2)_{y^iy^jy^k} = \frac{1}{2}(g_{ij})_{y^k} = \frac{1}{2}\sigma_{(i,j,k)}\{(\frac{\alpha_{ij}}{\alpha} - \frac{\beta_{ij}}{\beta})(\alpha\beta_k - \beta\alpha_k)\}.$$

The notation $\sigma_{(i,j,k)}$ denotes the summation of the cyclic permutation of indices i, j, k.

In Ichijyō and Hashiguchi (1995a), Ichijy$\bar{o}$ studied geometric properties of (a, b, J)-metrics and found a condition under which an (a, b, J)-metric is Berwaldian.

Theorem 2.4.

An (a, b, J)-manifold $(M, \alpha + \beta)$ is a Berwald space if and only if $\nabla_k b_{ij} = 0$, where ∇ is the Levi-Civita connection of α.

A Rizza manifold (M, F, J) is called a Kählerian Finsler manifold if it satisfies the following

$$J^i_{\ j|k} = 0,$$

where "|" denotes the h-covariant differentiation with respect to the Cartan connection of F. As an important class of Kählerian Finsler manifolds, we have the class of normal (a, b, J)-manifolds (Ichijyō and Hashiguchi (1995a)).

Let us consider an almost Hermitian manifold (M, h, J) with the Levi-Civita connection ∇. As an object of particular importance for nearly Kähler (M, h, J), we have the canonical Hermitian connection defined by

$$\bar{\nabla}_XY = \nabla_XY + \frac{1}{2}(\nabla_XJ)JY.$$

We know that $\bar{\nabla}$ is the unique Hermitian connection with totally skew-symmetric torsion given by

$$T(X, Y) = (\nabla_XJ)JY. \tag{15}$$

It is known that T vanishes if and only if (M, h, J) is a Kähler manifold (Nagy (2002b)).

A Rizza manifold (M, L, J) is called a nearly Kählerian Finsler manifold if the following holds

$$J^i_{\ j|k} + J^i_{\ k|j} = 0$$

If a nearly Kählerian Finsler manifold is Riemannian, then it is also called Tachibana manifold or a K-space (Ichijyō and Hashiguchi (1995b); Park and Lee (1993)). It was proved that every nearly normal (a, b, J)-manifold is a nearly Kählerian Finsler manifold (Ichijyō and Hashiguchi (1995b)).

3. Non-existence theorem on nearly normal (a, b, J)-manifold

Let us investigate some severe consequences of (13). We define the norm of 1-form $b = b_i(x)dx^i$ on a manifold M with respect to a Riemannain metric $\alpha = a_{ij}(x)dx^i \otimes dx^j$ by $||b||_\alpha := \sqrt{b^i b_i}$, where $b^i = a^{ir}b_r$. On a nearly normal (a, b, J)-manifold, contracting $\nabla_k b_i = \lambda_k J_i$ with b^i and using $b^i J_i = 0$, one gets $b^i \nabla_k b_i = 0$. Thus the norm of b with respect to α is a constant function, since $\nabla_k ||b||^2_\alpha = 2b^i \nabla_k b_i$.

Now, suppose that b and $b \circ J$ are cross-recurrent, i.e., for any vector field X and Y on M we have

$$(\nabla_X b)Y = \nabla_X(bY) - b(\nabla_X Y) = \lambda(X) b \circ J(Y), \tag{16}$$

$$(\nabla_X b \circ J)Y = \nabla_X(b \circ J)(Y) - b \circ J(\nabla_X Y) = -\lambda(X)b(Y). \tag{17}$$

Putting $J(Y)$ in palce of Y in (17) and using $J \circ J = -Id$, we get

$$-(\nabla_X b \circ J)J(Y) = \nabla_X(bY) + b \circ J(\nabla_X J(Y)) = \lambda(X)b(J(Y)). \tag{18}$$

Comparing (16) and (18), we obtain

$$b\big(T(X, Y)\big) = b\big((\nabla_X J)JY\big) = -b\big(J(\nabla_X J(Y) + \nabla_X Y\big) = 0, \tag{19}$$

which says $b \circ T = 0$. Summarizing up, we get the following.

Proposition 3.1.

If $(M, \alpha + \beta)$ is a nearly normal (a, b, J)-manifold, then the norm of the 1-form b with respect to α is a constant function on M. Moreover, b annihilates the torsion T, i.e., $b(T(X, Y)) = 0$ for any two vector fields X and Y on M, where T is given by (15).

In Ichijyō and Hashiguchi (1995a), Ichijyō proposed a problem on nearly Kähler Finsler manifolds:

"*Let (M, α, J) be a nearly Kählerian Riemannian manifold. Can we find a non-zero 1-form b on M such that the corresponding (a, b, J)-manifold $(M, \alpha + \beta)$ becomes a nearly Kählerian Finsler manifold?*"

We know that every nearly normal (a, b, J)-manifold is a nearly Kählerian Finsler manifold. So, it is natural to propose the following question:

"*Let* (M, α, J) *be a nearly Kählerian Riemannian manifold. Can we find a non-zero 1-form* b *on* M *such that the corresponding* (a, b, J)*-manifold* $(M, \alpha+\beta)$ *becomes a nearly normal* (a, b, J)*-manifold?*"

Let us consider the standard unit 6-dimensional sphere S^6 with its almost complex structure. We consider S^6 as a totally umbilical submanifold of the 7-dimensional Euclidean space $\mathbb{R}^7$. It is well known that the algebraic structure of octonions induces a so-called cross product "$\times$" on $\mathbb{R}^7$ (Gray (1966)). Choosing a unit normal vector field N on S^6, we define the canonical almost complex structure on S^6 by $J(X) := N \times X$. The following relation is a useful one,

$$\begin{aligned}(\nabla_X J)(Y) &= \nabla_X J(Y) - J(\nabla_X Y) = \nabla_X(N \times Y) - N \times \nabla_X Y \\ &= \nabla_X N \times Y + N \times \nabla_X Y - N \times \nabla_X Y \\ &= \nabla_X N \times Y \\ &= X \times Y,\end{aligned}$$

where we have used totally umbilicness of S^6 in $\mathbb{R}^7$. As a consequence, we get (14). Therefore S^6 is a nearly Kähler manifold. Now, we seek a desired 1-form b such that b and $b \circ J$ are cross-recurrent. Proposition 3.1 infers that b must be of constant length. On the other hand, every vector field on S^6 is zero at some point of S^6. Thus, we conclude that $b = 0$. Therefore, the only 1-form b that makes S^6 as a nearly normal (a, b, J)-manifold is null 1-form. Hence, we give a negative answer to Ichijy$\bar{o}$'s question, in the case of nearly normal.

Indeed, we can generalize above observation to any orientable compact manifold M with non-zero Euler characteristic. Hopf's well-known theorem states that in this case, every vector field on such a manifold vanishes at some point of the manifold. Due to the constancy of the length of b, the desired 1-form b must be zero, which is a contradiction.

The only four complete, homogeneous and simply connected examples of strictly nearly Kähler manifolds in dimension 6 are $S^3 \times S^3$, the complex projective space $\mathbb{C}P^3$, the flag manifold F^3 and the sphere S^6 (Butruille(2010)). Thus, all of theme are compact with (except $S^3 \times S^3$) non-zero Euler characteristics.

For the case $S^3 \times S^3$ we use the nearly Kähler structure which has been used and described in (Bolton et al. (2015)). As they considered the 3-sphere in $\mathbb{R}^4$ as the set of all unit quaternions. The vector fields X_1, X_2 and X_3 given by

$$\begin{aligned}X_1(p) &= pi = -x_2 + x_1 i + x_4 j - x_3 k, \\ X_2(p) &= pj = -x_3 - x_4 i + x_1 j + x_2 k, \\ X_3(p) &= pk = x_4 - x_3 i + x_2 j - x_1 k,\end{aligned}$$

at the point $p = x_1 + x_2 i + x_3 j + x_4 k$ form a basis of tangent vector fields. Thus a tangent vector in T_pS^3 can be expressed as $p\alpha$ where α is an imaginary quaternion. Using the quaternion relations $ij = k, jk = i$ and $ki = j$ one shows that the Lie brackets are given by $[X_i, X_j] = -2\varepsilon_{ijk}X_k$. Based on their notations ε_{ijk} is the Levi-Civita symbol. Using the natural identification $T(p,q)(S^3 \times S^3) \cong T_pS^3 \oplus T_qS^3$, they wrote a tangent vector at (p, q) as

$Z(p,q) = \Big(U(p,q), V(p,q)\Big)$ or simply $Z = (U, V)$. They defined the vector fields

$$\begin{aligned} E_1(p,q) &= (pi, 0), & F_1(p,q) &= (0, qi),\\ E_2(p,q) &= (pj, 0), & F_2(p,q) &= (0, qj),\\ E_3(p,q) &= -(pk, 0), & F_3(p,q) &= -(0, qk). \end{aligned}$$

These vector fields are mutually orthogonal with respect to the usual product metric on $S^3 \times S^3$. The Lie brackets are $[E_i, E_j] = -2\varepsilon_{ijk}E_k, [F_i, F_j] = -2\varepsilon_{ijk}F_k$ and $[E_i, F_j] = 0$. The almost complex structure J on $S^3 \times S^3$ is defined as $JZ(p,q) = \frac{1}{\sqrt{3}}\Big(2pq^{-1}VU, 2qp^{-1}U + V\Big)$ for $Z \in T(p,q)(S^3 \times S^3)$ (see (Butruille (2010))). We have

$$\begin{aligned} JE_i &= -\frac{1}{\sqrt{3}}(E_i + 2F_i),\\ JF_i &= \frac{1}{\sqrt{3}}(2E_i + F_i). \end{aligned}$$

Furthermore, they defined another metric g on $S^3 \times S^3$ by

$$\begin{aligned} g(Z, Z') &= \frac{1}{2}(\langle Z, Z'\rangle + \langle JZ, JZ'\rangle)\\ &= \frac{4}{3}(\langle U, U'\rangle + \langle V, V'\rangle) - \frac{2}{3}(\langle p^{-1}U, q^{-1}V'\rangle + \langle p^{-1}U', q^{-1}V\rangle), \end{aligned}$$

where $Z = (U, V), Z' = (U', V')$ and $\langle ., .\rangle$ is the product metric on $S^3 \times S^3$. By definition the almost complex structure is compatible with the metric g. Therefore one should obtain $g(E_i, E_j) = \frac{4}{3}\delta_{ij}, g(E_i, F_j) = -\frac{2}{3}\delta_{ij}$ and $g(F_i, F_j) = \frac{4}{3}\delta_{ij}$. Note that this metric differs up to a constant factor from the one introduced in (Butruille (2010)). They set everything up so that it equals the Hermitian metric associated with the usual metric. In (Butruille (2010)), the factor was chosen in such a way that the standard basis $E_1, E_2, E_3, F_1, F_2, F_3$ has volume 1. Using the Koszul formula, they express the following lemma.

Lemma 3.2. (Bolton et al. (2015))

The Levi-Civita connection ∇ on $S^3 \times S^3$ with respect to the metric g

$$\begin{aligned} \nabla_{E_i}E_j &= -\varepsilon_{ijk}E_k, & \nabla_{E_i}F_j &= \frac{\varepsilon_{ijk}}{3}(E_k - F_k),\\ \nabla_{F_i}E_j &= \frac{\varepsilon_{ijk}}{3}(F_k - E_k), & \nabla_{F_i}F_j &= -\varepsilon_{ijk}F_k. \end{aligned}$$

Then, the following has been obtained

$$\begin{aligned} (\nabla_{E_i}J)E_j &= -\frac{2}{3\sqrt{3}}\varepsilon_{ijk}(E_k + 2F_k), & (\nabla_{E_i}J)F_j &= -\frac{2}{3\sqrt{3}}\varepsilon_{ijk}(E_k - F_k),\\ (\nabla_{F_i}J)E_j &= -\frac{2}{3\sqrt{3}}\varepsilon_{ijk}(E_k - F_k), & (\nabla_{F_i}J)F_j &= -\frac{2}{3\sqrt{3}}\varepsilon_{ijk}(2E_k + F_k). \end{aligned}$$

As mentioned the torsion of Hermitian connection given by $T(X,Y) = (\nabla_X J)JY$ and according to relation (19) we have $boT = 0$. We compute torsion for different vector fields as follows.

$$\begin{aligned} T(E_i, E_j) &= -\frac{2}{3}\varepsilon_{ijk}E_k, \\ T(F_i, F_j) &= \frac{2}{9}\varepsilon_{ijk}(F_k - 4E_k), \\ T(F_i, E_j) &= -\frac{2}{9}\varepsilon_{ijk}(5E_k + F_3), \\ T(E_i, F_j) &= -\frac{2}{3}\varepsilon_{ijk}(E_k + F_k). \end{aligned} \tag{20}$$

Suppose that $\{E^i, F^i\}$ are dual of $\{E_i, F_i\}$. Let us consider a 1-form $b = b_1E^1 + b_2E^2 + b_3E^3 + b_4F^1 + b_5F^2 + b_6F^3$ making $S^3 \times S^3$ to a nearly normal (a, b, J)-manifold. Thus, we have

$$\begin{aligned} &bT(E_2, E_1) = -\frac{2}{3}b_3, && bT(E_2, E_3) = \frac{2}{3}b_1, && bT(E_3, E_1) = \frac{2}{3}b_2, \\ &bT(F_2, E_3) = \frac{10}{9}b_1 + \frac{2}{9}b_4, && bT(F_3, E_1) = \frac{10}{9}b_2 + \frac{2}{9}b_5, && bT(F_2, E_1) = -\frac{10}{9} - \frac{2}{9}b_6. \end{aligned} \tag{21}$$

Therefore b_i's should be zero, which one concludes that $b = 0$. Hence, the only 1-form b that makes $S^3 \times S^3$ as a nearly normal (a, b, J)-manifold is null 1-form. Finally, we express the following proposition.

Proposition 3.3.

Any complete homogeneous strictly 6-dimensional nearly Kähler manifold does not admit nearly normal (a, b, J)-structure.

It may be reasonable to consider non-comapct (nearly) Kähler manifolds. In the sequel, we are going to show that on every even dimensional Euclidean space $\mathbb{R}^{2n}$ with its canonical almost complex structure J and Euclidean metric α, one can always find the desired 1-form b making $\mathbb{R}^{2n}$ as a Kählerian Finsler (a, b, J)-manifold. Let us denote the Cartesian coordinates of $\mathbb{R}^{2n}$ by $(x^1, y^1...., x^n, y^n)$. Then J is given by

$$J\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial y^i}, \ J\left(\frac{\partial}{\partial y^i}\right) = -\frac{\partial}{\partial x^i}, \ \ i = 1, ..., n. \tag{22}$$

It is easy to see that $(\mathbb{R}^{2n}, \alpha, J)$ is a Kählerian manifold. For simplicity, let us denote the 1-form b by its components in Cartesian coordinates $b = (b_1, b_2, ..., b_{2n-1}, b_{2n})$. Thus, the components of the 1-form $b \circ J$ are given by $b \circ J = (-b_2, b_1, ..., -b_{2n}, b_{2n-1})$. Suppose that $\lambda = (\lambda_1, \lambda_2, ..., \lambda_{2n-1}, \lambda_{2n})$ is a constant vector in $\mathbb{R}^{2n}$ whose components are positive. If we put $u = (u^1, ..., u^{2n}) = (x^1, y^1...., x^n, y^n)$, then $\nabla_k b_i = \lambda_k J_i, \ \nabla_k J_i = -\lambda_k b_i$ if and only if b_i satisfies the following second order linear PDEs system,

$$\frac{\partial^2 b_i}{\partial u^j \partial u^k} + \lambda_j \lambda_k b_i = 0, \quad i, j, k = 1, 2, ..., 2n. \tag{23}$$

One can see that for every point $P = (A_1, B_1 ...., A_n, B_n) \in \mathbb{R}^{2n}$ the following 1-form b is a solution of (23),

$$b_{2k-1} = A_{2k-1}\cos(<\lambda, u>) + B_{2k-1}\sin(<\lambda, u>),$$
$$b_{2k} = A_{2k}\cos(<\lambda, u>) - B_{2k}\sin(<\lambda, u>), \qquad k = 1, ..., n, \tag{24}$$

where $<,>$ is the Euclidean inner product on $\mathbb{R}^{2n}$. Moreover, the norm of b with respect to α is the Euclidean norm of the fixed point P.

4. Generalized (a, b, J)-metrics

Let $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ be a Riemannian metric and $\beta = b_i(x)dx^i$ be a 1-form on a manifold M with $||\beta||_\alpha < 1$. Then $F = \alpha + \beta$ is called a Randers metric. Replacing the 1-form β with a symmetric quadratic form $\beta = b_{ij}dx^i \otimes dx^j$ of rank $0 \leq r < n$, we get a generalized Randers metric $F = \alpha + \beta$. Every (a, b, J)-metric is a generalized Randers metric. It is well known that the class of Randers metrics is an special case of a general class of Finsler metrics so-called (α, β)-metrics. A Finsler metric F is called an (α, β)-metric if it can be expressed as $F = \alpha\varphi(\frac{\beta}{\alpha})$, where $\varphi : (-b_0, b_0) \to \mathbb{R}$ is a positive smooth function satisfying some regularity conditions. Similarly, in order to extend the class of Rizza manifolds introduced by Ichijyō, we define generalized (a, b, J)-metrics as follows.

Definition 4.1.

Consider an (a, b, J)-metric $F = \alpha + \beta$. Let $\Psi : (-b_0, b_0) \to \mathbb{R}$ be a positive smooth function. Then, a Finsler metric in the form $F = \alpha\Psi(\frac{\beta}{\alpha})$ is called a generalized (a, b, J)-metric.

One can compute the fundamental tensor of generalized (a, b, J)-metric $F = \alpha\Psi(s)$, where $s := \frac{\beta}{\alpha}$ as follows

$$\begin{aligned} g_{ij} = & \left(\Psi^2 - s\Psi\Psi'\right) a_{ij} + \frac{1}{s}\Psi\Psi' b_{ij} - s\left(-s(\Psi\Psi'' + \Psi'\Psi') + \Psi\Psi'\right)\alpha_i\alpha_j \\ & + \left(-s(\Psi\Psi'' + \Psi'\Psi') + \Psi\Psi'\right)(\alpha_i\beta_j + \alpha_j\beta_i) \\ & + \left(-\frac{1}{s}\Psi\Psi' + \Psi\Psi'' + \Psi'\Psi'\right)\beta_i\beta_j \end{aligned} \tag{25}$$

To find the inverse of (g_{ij}), we need the following lemma.

Lemma 4.2. (Lee (2003))

Let (P_{ij}) be a real symmetric non-singular matrix with the inverse (P^{ij}). And let $(Q_{ij}) = (P_{ij} \pm c_i c_j)$ with $1 \pm c^2 \neq 0$ and $c^2 := c_i P^{ij} c_j$. Then the matrix (Q_{ij}) is non-singular and its inverse is $(Q^{ij}) = (P^{ij} \mp \frac{1}{1 \pm c^2} c^i c^j)$ where $c^i = P^{ij} c_j$ and $det(Q_{ij}) = (1 \pm c^2) det(P_{ij})$.

Proposition 4.3.

For the fundamental tensor (g_{ij}) of a generalized (a, b, J)-metric $F = \alpha\Psi(\frac{\beta}{\alpha})$, the determinant of

(g_{ij}) is given by

$$det(g_{ij}) = \Psi^{n+1}\left(\Psi - s\Psi'\right)^{n-3}\left((\Psi - s\Psi') + (b^2 - s^2)\Psi''\right)\left(\Psi + (\frac{b^2}{s} - s)\Psi'\right)det(a_{ij}).$$

Proof:

First, we set

$$g_{ij} = \rho\left\{a_{ij} + \frac{\rho_0}{\rho}b_{ij} + \frac{\rho_1}{\rho}\alpha_i\alpha_j + \frac{\rho_2}{\rho}(\alpha_i\beta_j + \alpha_j\beta_i) + \frac{\rho_3}{\rho}\beta_i\beta_j\right\},$$

where

$$\begin{aligned}
\rho &:= \Psi(\Psi - s\Psi'),\\
\rho_0 &:= \frac{1}{s}\Psi\Psi',\\
\rho_1 &:= -s(-s(\Psi\Psi'' + \Psi'\Psi') + \Psi\Psi'),\\
\rho_2 &:= -s(\Psi\Psi'' + \Psi'\Psi') + \Psi\Psi',\\
\rho_3 &:= -\frac{1}{s}\Psi\Psi' + \Psi'\Psi' + \Psi\Psi''.
\end{aligned}$$

We want to simplify the formula and use the Lemma to calculate the determinants. We have the following equalities,

$$\begin{aligned}
\frac{\rho_3 - (\frac{\rho_2}{\rho_1})^2\rho_1}{\rho}\beta_i\beta_j + \frac{\rho_1}{\rho}(\alpha_i + \frac{\rho_2}{\rho_1}\beta_i)(\alpha_j + \frac{\rho_2}{\rho_1}\beta_j) &= \frac{\rho_3}{\rho}\beta_i\beta_j - \frac{(\frac{\rho_2}{\rho_1})^2\rho_1}{\rho}\beta_i\beta_j\\
&+ \frac{\rho_1}{\rho}\alpha_i\alpha_j + \frac{\rho_1}{\rho}(\frac{\rho_2}{\rho_1})^2\beta_i\beta_j + \frac{\rho_2}{\rho}\alpha_i\beta_j + \frac{\rho_2}{\rho}\alpha_j\beta_i.
\end{aligned}$$

g_{ij} can be written as follows,

$$g_{ij} = \rho\left\{a_{ij} + \eta b_{ij} + \delta\beta_i\beta_j + \mu Y_iY_j\right\}, \tag{26}$$

where $Y_i = \alpha_i + \epsilon\beta_i$ and

$$\begin{aligned}
\delta &:= \frac{\rho_3 - \epsilon^2\rho_1}{\rho} = 0,\\
\epsilon &:= \frac{\rho_2}{\rho_1} = -\frac{1}{s},\\
\mu &:= \frac{\rho_1}{\rho} = \frac{s\left\{s(\Psi\Psi'' + \Psi'\Psi') - \Psi\Psi'\right\}}{\Psi(\Psi - s\Psi')},\\
\eta &:= \frac{\rho_0}{\rho} = \frac{\Psi'}{s(\Psi - s\Psi')}.
\end{aligned}$$

Now we check whether Equation (26) lead us to Equation (25).

$$\begin{aligned}
g_{ij} &= \Psi(\Psi - s\Psi')\Bigg\{a_{ij} + \frac{\Psi'}{s(\Psi - s\Psi')}b_{ij} + 0\beta_i\beta_j \\
&\quad + \frac{s\{s(\Psi\Psi'' + \Psi'\Psi') - \Psi\Psi'\}}{\Psi(\Psi - s\Psi')}(\alpha_i + (-\frac{1}{s}\beta_i))(\alpha_j + (-\frac{1}{s}\beta_j))\Bigg\} \\
&= \Psi(\Psi - s\Psi')a_{ij} + \frac{1}{s}\Psi\Psi' b_{ij} + s\{s(\Psi\Psi'' + \Psi'\Psi') - \Psi\Psi'\}(\alpha_i\alpha_j + \frac{1}{s^2}\beta_i\beta_j - \frac{1}{s}\alpha_i\beta_j - \frac{1}{s}\alpha_j\beta_i) \\
&= \Psi(\Psi - s\Psi')a_{ij} + \frac{1}{s}\Psi\Psi' b_{ij} + (s^2(\Psi\Psi'' + \Psi'\Psi' - s\Psi\Psi'))\alpha_i\alpha_j \\
&\quad + ((\Psi\Psi'' + \Psi'\Psi') - \frac{1}{s}\Psi\Psi')\beta_i\beta_j - \{s(\Psi\Psi'' + \Psi'\Psi') - \Psi\Psi'\}\alpha_i\beta_j \\
&\quad - \{s(\Psi\Psi'' + \Psi'\Psi') - \Psi\Psi'\}\alpha_j\beta_i,
\end{aligned}$$

and we see that it is exactly Equation (25).

First, we express some relationships that are used in the calculations: $\alpha_i = \frac{\partial\alpha}{\partial y^i}, \beta_i = \frac{\partial\beta}{\partial y^i}$. We set $\alpha^i = a^{ir}\alpha_r, \beta^i = a^{ir}\beta_r, b^i = a^{ij}b_j, J^i = a^{ij}J_j$. Hence,

$$\begin{aligned}
\alpha^i &= a^{ir}\frac{y_r}{\alpha} = \frac{y^i}{\alpha}, \\
\beta^i &= \frac{a^{ir}b_{rs}y^s}{\beta} = \frac{b^i_s y^s}{\beta}, \\
\alpha_i\alpha^i &= \frac{y_i}{\alpha}\frac{y^i}{\alpha} = \frac{\alpha^2}{\alpha^2} = 1, \\
\alpha^i\beta_i &= \frac{y^i}{\alpha}\frac{b_{ij}y^j}{\beta} = \frac{b_{ij}y^i y^j}{\alpha\beta} = \frac{\beta^2}{\alpha\beta} = \frac{\beta}{\alpha}, \\
\alpha_i\beta^i &= \frac{y_i}{\alpha}\frac{a^{ir}b_{rs}y^s}{\beta} = \frac{b_{rs}y^r y^s}{\alpha\beta} = \frac{\beta^2}{\alpha\beta} = \frac{\beta}{\alpha}.
\end{aligned}$$

Also,

$$\begin{aligned}
&J_r = b_i J^i_r, && b^{ij}b_{ij} = 2b^4, \quad b^i_j = a^{ir}b_{rj}, && b^i J_i = b_i J^i = 0, \\
&J^j J_j = b^2, && b^{ij} = a^{jr}b^i_r, \quad b^s b_s = a^{rs}b_r b_s = b^2, && b^i_j = b^i b_j + J^i J_j, \\
&y_i y^i = a_{ij}y^i y^j = \alpha^2, && a^{ij}b_{ij} = 2b^2, \quad b^{is}b_{js} = b^2 b^i_j, && b^{ij} = b^i b^j + J^i J^j,
\end{aligned}$$

$$b^{ij}\alpha_i\alpha_j = a^{jr}b^i_r\alpha_i\alpha_j = a^{jr}b^i_r\frac{y_i}{\alpha}\frac{y_j}{\alpha} = \frac{b^i_r y_i y^r}{\alpha^2} = \frac{b^i_r a_{is}y^s y^r}{\alpha^2} = \frac{b_{rs}y^r y^s}{\alpha^2} = \frac{\beta^2}{\alpha^2},$$

$$b^{ij}\alpha_i\beta_i = b^{ij}\frac{y_i}{\alpha}\frac{b_{js}y^s}{\beta} = \frac{a^{jr}b^i_r b_{js}y_i y^s}{\alpha\beta} = \frac{b^i_r b^r_s y_i y^s}{\alpha\beta} = \frac{b^2 b^i_s a_{pi}y^p y^s}{\alpha\beta} = \frac{b^2 b_{sp}y^p y^s}{\alpha\beta} = \frac{b^2\beta^2}{\alpha\beta} = \frac{b^2\beta}{\alpha},$$

$$\beta_i\beta^i = \frac{b_{ij}y^j}{\beta}\frac{a^{ir}b_{rs}y^s}{\beta} = \frac{b_{ij}b^i_s y^j y^s}{\beta^2} = \frac{b^2 b_{js}y^j y^s}{\beta^2} = \frac{b^2\beta^2}{\beta^2} = b^2,$$

$$b^{ij}\beta_i\beta_j = a^{jr}b^i_r\beta_i\beta_j = b^i_r\beta_i\beta^r = b^i_r\frac{b_{ip}y^p}{\beta}\frac{(a^{ir}b_{is}y^s = b^r_s y^s)}{\beta}$$
$$= \frac{b^2 b^i_s b_{ip}y^s y^p}{\beta^2} = \frac{b^2(b^2 b_{sp})y^s y^p}{\beta^2} = \frac{b^4\beta^2}{\beta^{\beta^2}} = b^4.$$

To calculate the determinant we use the lemma. Attaining the formula of the determinant is done in three stages.

Let's first assume $\tilde{A_{ij}} := a_{ij} + \eta b_i b_j$. The inverse of $\tilde{A_{ij}}^{-1} = \tilde{A}^{ij}$ is given as follows,

$$\tilde{c}_i = \sqrt{\eta}b_i, \quad \tilde{c}^i = \sqrt{\eta}b^i, \quad \tilde{c}^2 = \tilde{c}_i a^{ij}\tilde{c}_j = \sqrt{\eta}b_i a^{ij}\sqrt{\eta}b_j = \eta b_i b^i = \eta b^2,$$

$$\tilde{A}^{ij} = a^{ij} - \tau b^i b^j,$$

where

$$\tau := \frac{\eta}{1+\eta b^2} = \frac{\Psi'}{s\Psi + (b^2 - s^2)\Psi'},$$

and $\det(\tilde{A_{ij}}) = (1+\tilde{c}^2)\det a_{ij} = (1+\eta b^2)\det(a_{ij})$.

Now we set $A_{ij} = \tilde{A_{ij}} + \eta J_i J_j$ and again use Lemma 4.2,

$$\tilde{\tilde{c}}_i = \sqrt{\eta}J_i, \quad \tilde{\tilde{c}}^i = \sqrt{\eta}J^i, \quad \tilde{\tilde{c}}^2 = \eta J_i(\tilde{A}^{ij})J_j = \eta J_i(a^{ij} - \tau b^i b^j)J_j = \eta J_i(J^i - \tau b^i \underbrace{b^j J_j}_{0}) = \eta b^2,$$

$$A^{ij} = \tilde{A}^{ij} - \frac{\tilde{\tilde{c}}^i\tilde{\tilde{c}}^j}{1+\tilde{\tilde{c}}^2} = a^{ij} - \tau b^i b^j - \eta\frac{J^i J^j}{1+\eta b^2} = a^{ij} - \tau b^i b^j - \tau J^i J^j = a^{ij} - \tau b^{ij}.$$

Therefore, $A^{ij} = a^{ij} - \tau b^{ij}$ and

$$\det(A_{ij}) = (1+\tilde{\tilde{c}}^2)\det(\tilde{A}_{ij}) = (1+\eta b^2)^2\det(a_{ij}).$$

Next, let's put $Q_{ij} = A_{ij} + \delta\beta_i\beta_j$ and utilize Lemma 4.2,

$$\tilde{\tilde{\tilde{c}}}_i = \sqrt{\delta}\beta_i,$$
$$\tilde{\tilde{\tilde{c}}}^i = A^{ij}\tilde{\tilde{\tilde{c}}}_j = \sqrt{\delta}(a^{ij} - \tau b^{ij})\beta_j = \sqrt{\delta}(\beta^i - \tau\underbrace{b^{ij}\beta_j}_{b^2\beta^i}) = \sqrt{\delta}(1-\tau b^2)\beta^i,$$
$$\tilde{\tilde{\tilde{c}}}^2 = \tilde{\tilde{\tilde{c}}}_i\tilde{\tilde{\tilde{c}}}^i = \delta(1-\tau b^2)\beta_i\beta^i = \delta b^2(1-\tau b^2),$$

$$Q^{ij} = A^{ij} - \frac{\tilde{\tilde{\tilde{c}}}^i\tilde{\tilde{\tilde{c}}}^j}{1+\tilde{\tilde{\tilde{c}}}^2} = (a^{ij} - \tau b^{ij}) - \frac{\delta(1-\tau b^2)^2\beta^i\beta^j}{1+\delta b^2(1-\tau b^2)}.$$

We put $\sigma := \frac{\delta(1-\tau b^2)^2}{1+\delta(1-\tau b^2)b^2}$. Therefore,

$$Q^{ij} = a^{ij} - \tau b^{ij} - \sigma\beta^i\beta^j,$$

$$\det Q_{ij} = (1+\delta b^2(1-\tau b^2))(1+\eta b^2)^2\det a_{ij}.$$

For the third step, we set $P_{ij} = Q_{ij} + \mu Y_i Y_j$ and using Lemma 4.2 we get

$$\hat{c}_i = \sqrt{\mu}Y_i, \quad \hat{c}^i = \sqrt{\mu}Y^i,$$

$$\begin{aligned}\hat{c}^i &= \sqrt{\mu}Q^{ij}Y_j = \sqrt{\mu}(a^{ij} - \tau b^{ij} - \sigma\beta^i\beta^j)(\alpha_j + \epsilon\beta_j)\\ &= \sqrt{\mu}(\alpha^i - \tau \underbrace{b^{ij}\alpha_j}_{\frac{\beta}{\alpha}\beta^i} - \sigma\beta^i \underbrace{\beta^j\alpha_j}_{\frac{\beta}{\alpha}} + \epsilon\beta^i - \tau\epsilon \underbrace{b^{ij}\beta_j}_{b^2\beta^i} - \sigma\epsilon\beta^i \underbrace{\beta^j\beta_j}_{b^2})\\ &= \sqrt{\mu}(\alpha^i + \beta^i(-\tau s - \sigma s + \epsilon - \tau\epsilon b^2 - \sigma\epsilon b^2)).\end{aligned}$$

We place $\sigma' := -\tau s - \sigma s + \epsilon - \tau\epsilon b^2 - \sigma\epsilon b^2$. Then,

$$\hat{c}^i = \sqrt{\mu}(\alpha^i + \sigma'\beta^i),$$

we obtain

$$\begin{aligned}\hat{c}^2 &= \mu(\alpha_i + \epsilon\beta_i)(\alpha^i + \sigma'\beta^i) = \mu(1 + \sigma'\alpha_i\beta^i + \epsilon\alpha^i\beta_i + \epsilon\sigma'\beta_i\beta^i)\\ &= \mu(1 + \sigma'\frac{\beta}{\alpha} + \epsilon\frac{\beta}{\alpha} + \epsilon\sigma' b^2) = \mu(1 + \sigma' s + \epsilon s + \epsilon\sigma' b^2)\\ &= \mu(\sigma' s - \frac{1}{s}b^2\sigma') = \mu((s - \frac{b^2}{s})\sigma'),\end{aligned}$$

$$P^{ij} = Q^{ij} - \frac{\hat{c}^i\hat{c}^j}{1 + \hat{c}^2} = a^{ij} - \tau b^{ij} - \sigma\beta^i\beta^j - \frac{\mu(\alpha^i + \sigma'\beta^i)(\alpha^j + \sigma'\beta^j)}{1 + \mu(s - \frac{b^2}{s})\sigma'}.$$

We set $\sigma'' := \frac{\mu}{1+\mu((s-\frac{b^2}{s})\sigma')}$. Hence, by computing the following product,

$$(\alpha^i + \sigma'\beta^i)(\alpha^j + \sigma'\beta^j) = \alpha^i\alpha^j + {\sigma'}^2\beta^i\beta^j + \sigma'\alpha^i\beta^j + \sigma'\alpha^j\beta^i,$$

we gain

$$P^{ij} = a^{ij} - \tau b^{ij} - (\sigma + \sigma''{\sigma'}^2)\beta^i\beta^j - \sigma''\alpha^i\alpha^j - \sigma''\sigma'(\alpha^j\beta^i + \alpha^i\beta^j),$$

and according to Lemma 4.2 we have

$$\det P_{ij} = (1 + \hat{c}^2)\det Q_{ij} = (1 + \mu((s - \frac{b^2}{s})\sigma'))(1 + \delta b^2(1 - \tau b^2))(1 + \eta b^2)^2 \det a_{ij}.$$

Therefore, inverse of fundamental tensor of generalized (a, b, J)-metric is as follows,

$$g^{ij} = \frac{1}{\rho}P^{ij} = \rho^{-1}\{a^{ij} - \tau b^{ij} - \sigma\beta^i\beta^j - \sigma'' Y^i Y^j\}, \tag{27}$$

$$g_{ij} = \rho P_{ij}, \quad \det g_{ij} = \rho^n \det P_{ij},$$

$$\det g_{ij} = \rho^n(1 + \mu((s - \frac{b^2}{s})\sigma'))(1 + \delta b^2(1 - \tau b^2))(1 + \eta b^2)^2 \det a_{ij}. \tag{28}$$

Now we compute $\tau, \delta, \sigma', \mu$ and η in the above relation base on Ψ and then replace them in it,

$$\epsilon = -\frac{1}{s},$$

$$\mu = \frac{s\{s(\Psi\Psi'' + \Psi'\Psi') - \Psi\Psi'\}}{\Psi(\Psi - s\Psi')},$$

$$\eta = \frac{\Psi'}{s(\Psi - s\Psi')},$$

$$1 + \eta b^2 = \frac{s\Psi + (b^2 - s^2)\Psi'}{s(\Psi - s\Psi')},$$

$$\tau = \frac{\Psi'}{s\Psi + (b^2 - s^2)\Psi'},$$

$$\delta = 0,$$

$$\sigma = 0,$$

$$\sigma' = \frac{-\Psi}{s\Psi + (b^2 - s^2)\Psi'},$$

$$\sigma'' = \frac{(s\Psi\Psi'' + s\Psi'\Psi' - \Psi\Psi')(s\Psi + (b^2 - s^2)\Psi)}{\Psi^2((\Psi - s\Psi') + (b^2 - s^2)\Psi'')},$$

$$1 + \mu(\frac{-b^2 + s^2}{s})\sigma' = 1 + \frac{s(s(\Psi\Psi'' + \Psi'\Psi') - \Psi\Psi')}{\Psi(\Psi - s\Psi')}\frac{\Psi}{s\Psi + (b^2 - s^2)\Psi'}\frac{b^2 - s^2}{s}$$
$$= \frac{s\Psi(\Psi - s\Psi' + (b^2 - s^2)\Psi'')}{(\Psi - s\Psi')(s\Psi + (b^2 - s^2)\Psi')}.$$

Now replace these in (28),

$$\det g_{ij} = \Psi^n(\Psi - s\Psi')^n \frac{s\Psi(\Psi - s\Psi' + (b^2 - s^2)\Psi'')}{(\Psi - s\Psi')(s\Psi + (b^2 - s^2)\Psi')}\frac{(s\Psi + (b^2 - s^2)\Psi')^2}{s^2(\Psi - s\Psi')^2}.$$

Finally, by simplification, we arrive at the following formula,

$$\det(g_{ij}) = \Psi^{n+1}(\Psi - s\Psi')^{n-3}\left((\Psi - s\Psi') + (b^2 - s^2)\Psi''\right)\left(\Psi + (\frac{b^2}{s} - s)\Psi'\right)\det(a_{ij}).$$ ■

One can see that for $\Psi = 1 + s, \Psi' = 1, \Psi'' = 0$ the result is the same as the formula which has been gotten in (Proposition 3.1, Lee (2003)).

Lemma 4.4.

$F = \alpha\Psi(\frac{\beta}{\alpha})$ is a Finsler metric, for any Riemannian metric α and symmetric quadratic form β if and only if $\Psi = \Psi(s)$ satisfies the following conditions,

$$\Psi(s) > 0, \quad (\Psi - s\Psi') + (b^2 - s^2)\Psi'' > 0, \quad \Psi + (\frac{b^2}{s} - s)\Psi' > 0, \tag{29}$$

where $\Psi' = \frac{d\Psi}{ds}$.

Proof:

Assume that (29) is satisfied. Then by taking $b = s$ in (29), we see that the following inequality holds,

$$\Psi - s\Psi' > 0.$$

Consider the following family of functions,

$$\Psi_\epsilon(s) := 1 - \epsilon + \epsilon\Psi(s).$$

Let $F_\epsilon := \alpha\Psi_\epsilon(\frac{\beta}{\alpha})$ and $g^\epsilon_{ij} := \frac{1}{2}[F^2_\epsilon]_{y^i y^j}(y), 0 \leq \epsilon \leq 1$. We have

$$\Psi_\epsilon - s{\Psi_\epsilon}' = 1 - \epsilon + \epsilon(\Psi - s\Psi') > 0,$$

$$(\Psi_\epsilon - s{\Psi_\epsilon}') + (b^2 - s^2){\Psi_\epsilon}'' = 1 - \epsilon + \epsilon\left((\Psi - s\Psi') + (b^2 - s^2)\Psi''\right) > 0,$$

$$\Psi_\epsilon + (\frac{b^2}{s} - s){\Psi_\epsilon}' = 1 - \epsilon + \epsilon(\Psi + (\frac{b^2}{s} - s)\Psi') > 0.$$

For $\epsilon = 0$, $F_\epsilon = \alpha$ since $det({g_{ij}}^\epsilon)$ is continuous for ϵ, all the eigenvalues of $({g_{ij}}^\epsilon)$ are positive by intermediate value theorem. Therefore, all the eigenvalues of (g_{ij}) are positive.

Conversely, suppose that $F = \alpha\Psi(\frac{\beta}{\alpha})$ is a regular Finsler metric. we can always find a vector $y \in \mathcal{D}$ such that $\beta(x_0, y) = s\alpha(x_0, y)$. By assumption, $F(x_0, y) = \alpha\Psi(s) > 0$, we conclude that $\Psi(s) > 0$. By another assumption, $det(g_{ij}(x_0, y)) > 0$, we conclude from (29) that

$$\Psi(s) - s\Psi'(s) \neq 0.$$

provided that $n > 3$. Since $\Psi(0) > 0$, and s arbitrary we must have

$$\Psi(s) - s\Psi'(s) > 0.$$

Now by (29), we conclude that

$$\Psi(s) - s\Psi'(s) + (b^2 - s^2)\Psi''(s) > 0, \quad \Psi + (\frac{b^2}{s} - s)\Psi' > 0.$$

If $n = 3$, we still get the above inequality from (29). This proves the lemma. ■

Here, we state that every generalized (a, b, J)-metric is a Rizza manifold.

Proposition 4.5.

A generalized (a, b, J)-metric $F = \alpha\Psi(\frac{\beta}{\alpha})$ is a Rizza manifold.

Lemma 4.6.

Mean Cartan torsion of a generalized (a, b, J)-metric $F = \alpha\Psi(\frac{\beta}{\alpha})$ is given by

$$I_i = \frac{1}{2\alpha}\left\{(n+1)\frac{\Psi'}{\Psi} - (n-3)\frac{s\Psi''}{\Psi - s\Psi'} + \frac{(b^2 - s^2)\Psi''' - 3s\Psi''}{(\Psi - s\Psi') + (b^2 - s^2)\Psi''} + \frac{(\frac{b^2}{s} - s)\Psi'' - \frac{b^2}{s^2}\Psi'}{\Psi + (\frac{b^2}{s} - s)\Psi'}\right\} h_i,$$

where $h_i = \frac{b_{ki}y^k}{\beta} - \frac{y_i}{\alpha}s = \beta_i - \frac{y_i}{\alpha}s$.

Proof:

We know that $I_i = g^{jk}C_{ijk} = \frac{\partial}{\partial y^i}\left(Ln\sqrt{det(g_{jk})}\right)$. A direct computation shows that $\frac{\partial s}{\partial y^i} = \frac{h_i}{\alpha}$ where $h_i := \frac{b_{ki}y^k}{\beta} - \frac{y_i}{\alpha}s = \beta_i - \frac{y_i}{\alpha}s$. Thus

$$I_i = \frac{\partial}{\partial y^i}\left\{\frac{n+1}{2}Ln\Psi + \frac{n-3}{2}Ln(\Psi - s\Psi') + \frac{1}{2}Ln\left[(\Psi - s\Psi') + (b^2 - s^2)\Psi''\right]\right.$$
$$\left. + \frac{1}{2}Ln\left(\Psi + (\frac{b^2}{s} - s)\Psi'\right) + \frac{1}{2}Ln(det(a_{ij}))\right\}$$

Finally, the desired result is obtained. ■

Therefore, mean Cartan torsion of an (a, b, J)-metric is $I_i = (\frac{n+1}{2(\alpha+\beta)} - \frac{b^2\alpha^2}{\beta(\beta+b^2\alpha)})(\beta_i - \frac{y_i}{\alpha}\frac{\beta}{\alpha})$.

To state our result, we shall first introduce some notations. Let $\nabla_j b_i$ and $\nabla_j J_i$ denote the horizontal covariant derivative of $b_i dx^i$ and $J_i dx^i$ with respect to $\alpha = \sqrt{a_{ij}y^iy^j}$, respectively. Let

$$r_{ij} := \frac{1}{2}\left(\nabla_j b_i + \nabla_i b_j\right), \quad s_{ij} := \frac{1}{2}\left(\nabla_j b_i - \nabla_i b_j\right),$$

$$\tilde{r}_{ij} := \frac{1}{2}\left(\nabla_j J_i + \nabla_i J_j\right), \quad \tilde{s}_{ij} := \frac{1}{2}\left(\nabla_j J_i - \nabla_i J_j\right),$$

$$r_{00} := r_{ij}y^iy^j = \nabla_i b_j y^i y^j, \quad \tilde{r}_{00} := \tilde{r}_{ij}y^iy^j = \nabla_i J_j y^i y^j,$$

$$s^i_0 = s^i_j y^j, \quad s_0 := s_i y^i, \quad \tilde{s}^i_0 = \tilde{s}^i_j y^j, \quad \tilde{s}_0 := \tilde{s}_i y^i, \quad b^2 := a^{ij}b_ib_j.$$

It is easy to see that

$$s_{ij} + s_{ji} = 0, \quad s_{00} := s_{ij}y^iy^j = 0, \quad \tilde{s}_{ij} + \tilde{s}_{ji} = 0, \quad \tilde{s}_{00} := \tilde{s}_{ij}y^iy^j = 0.$$

Let $G^i = G^i(x, y)$ and $\bar{G}^i = \bar{G}^i(x, y)$ denote the spray coefficients of F and α, respectively, in the same coordinate system. Put $b_0 = b_iy^i$ and $J_0 = J_iy^i$. According to (1) and considering (27) spray coefficients G^i are given by

$$G^i = G^i_\alpha + \frac{\nabla_k F y^k}{2F}y^i + \frac{F}{2}g^{il}\{(\nabla_k F)_{.l}y^k - \nabla_l F\}, \tag{30}$$

where $\nabla_i F$ denote the covariant derivatives of F with respect to α and $(\nabla_i F)_{.j} = [\nabla_i F]_{y^j}$.

Lemma 4.7.

Spray coefficients of a generalized (a, b, J)-metric $F = \alpha\Psi(\frac{\beta}{\alpha})$ are given by

$$G^i = \bar{G}^i + Py^i + Q^i,$$

where

$$P = \Xi s_0 + (\Theta_1 + Q)(b_0 r_{00} + J_0\tilde{r}_{00}) + \Xi(s_{l0}J^l + \tilde{s}_{l0}b^l_0)\frac{J_0}{b_0},$$

$$Q^i = \left(\Theta_2 b^i + \Theta_3 J^i + \Theta_4 b^i_0 + \Theta_5\right) r_{00} + \left((\Theta_6 + \Theta_8) b^i_0 + \Theta_7 J^i + \Theta_9\right) \tilde{r}_{00} + \Theta_{10} b^i_0 + \Theta_{11},$$

where $\Xi = \frac{b_0\Psi'(s\Psi\Psi' + s\Psi'\Psi' - \Psi\Psi')}{\beta^2\Psi(\Psi - s\Psi')((\Psi - s\Psi') - (b^2 - s^2)\Psi'')}$ and Θ_is are given in the Appendix.

In (Ichijyō and Hashiguchi (1994)), Ichijy$\bar{o}$ says irrationality of $\lambda = \frac{\beta}{\alpha}$ in terms of y yields that a generalized Randers space is a Berwald space if and only if $\nabla_k b_{ij} = 0$. Here, we extend Ichijy$\bar{o}$'s result to generalized (a, b, J)-metrics.

Theorem 4.8.

Let $F = \alpha\Psi(\frac{\beta}{\alpha})$ be a generalized (a, b, J)-metric. Suppose the $\lambda = \frac{s(\Psi - s\Psi')}{\Psi'}$ is an irrational function of y. Then, (M, F) is a Berwald space if and only if two 1-forms b_i and J_i are cross-recurrent.

Proof:

Suppose F is a Berwald metric. We denote the h-covariant differentiation with respect to the Berwald connection of F with ";". We have

$$F_{;k} = \alpha_{;k}\Psi(s) + \Psi'(s)(\frac{\beta_{;k}\alpha - \beta\alpha_{;k}}{\alpha}). \tag{31}$$

On the other hand, it is well known that $F_{;k} = 0$. Multiplying (31) with α, we obtain

$$\alpha\Psi\alpha_{;k} + \Psi'\alpha\beta_{;k} - \beta\Psi'\alpha_{;k} = 0. \tag{32}$$

Substituting $\alpha_{;k}$ and $\beta_{;k}$ in (32), we rewrite as following

$$\lambda a_{ij;k} y^i y^j + b_{ij;k} y^i y^j = 0. \tag{33}$$

(M, F) is a Berwald space. Thus, the horizontal Christoffel coefficients $G^i{}_{jk}$ of the Berwald connection $B\Gamma$ are functions of position alone. Consequently, $a_{ij;k} y^i y^j$ and $b_{ij;k} y^i y^j$ become polynomials of y. Now, the irrationality of λ and (33) infer that $a_{ij;k} y^i y^j = 0$ and $b_{ij;k} y^i y^j = 0$, that is, $a_{ij;k} = 0$ and $b_{ij;k} = 0$. The former implies that the Levi-Civita connection of α and the Berwald connection of F coincide. Therefore, we have $\nabla_k b_{ij} = 0$, which is equivalent to b_i and J_i are cross-recurrent. The converse is also true. In fact, if $\nabla_k b_{ij} = 0$ is satisfied, then $\nabla_k F = \frac{1}{\alpha}\nabla_k\beta\Psi' = 0$ it follows from Theorem 4.1.3 in (Chern and Shen (2004)) that F and α are affinely equivalent. Hence, F is Berwaldian. ■

Remark 4.9.

If in Theorem 4.8 we put $\Psi(s) = 1 + s$, then we get Ichijyō's assumption on generalized Randers metrics.

We call a generalized (a, b, J)-metric $F = \alpha\Psi(\frac{\beta}{\alpha})$ normal if (12) holds. It follows from (12) that β is parallel with respect to α. Thus, Theorem 4.8 implies that every normal generalized (a, b, J)-metric is Berwaldian.

We call also a generalized (a, b, J)-metric $F = \alpha\Psi(\frac{\beta}{\alpha})$ nearly normal if b_i and J_i are cross-recurrent and the following holds ,

$$\nabla_k J^i_j + \nabla_j J^i_k = 0. \tag{34}$$

Thus every nearly normal generalized (a, b, J)-metric is a Berwald space. Thus, the horizontal covariant differentiation with respect to Cartan and Berwald connections coincide with the one with respect to Levi-Civita connection of Riemannian metric α. Therefore, $\nabla_k J^i_j = J^i_{j|k}$. Hence, we get the following proposition.

Proposition 4.10.

Every normal (nearly normal) generalized (a, b, J)-metric is a Kählerian (nearly Kählerian) Finsler metric.

In Theorem 4.8, we characterize those (a, b, J)-metrics $F = \alpha + \beta$ whose Riemannian parts α is affinely equivalent to F. Now, we deal with projectively equivalency of F and α.

Proposition 4.11.

Let $F = \alpha + \beta$ be an (a, b, J)-metric. Then F is projectively equivariant to α if and only if for $l = 1, ..., n$

$$\sigma_{(ijkm)}\Big(-(\nabla_k b_{ij})b_{ml} + (\nabla_k b_{lj})b_{im}y^i + (\nabla_k b_{il})b_{jm} - (\nabla_k b_{lj})b_{im}\Big) = 0, \tag{35}$$

where $\sigma_{(ijkm)}$ denotes the summation over all permutations of indices i, j, k and m.

Proof:

By (6), it suffices to prove that $(\nabla_k F)_{.l}y^k - \nabla_l F = 0$ is equivalent to (35). One can see

$$\begin{aligned}(\nabla_k F)_{.l}y^k - \nabla_l F &= (\nabla_k \beta)_{.l}y^k - \nabla_l \beta\\ &= \frac{1}{2\beta}\left\{(\nabla_k b_{ij})(\frac{-b_{ml}y^m}{\beta^2})y^iy^jy^k + (\nabla_k b_{lj})y^jy^k + (\nabla_k b_{il})y^iy^k - (\nabla_l b_{ij})y^iy^j\right\}.\end{aligned} \tag{36}$$

Suppose that F is projectively related to α. Thus, (36) implies that

$$(\nabla_k b_{ij})(\frac{-b_{ml}y^m}{\beta^2})y^iy^jy^k + (\nabla_k b_{lj})y^jy^k + (\nabla_k b_{il})y^iy^k - (\nabla_l b_{ij})y^iy^j = 0. \tag{37}$$

Multiplying (37) by β^2, we get a polynomial equation of degree four in terms of y as follows

$$S_{ijkml}y^iy^jy^ky^m = 0, \tag{38}$$

in which $S_{ijkml} := -b_{ml}\nabla_k b_{ij} + b_{im}\nabla_k b_{lj} + b_{jm}\nabla_k b_{il} - b_{km}\nabla_l b_{ij}$ are functions independent of direction. We get (35) from (37). This completes the proof. ■

It is natural to study the projectively equivalency problem for generalized (a, b, J)-metrics.

Theorem 4.12.

Let $F = \alpha\Psi(\frac{\beta}{\alpha})$ be a generalized (a, b, J)-metric. Suppose that $\frac{\Psi''}{s\Psi'}$ is an irrational function of y. Then F is projectively equivalent to α if and only if (35) and $\sigma_{ijk}\nabla_k b_{ij} = 0$ are satisfied.

Proof:

To use for the case $F = \alpha\Psi(\beta/\alpha)$, We need the following

$$(\nabla_k F)_{.l} y^k - \nabla_l F = (\nabla_k \beta)_{.l} \Psi'(s) y^k + \nabla_k \beta \frac{\beta_{.l}\alpha - \beta\alpha_{.l}}{\alpha^2} \Psi''(s) y^k - \nabla_l \beta \Psi'(s).$$

Suppose that F is projectively related to α. Then Rapcsak's Theorem implies that $(\nabla_k F)_{.l} y^k - \nabla_l F = 0$. Therefore, we have

$$(\nabla_k \beta)_{.l} y^k - \nabla_l \beta + \frac{1}{\alpha} \nabla_k \beta (\beta_{.l} - s\alpha_{.l}) y^k \frac{\Psi''}{\Psi'} = 0. \tag{39}$$

Multiplying (39) with $\alpha^4\beta^2$ we get

$$\alpha^4 \left\{ -(\nabla_k b_{ij}) b_{ml} y^m y^i y^j y^k + (\nabla_k b_{lj}) y^j y^k \beta^2 + (\nabla_k b_{il}) y^i y^k \beta^2 - (\nabla_l b_{ij}) y^i y^j \beta^2 \right\}$$
$$+ \beta^2 (\nabla_k b_{ij}) y^k y^i y^j (\alpha^2 b_{l0} - \beta^2 y_l) \frac{\Psi''}{s\Psi'} = 0. \tag{40}$$

All terms appeared in (40) except $\frac{\Psi''}{s\Psi'}$, are rational functions of y. Therefore, (40) is equivalent to following

$$-(\nabla_k b_{ij}) b_{ml} y^m y^i y^j y^k + (\nabla_k b_{lj}) y^j y^k \beta^2 + (\nabla_k b_{il}) y^i y^k \beta^2 - (\nabla_l b_{ij}) y^i y^j \beta^2 = 0, \tag{41}$$

$$(\nabla_k b_{ij}) y^k y^i y^j (\alpha^2 b_{l0} - \beta^2 y_l) = 0. \tag{42}$$

As we saw in the previous theorem, (41) is equivalent to (35). We claim that $\alpha^2 b_{l0} - \beta^2 y_l$ is non-zero. Otherwise, we have

$$(b_{li} a_{ts} - b_{ts} a_{li}) y^i y^t y^s = 0. \tag{43}$$

Therefore,

$$(b_{li} a_{ts} - b_{ts} a_{li}) + (b_{lt} a_{is} - b_{is} a_{lt}) + (b_{ls} a_{ti} - b_{ti} a_{ls}) = 0. \tag{44}$$

If we denote the inverse of (a_{ts}) by (a^{ts}) and contract (44) with a^{ts}, we obtain

$$n b_{li} = \mu a_{li}, \tag{45}$$

where $\mu = a^{ts} b_{ts}$ and we have used the symmetry of β. One can see (45) contradicts β being of rank 2. Hence, $\nabla_k b_{ij} y^k y^i y^j = 0$ which is equivalent to $\nabla_k b_{ij} + \nabla_j b_{ik} + \nabla_i b_{kj} = 0$. ■

Ichijyō in (Ichijyō (1994)) proved that a Kählerian Finsler manifold is a Landsberg manifold. We are going to generalize this fact to nearly Kählerian Finsler manifolds. First, we prove that the Berwlad curvature of a nearly Kähler Finsler manifold and its almost complex structure have a delicate relation. Let us recall two important identities,

$$(a)\ g_{ij;k} = -2L_{ijk}, \quad (b)\ F^i_{jk} = G^i_{jk} - L^i_{jk}, \tag{46}$$

where ";" stands for the h-covariant derivative with respect to the Berwald connection $B\Gamma = (G^i_{jk}, G^i_j, 0)$ and F^i_{jk} are given by (2).

Proposition 4.13.

Let (M, F, J) be a nearly Kähler Finsler manifold. Then

$$(a) J^r_m L^i_{rj} + y^k J^r_k \frac{\partial L^i_{rj}}{\partial y^m} = 0, \qquad (b) J^r_j L^i_{rm} - 2J^i_r L^r_{mj} + J^r_m L^i_{rj} + y^k J^r_k B^i_{rjm} = 0. \tag{47}$$

Proof:

Using (3), we rewrite $J^i_{j|k} + J^i_{k|j} = 0$ as follows,

$$\partial_k J^i_j + J^r_j F^i_{rk} + \partial_j J^i_k + J^r_k F^i_{rj} - 2J^i_r F^r_{jk} = 0. \tag{48}$$

We multiply (48) by y^k and obtain

$$y^k \partial_k J^i_j + J^r_j G^i_r + y^k \partial_j J^i_k - 2J^i_r G^r_j + y^k J^r_k F^i_{rj} = 0, \tag{49}$$

where we have used $y^k F^r_{kj} = G^r_j$. Differentiating (49) with respect to y^m, we get

$$\partial_m J^i_j + \partial_j J^i_m + J^r_j G^i_{rm} + J^r_m G^i_{rj} - 2J^i_r G^r_{jm} - J^r_m L^i_{rj} + y^k J^r_k \frac{\partial F^i_{rj}}{\partial y^m} = 0. \tag{50}$$

Using (5) , (50) and (46b), we reformulate (50) as follows,

$$J^i_{j;m} + J^i_{m;j} = J^r_m L^i_{rj} + y^k J^r_k \frac{\partial L^i_{rj}}{\partial y^m} - y^k J^r_k B^i_{rjm}. \tag{51}$$

Transvecting (49) with y^j yields

$$y^j y^k \partial_k J^i_j + y^k J^r_k G^i_r - 2J^i_r G^r = 0, \tag{52}$$

where we have used $y^j G^r_j = 2G^r$. Differentiating (55) with respect to y^j and y^m, respectively, leads us to

$$\partial_m J^i_j + \partial_j J^i_m + J^r_j G^i_{rm} - 2J^i_r G^r_{mj} + J^r_m G^i_{rj} + y^k J^r_k B^i_{rjm} = 0, \tag{53}$$

where we have used $y^j B^i_{jkl} = 0$. One can rewrite (53) as follows,

$$J^i_{j;m} + J^i_{m;j} = -y^k J^r_k B^i_{rjm}. \tag{54}$$

Comparing (51) with (54), we get (47a).

Now we multiply (49) with y^j and get the following,

$$2y^j y^k \partial_k J^i_j + y^k J^r_k G^i_r - 2y^j J^i_r G^r_j + y^j J^r_j G^i_r = 0. \tag{55}$$

First differentiating (55) with respect to y^s and then with respect to y^t , by using (46) we achieve

$$J^r_j L^i_{rm} - 2J^i_r L^r_{mj} + J^r_m L^i_{rj} + y^k J^r_k B^i_{rjm} = 0. \tag{56}$$

■

Theorem 4.14.

Let (M, F, J) be a nearly Kähler Finsler manifold satisfying (57). Then F is a Landsberg metric

$$J^p_r L^r_{ik} = J^r_i L^p_{rk}. \tag{57}$$

Proof:

Using (57), we have

$$J^p_{i|m} = J^p_{i;m} + J^p_r L^r_{im} - J^r_i L^p_{rm} = J^p_{i;m}. \tag{58}$$

Thus, by (54) and $J^i_{j|m} + J^i_{m|j} = 0$, we have $y^k J^r_k B^i_{rjm} = 0$ and concequently (47b) reduces to the following,

$$J^r_j L^i_{rm} + J^r_m L^i_{rj} = 2J^i_r L^r_{jm}. \tag{59}$$

By contracting (59) with J^k_i we get

$$J^k_i J^r_j L^i_{rm} + J^k_i J^r_m L^i_{rj} = -2L^k_{jm}. \tag{60}$$

Transvecting (60) with y^j and using $L^k_{jm} y^j = L^i_{rj} y^j = 0$, we get

$$y^j J^k_i J^r_j L^i_{rm} = 0. \tag{61}$$

Taking vertical differentiating with respect to y^l from (61) and taking into account (47a) implies that

$$J^k_i J^r_l L^i_{rs} = 0. \tag{62}$$

Substituting (62) into (60) completes the proof. ■

5. Conclusion

In this paper, we introduced generalized (a, b, J) -manifolds. A partial negative answer to Ichijyō's problem on nearly Kähler Finsler manifolds was given. The condition under which generalized (a, b, J) - manifolds are Berwaldian was obtained. Finally, we proved that under a mild assumption a nearly Kähler Finsler manifold is Landsbergian.

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APPENDIX

$$\Theta_1 = \left\{ \frac{\Psi'}{2\alpha\beta\Psi} + \frac{(s\Psi\Psi'' + s\Psi'\Psi' - \Psi\Psi')[\Psi'' s(s\Psi + (b^2 - s^2)\Psi') + \alpha\Psi'(\frac{b^2}{\beta}(-\frac{s}{\beta} + \frac{1}{\alpha})) - \frac{\Psi''}{\alpha}(b^2 - \frac{s}{\alpha\beta} b_0^l y_l)]}{2\alpha\beta\Psi(\Psi - s\Psi')((\Psi - s\Psi') + (b^2 - s^2)\Psi'')} \right\}$$

$$\Theta_2 = \frac{1}{2(\Psi - s\Psi')} \left[\alpha\Psi' \left((-\frac{b_0^2}{\beta^3} + \frac{1}{\beta}) - \frac{\Psi'}{s\Psi + (b^2 - s^2)\Psi'}(-\frac{b_0^2 b^2}{\beta^3} + \frac{b^2}{\beta}) \right) + \frac{b_0^2 \Psi''}{\beta^2} \left(1 - \frac{b^2\Psi'}{s\Psi + (b^2 - s^2)\Psi'} \right) \right]$$

$$\Theta_3 = \frac{b_0 J_0}{2(\Psi - s\Psi')\beta^2} \left[(\frac{\alpha\Psi'}{\beta} - \Psi'') \left(-1 + \frac{\Psi' b^2}{s\Psi + (b^2 - s^2)\Psi'} \right) \right]$$

$$\Theta_4 = \frac{1}{2(\Psi - s\Psi')} \times$$

$$\left[\frac{\alpha(s\Psi\Psi'' + s\Psi'\Psi' - \Psi\Psi')}{\Psi((\Psi - s\Psi') + (b^2 - s^2)\Psi'')}(\frac{b_0}{\alpha\beta^2} - \frac{b_0 s}{\beta 3}) - \frac{\Psi''(s\Psi\Psi'' + s\Psi'\Psi' - \Psi\Psi')}{(s\Psi + (b^2 - s^2\Psi'))((\Psi - s\Psi') + (b^2 - s^2)\Psi''}(-\frac{b_0 s}{\alpha\beta^3 b_0^l y_l} + \frac{b_0 b^2}{\beta^2}) \right]$$

$$\Theta_5 = \frac{\Psi''}{2\alpha(\Psi - s\Psi')} \left[\frac{s b_0 \Psi'}{s\Psi + (b^2 - s^2)\Psi'} b^{il} y_l - \frac{b_0(s\Psi\Psi'' + s\Psi'\Psi' - \Psi\Psi')(s\Psi + (b^2 - s^2)\Psi'}{\alpha\Psi^2((\Psi - s\Psi') + (b^2 - s^2)\Psi'')} \right]$$

$$\Theta_6 = \frac{1}{2(\Psi - s\Psi')} \left[\alpha\Psi' \left(-\frac{b_0 J_0}{\beta^3} - \frac{\Psi'}{s\Psi + (b^2 - s^2)\Psi'}(-\frac{b_0 J_0 b^2}{\beta^3} + \frac{b^2}{\beta}) \right) - \Psi'' \frac{b_0 J_0}{\beta^2} \left(1 + \frac{b^2\Psi'}{s\Psi + (b^2 - s^2)\Psi'} \right) \right]$$

$$\Theta_7 = \frac{1}{2(\Psi - s\Psi')} \left[\alpha\Psi' \left((-\frac{J_0^2}{\beta^3}) + \frac{1}{\beta} + \frac{\Psi'}{s\Psi + (b^2 - s^2)\Psi'} \frac{J_0^2 b^2}{\beta^3} \right) - \Psi'' \frac{J_0^2}{\beta^2} \left(1 + \frac{b^2\Psi'}{s\Psi + (b^2 - s^2)\Psi'} \right) \right]$$

$$\Theta_8 = \frac{1}{2(\Psi - s\Psi')} \times$$

$$\left[\frac{\alpha(s\Psi\Psi'' + s\Psi'\Psi' - \Psi\Psi')}{\Psi((\Psi - s\Psi') + (b^2 - s^2)\Psi''}(\frac{J_0}{\alpha\beta^2} - \frac{J_0 s}{\beta^3}) - \frac{\Psi''(s\Psi\Psi'' + s\Psi'\Psi' - \Psi\Psi')}{(s\Psi + (b^2 - s^2)\Psi')((\Psi - s\Psi') + (b^2 - s^2)\Psi'')}(-\frac{J_0 s}{\alpha\beta^3} b_0^l y_l + \frac{J_0 b^2}{\beta^2}) \right]$$

$$\Theta_9 = \frac{\Psi''}{2\alpha(\Psi - s\Psi')} \left[\frac{s J_0 \Psi'}{s\Psi + (b^2 - s^2)\Psi'} b^{il} y_l - \frac{J_0(s\Psi\Psi'' + s\Psi'\Psi' - \Psi\Psi')(s\Psi + (b^2 - s^2)\Psi'}{\alpha\Psi^2((\Psi - s\Psi') + (b^2 - s^2)\Psi'')} \right]$$

$$\Theta_{10} = \frac{-\alpha\Psi'(s\Psi\Psi'' + s\Psi'\Psi' - \Psi\Psi')}{(\Psi - s\Psi')\beta^3(s\Psi + (b^2 - s^2)\Psi')((\Psi - s\Psi') + (b^2 - s^2)\Psi'')} \left[b_0^2 s_0 + s_{l0} J^l J_0 b_0 + \tilde{s}_{l0} b_0^l J_0 \right] b_0^i$$

$$\Theta_{11} = \frac{\alpha\Psi'}{(\Psi - s\Psi'\beta)}(b_0 s_0^i + J_0 \tilde{s}_0^i) - \frac{\alpha\Psi'\Psi'}{\beta(\Psi - s\Psi')(s\Psi + (b^2 s^2)\Psi')} \left((b_0 s_0) b^i + (b_0 s_{l0} J^l) J^i + \tilde{s}_{l0} J_0 b^{il} \right)$$