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Adjacent vertex-distinguishing proper edge-coloring of strong product of graphs

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Abstract

Let G be a finite, simple, undirected and connected graph. The adjacent vertex-distinguishing proper edge-coloring is the minimum number of colors required for a proper edge-coloring of G, in which no two adjacent vertices are incident to edges colored with the same set of colors. The minimum number of colors required for an adjacent vertex-distinguishing proper edge-coloring of G is called the adjacent vertex-distinguishing proper edge-chromatic index. In this paper, I compute adjacent vertex-distinguishing proper edge-chromatic index of strong product of graphs.

Keywords: Edge-coloring; adjacent vertex-distinguishing proper edge-coloring; strong product

MSC 2010 No.: 05C15; 05C38

1. Introduction

Let G be a finite, simple, undirected and connected graph. Denote by $V(G)$ and $E(G)$ the set of vertices and edges of G, respectively. Let $\Delta(G)$ denote the maximum degree of G. A *proper edge-coloring* σ is a mapping from $E(G)$ to the set of colors such that any two adjacent edges receive distinct colors. For any vertex v of G, let $S_\sigma(v)$ denote the set of the colors of all edges incident to v. A proper edge-coloring σ is said to an *adjacent vertex-distinguishing* (AVD) if $S_\sigma(u) \neq S_\sigma(v)$, for every adjacent vertices u and v. The minimum number of colors required for an adjacent vertex-distinguishing proper edge-coloring of G, denoted by $\chi'_{as}(G)$, is called the *adjacent vertex-distinguishing proper edge-chromatic index* (*AVD proper edge-chromatic index*) of G. Thus, $\chi'_{as}(G) \geq \chi'(G)$.

Conjecture 1.1. [Zhang et al. (2002)]

For any connected graph G ($|V(G)| \geq 6$), there is $\chi'_{as}(G) \leq \Delta(G) + 2$. If H is a subgraph of G, it is interesting that $\chi'_{as}(H) \leq \chi'_{as}(G)$ is not always true (Zhang et al. (2008)). Let $K_{m,n}$ be the complete bipartite graph, then $\chi'_{as}(K_{2,3}) = 3$ and $K_{2,3} - e$ for any edge, then $\chi'_{as}(K_{2,3} - e) = 4$. Deletion of an edge of a graph may also decrease the coloring number of the graph. Let $n \geq 3$ then $\chi'_{as}(K_{1,n}) = n$ and $\chi'_{as}(K_{1,n} - e) = n - 1$.

Zhang et al. (2008) proved: if G has n components G_i, $1 \leq i \leq n$, with at least three vertices in each, then $\chi'_{as}(G) = max_{1 \leq i \leq n}\{\chi'_{as}(G_i)\}$. So we consider only connected graphs. For a tree T with $|V(T)| \geq 3$, if any two vertices of maximum degree are nonadjacent, then $\chi'_{as}(T) = \Delta(T)$; if T has two vertices of maximum degree which are adjacent, then $\chi'_{as}(T) = \Delta(T) + 1$. For cycle C_n we have $\chi'_{as}(C_n) = 3$ for $n \equiv 0$ (mod 3), otherwise $\chi'_{as}(C_n) = 4$ for $n \not\equiv 0$ (mod 3) and $n \neq 5$; $\chi'_{as}(C_n) = 5$ for $n = 5$. For the complete bipartite graph $K_{m,n}$ $(1 \leq m \leq n)$, we have $\chi'_{as}(K_{m,n}) = n$ if $m < n$, and $\chi'_{as}(K_{m,n}) = n + 2$ if $m = n \geq 2$. For the complete graph K_n $(n \geq 3)$, we have $\chi'_{as}(K_n) = n$ for $n \equiv 1 \pmod 2$; and $\chi'_{as}(K_n) = n + 1$ for $n \equiv 0 \pmod 2$. If G is a graph with two adjacent maximum degree vertices, then $\chi'_{as}(G) \geq \Delta(G) + 1$. If G is a graph such that the degree of any two adjacent vertices is different, then $\chi'_{as}(G) \geq \Delta(G)$.

The *wheel* $W_n = C_n \vee K_1$ is the graph obtained by joining each vertex of C_n to the vertex of K_1. The *friendship graph* F_n constructed joining of n triangles with a common vertex.

Shiu (2011) proved: for $n \geq 3$, we have $\chi'_{as}(F_{n-1}) = n$, if $n = 3, 4$ and $\chi'_{as}(F_{n-1}) = n - 1$, for $n \geq 5$. For $n \geq 3$, we have $\chi'_{as}(W_n) = 5$, if $n = 3$, and $\chi'_{as}(W_n) = n$, for $n \geq 4$. Hatami (2005) proved: if G is a graph with no isolated edges and maximum degree $\Delta(G) > 10^{20}$, then $\chi'_{as}(G) \leq \Delta + 300$. Balister et al. (2007) proved: if G is a k-chromatic graph with no isolated edges, then $\chi'_{as}(G) \leq \Delta(G) + O(\log k)$. Axenovich et al. (2016) proved: $dis[G] \leq \Delta(G)^2 - \Delta(G) + 1$, where $dis[G]$ is the smallest integer n such that there is a closed distinguishing labeling of G using labels from $\{1,2,3, \dots, n\}$. This result is sharp. Zhang et al. (2002) proved: any graph G having maximum degree Δ and no isolated edges, then $\chi'_{as}(G) \leq \frac{5}{2}(\Delta(G) + 2)$.

This concept has been studied in many papers such as Zhang et al. (2002), Hatami (2005), Baril (2006), Balister et al. (2007), Wang et al. (2010) , Shiu et al. (2011), Zhang et al. (2002), Axenovich et al. (2016) and Omai et al. (2017).

In this paper, we compute adjacent vertex-distinguishing edge-chromatic index of strong product of graphs and verifying the conjecture for strong product of graphs. We refer the book (2001) for graph theoretical notation and terminology.

The *strong product* of graphs G and H is the simple graph $G \boxtimes H$ with vertex set $V(G) \times V(H)$ and $(x_1, x_2)(y_1, y_2) \in E(G \boxtimes H)$, whenever $x_1 y_1 \in E(G)$ and $x_2 = y_2$, (or) $x_1 = y_1$ and $x_2 y_2 \in E(H)$, (or) $x_1 y_1 \in E(G)$ and $x_2 y_2 \in E(H)$.

We take some notation assumption,

$$V(P_m) = V(C_m) = \{v_1, v_2, v_3, \dots, v_m\},$$
$$E(P_m) = \{v_i v_{i+1} : i \in \{1,2,3, \dots, m-1\}\},$$
$$E(C_m) = E(P_m) \cup \{v_m v_1\}, \text{ in } C_m, v_{m+1} = v_1 \text{ and } v_0 = v_m.$$
$$V(W_n) = \{v_0, v_1, v_2, \dots, v_n\},$$
$$E(W_n) = \{v_i v_{i+1} : i \in \{1,2,3, \dots, n-1\} \cup \{v_n v_1\} \cup \{v_0 v_i : i \in \{1,2,3, \dots, n\}\}.$$

In the product graphs $P_r \boxtimes P_s$, $P_r \boxtimes C_s$, $P_m \boxtimes W_n$ and $C_m \boxtimes W_n$ denote the vertex (v_i, v_j) by $v_{i,j}$.

Proposition 1.1.

Every graph G with two adjacent vertices of degree $\Delta(G)$ satisfies $\chi'_{as}(G) \geq \Delta(G)+1$.

Proof:

Let x and y be two adjacent vertices of degree $\Delta(G)$. Suppose $\chi'_{as}(G) = \Delta(G)$, then $S_\sigma(x) = S_\sigma(y)$, a contradiction. ■

2. AVD proper edge-coloring for strong product of graphs

Theorem 2.1.

(i) $\chi'_{as}(P_2 \boxtimes W_3) = 9$.
(ii) If $n \geq 4$, then $\chi'_{as}(P_2 \boxtimes W_n) = 2(n+1)$.
(iii) $\chi'_{as}(P_3 \boxtimes W_3) = 12$.
(iv) If $n \geq 4$, then $\chi'_{as}(P_3 \boxtimes W_n) = 3n+2$.
(v) If $m \geq 4$ and $n \geq 3$, then $\chi'_{as}(P_m \boxtimes W_n) = 3(n+1)$.

Proof:

(i) Since $P_2 \boxtimes W_3 \cong K_8$ and $\chi'_{as}(K_8) = 9$, then $\chi'_{as}(P_2 \boxtimes W_3) = 9$.

(ii) For $n \geq 4$. Since $\Delta(P_2 \boxtimes W_n) = 2n+1$, by Proposition 1.1, $\chi'_{as}(P_2 \boxtimes W_n) \geq 2n+2$. To show $\chi'_{as}(P_2 \boxtimes W_n) \leq 2n+2$, define $\sigma : E(P_2 \boxtimes W_n) \to \{1, 2, 3, \dots, 2n+2\}$ as follows:
for $i \in \{0,1,2, \dots, n\}$,

$$\sigma(v_{i,1} v_{i,2}) = 1; \quad \sigma(v_{i,1} v_{i+1,2}) = 3; \quad \sigma(v_{i,2} v_{i+1,1}) = 4;$$

and

$$\sigma(v_{0,1} v_{1,1}) = \sigma(v_{0,2} v_{1,2}) = 2;$$

for $i \in \{1,2,3, \dots, n\}$,

$$\sigma(v_{0,1}v_{i,1}) = \sigma(v_{0,2}v_{i,2}) = 2i+1; \ \sigma(v_{0,1}v_{i,2}) = \sigma(v_{0,2}v_{i,1}) = 2(i+1);$$

for $i \in \{1,2,3,\dots,n-2\}$,

$$\sigma(v_{i,1}v_{i+1,1}) = 2i+5; \ \sigma(v_{i,2}v_{i+1,2}) = 2i+6;$$

and

$$\sigma(v_{n-1,1}v_{n,1}) = \sigma(v_{n-1,2}v_{n,2}) = 2; \ \sigma(v_{1,1}v_{n,1}) = \sigma(v_{1,2}v_{n,2}) = 5; \ \sigma(v_{1,1}v_{n,2}) = \sigma(v_{1,2}v_{n,1}) = 6.$$

First, one can see that, by construction, σ is a proper edge-coloring. It remains to show that σ is an AVD proper edge-coloring. We compare the sets of colors of adjacent vertices of the same degree.

The induced vertex-color sets are:

$$S_\sigma(v_{0,1}) = \{1,2,3,\ldots,2n+2\} \setminus \{4\}; \ S_\sigma(v_{0,2}) = \{1,2,3,\ldots,2n+2\} \setminus \{3\};$$
$$S_\sigma(v_{1,1}) = \{1,2,3,4,5,6,7\}; \ S_\sigma(v_{1,2}) = \{1,2,3,4,5,6,8\};$$

for $i \in \{2,3,4,\dots,n-2\}$,

$$S_\sigma(v_{i,1}) = \{1,3,4,2i+1,2i+2,2i+3,2i+5\};$$
$$S_\sigma(v_{i,2}) = \{1,3,4,2i+1,2i+2,2i+4,2i+6\};$$

and

$$S_\sigma(v_{n-1,1}) = \{1,2,3,4,2n-1,2n,2n+1\}; \ S_\sigma(v_{n-1,2}) = \{1,2,3,4,2n-1,2n,2n+2\};$$
$$S_\sigma(v_{n,1}) = \{1,2,4,5,6,2n+1,2n+2\}; \ S_\sigma(v_{n,2}) = \{1,2,3,5,6,2n+1,2n+2\}.$$

Observe that σ is an AVD proper edge-coloring of $P_2 \boxtimes W_n$. Hence, $\chi'_{as}(P_2 \boxtimes W_n) = 2(n+1)$.

(iii) Since $\Delta(P_3 \boxtimes W_3) = 11$, by Proposition 1.1, $\chi'_{as}(P_3 \boxtimes W_3) \geq 12$. To show $\chi'_{as}(P_3 \boxtimes W_3) \leq 12$, define $\sigma : E(P_3 \boxtimes W_3) \to \{1,2,3,\ldots,12\}$ as follows:

$$\sigma(v_{0,1}v_{0,2}) = \sigma(v_{1,1}v_{1,2}) = \sigma(v_{2,1}v_{2,2}) = \sigma(v_{3,1}v_{3,2}) = 1;$$
$$\sigma(v_{0,2}v_{0,3}) = \sigma(v_{1,2}v_{1,3}) = \sigma(v_{2,2}v_{2,3}) = \sigma(v_{3,2}v_{3,3}) = 2;$$
$$\sigma(v_{0,1}v_{1,1}) = \sigma(v_{0,2}v_{1,2}) = \sigma(v_{0,3}v_{1,3}) = \sigma(v_{2,1}v_{3,1}) = \sigma(v_{2,2}v_{3,2}) = \sigma(v_{2,3}v_{3,3}) = 3;$$
$$\sigma(v_{0,1}v_{1,2}) = \sigma(v_{0,2}v_{1,3}) = \sigma(v_{2,1}v_{3,2}) = \sigma(v_{2,2}v_{3,3}) = 4;$$
$$\sigma(v_{0,2}v_{1,1}) = \sigma(v_{0,3}v_{1,2}) = \sigma(v_{2,2}v_{3,1}) = \sigma(v_{2,3}v_{3,2}) = 5;$$
$$\sigma(v_{0,1}v_{2,1}) = \sigma(v_{0,2}v_{2,2}) = \sigma(v_{0,3}v_{2,3}) = 6;$$
$$\sigma(v_{0,1}v_{2,2}) = \sigma(v_{0,2}v_{2,3}) = \sigma(v_{1,1}v_{3,2}) = \sigma(v_{1,2}v_{3,3}) = 7;$$
$$\sigma(v_{0,2}v_{2,1}) = \sigma(v_{0,3}v_{2,2}) = \sigma(v_{1,2}v_{3,1}) = \sigma(v_{1,3}v_{3,2}) = 8;$$
$$\sigma(v_{0,1}v_{3,1}) = \sigma(v_{0,2}v_{3,2}) = \sigma(v_{0,3}v_{3,3}) = \sigma(v_{1,1}v_{2,1}) = \sigma(v_{1,2}v_{2,2}) = \sigma(v_{1,3}v_{2,3}) = 9;$$
$$\sigma(v_{0,1}v_{3,2}) = \sigma(v_{0,2}v_{3,3}) = \sigma(v_{1,1}v_{2,2}) = \sigma(v_{1,2}v_{2,3}) = 10;$$
$$\sigma(v_{0,2}v_{3,1}) = \sigma(v_{0,3}v_{3,2}) = \sigma(v_{1,2}v_{2,1}) = \sigma(v_{1,3}v_{2,2}) = 11;$$
$$\sigma(v_{1,1}v_{3,1}) = \sigma(v_{1,2}v_{3,2}) = \sigma(v_{1,3}v_{3,3}) = 12.$$

First, one can see that, by construction, σ is a proper edge-coloring. It remains to show that σ is an AVD proper edge-coloring. We compare the sets of colors of adjacent vertices of the same degree.

The induced vertex-color sets are:

$$S_\sigma(v_{0,1})=\{1,3,4,6,7,9,10\};\ S_\sigma(v_{1,1})=\{1,3,5,7,9,10,12\};$$
$$S_\sigma(v_{2,1})=\{1,3,4,6,8,9,11\};\ S_\sigma(v_{3,1})=\{1,3,5,8,9,11,12\};$$
$$S_\sigma(v_{0,2})=\{1,2,3,4,5,6,7,8,9,10,11\};\ S_\sigma(v_{1,2})=\{1,2,3,4,5,7,8,9,10,11,12\};$$
$$S_\sigma(v_{2,2})=\{1,2,3,4,5,6,7,8,9,10,11\};\ S_\sigma(v_{3,2})=\{1,2,3,4,5,7,8,9,10,11,12\};$$
$$S_\sigma(v_{0,3})=\{2,3,5,6,8,9,11\};\ S_\sigma(v_{1,3})=\{2,3,4,8,9,11,12\};$$
$$S_\sigma(v_{2,3})=\{2,3,5,6,7,9,10\};\ S_\sigma(v_{3,1})=\{2,3,4,7,9,10,12\}.$$

Observe that σ is an AVD proper edge-coloring of $P_3 \boxtimes W_3$. Hence, $\chi'_{as}(P_3 \boxtimes W_3)$=12.

(iv) For $n \geq 4$. Since $\Delta(P_3 \boxtimes W_n)=3n+2$. To show $\chi'_{as}(P_3 \boxtimes W_n)=3n+2$,

define σ: $E(P_3 \boxtimes W_n) \rightarrow \{1,2,3,\ldots,3n+2\}$ as follows:

for $i \in \{0,1,2,\ldots,n\}$,

$$\sigma(v_{i,1}v_{i,2})=1;\ \sigma(v_{i,2}v_{i,3})=2;$$

for $i \in \{1,2,3,\ldots,n\}$,

$$\sigma(v_{0,1}v_{i,1})=\sigma(v_{0,2}v_{i,2})=\sigma(v_{0,3}v_{i,3})=3i;$$
$$\sigma(v_{0,1}v_{i,2})=\sigma(v_{0,2}v_{i,3})=3i+1;\ \sigma(v_{0,2}v_{i,1})=\sigma(v_{0,3}v_{i,2})=3i+2;$$
$$\sigma(v_{1,1}v_{n,1})=\sigma(v_{1,2}v_{n,2})=\sigma(v_{1,3}v_{n,3})=6;$$
$$\sigma(v_{1,1}v_{n,2})=\sigma(v_{1,2}v_{n,3})=7;\ \sigma(v_{1,2}v_{n,1})=\sigma(v_{1,3}v_{n,2})=8;$$

for $i \in \{1,2,3,\ldots,n-2\}$,

$$\sigma(v_{i,1}v_{i+1,1})=\sigma(v_{i,2}v_{i+1,2})=\sigma(v_{i,3}v_{i+1,3})=3i+6;$$
$$\sigma(v_{i,1}v_{i+1,2})=\sigma(v_{i,2}v_{i+1,3})=3i+7;\ \sigma(v_{i,2}v_{i+1,1})=\sigma(v_{i,3}v_{i+1,2})=3i+8;$$
$$\sigma(v_{n-1,1}v_{n,1})=\sigma(v_{n-1,2}v_{n,2})=\sigma(v_{n-1,3}v_{n,3})=3;$$
$$\sigma(v_{n-1,1}v_{n,2})=\sigma(v_{n-1,2}v_{n,3})=4;\ \sigma(v_{n-1,2}v_{n,1})=\sigma(v_{n-1,3}v_{n,2})=5.$$

First, one can see that, by construction, σ is a proper edge-coloring. It remains to show that σ is an AVD proper edge-coloring. I compare the sets of colors of adjacent vertices of the same degree.

The induced vertex-color sets are:

$$S_\sigma(v_{0,1})=\{1\}\cup\{3,6,9,\ldots,3n\}\cup\{4,7,10,\ldots,3n+1\};$$
$$S_\sigma(v_{0,2})=\{1,2,3,\ldots,3n+2\};$$
$$S_\sigma(v_{0,3})=\{2\}\cup\{3,6,9,\ldots,3n\}\cup\{5,8,11,\ldots,3n+2\};$$
$$S_\sigma(v_{1,1})=\{1,3,5,6,7,9,10\};$$

$$S_\sigma(v_{1,2}) = \{1,2,3,4,5,6,7,8,9,10,11\};$$
$$S_\sigma(v_{1,3}) = \{2,3,4,6,8,9,11\};$$

for $i \in \{2,3,4,\dots,n-2\}$,

$$S_\sigma(v_{i,1}) = \{1,3i,3i+2,3i+3,3i+5,3i+6,3i+7\};$$
$$S_\sigma(v_{i,2}) = \{1,2,3i,3i+1,3i+2,3i+3,3i+4,3i+5,3i+6,3i+7,3i+8\};$$
$$S_\sigma(v_{i,3}) = \{2,3i,3i+1,3i+3,3i+4,3i+6,3i+8\};$$

and

$$S_\sigma(v_{n-1,1}) = \{1,3,4,3n-3,3n-1,3n,3n+2\};$$
$$S_\sigma(v_{n-1,2}) = \{1,2,3,4,5,3n-3,3n-2,3n-1,3n,3n+1,3n+2\};$$
$$S_\sigma(v_{n-1,3}) = \{2,3,5,3n-3,3n-2,3n,3n+1\};$$
$$S_\sigma(v_{n,1}) = \{1,3,5,6,8,3n,3n+2\};$$
$$S_\sigma(v_{n,2}) = \{1,2,3,4,5,6,7,8,3n,3n+1,3n+2\};\ S_\sigma(v_{n,3}) = \{2,3,4,6,7,3n,3n+1\}.$$

Observe that σ is an AVD proper edge-coloring of $P_3 \boxtimes W_n$. Hence, $\chi'_{as}(P_3 \boxtimes W_n)=3n+2$.

(v) Let $m \ge 4$ and $n \ge 3$, since $\Delta(P_m \boxtimes W_n) = 3n+2$, by Proposition 1.1, $\chi'_{as}(P_m \boxtimes W_n) \ge 3n+3$. To show $\chi'_{as}(P_m \boxtimes W_n) \le 3n+3$, define $\sigma : E(P_m \boxtimes W_n) \to \{1,2,3,\dots,3n+3\}$ as follows:

for $i \in \{0,1,2,\dots,n\}$ and $j \in \{1,2,3,\dots,m\}$,

$$\sigma(v_{i,j}v_{i,j+1}) = \begin{cases} 3 \text{ if } j \equiv 0 \pmod 3, \\ 1 \text{ if } j \equiv 1 \pmod 3, \\ 2 \text{ if } j \equiv 2 \pmod 3; \end{cases}$$

for $i \in \{1,2,3,\dots,n\}$ and $j \in \{2,3,4,\dots,m\}$, $\sigma(v_{0,j}v_{i,j-1})=3i+2$;
for $i \in \{1,2,3,\dots,n\}$ and $j \in \{1,2,3,\dots,m\}$, $\sigma(v_{0,j}v_{i,j})=3i+3$;
for $i \in \{1,2,3,\dots,n\}$ and $j \in \{1,2,3,\dots,m-1\}$, $\sigma(v_{0,j}v_{i,j+1}) = 3i+1$;
for $i \in \{1,2,3,\dots,n-2\}$ and $j \in \{2,3,4,\dots,m\}$, $\sigma(v_{i,j}v_{i+1,j-1})=3i+8$;
for $i \in \{1,2,3,\dots,n-2\}$ and $j \in \{1,2,3,\dots,m\}$, $\sigma(v_{i,j}v_{i+1,j})=3i+9$;
for $i \in \{1,2,3,\dots,n-2\}$ and $j \in \{1,2,3,\dots,m-1\}$, $\sigma(v_{i,j}v_{i+1,j+1}) = 3i+7$;
for $j \in \{2,3,4,\dots,m\}$, $\sigma(v_{n-1,j}v_{n,j-1})=5$; for $j \in \{1,2,3,\dots,m\}$, $\sigma(v_{n-1,j}v_{n,j})=6$;
for $j \in \{1,2,3,\dots,m-1\}$, $\sigma(v_{n-1,j}v_{n,j+1})=4$; for $j \in \{2,3,4,\dots,m\}$, $\sigma(v_{1,j}v_{n,j-1})=8$;
for $j \in \{1,2,3,\dots,m\}$, $\sigma(v_{1,j}v_{n,j})=9$; for $j \in \{1,2,3,\dots,m-1\}$, $\sigma(v_{1,j}v_{n,j+1})=7$.

First, one can see that, by construction, σ is a proper edge-coloring. It remains to show that σ is an AVD proper edge-coloring. I compare the sets of colors of adjacent vertices of the same degree.

The induced vertex-color sets are:

$$S_\sigma(v_{0,1}) = \{1\} \cup \{4,7,10,\dots,3n+1\} \cup \{6,9,12,\dots,3n+3\};$$

for $j \in \{2,3,4,\dots,m-1\}$,

$$S_\sigma(v_{0,j}) = \begin{cases} \{1,2,3,\dots,3n+3\}\backslash\{1\} \text{ if } j \equiv 0 \pmod 3, \\ \{1,2,3,\dots,3n+3\}\backslash\{2\} \text{ if } j \equiv 1 \pmod 3, \\ \{1,2,3,\dots,3n+3\}\backslash\{3\} \text{ if } j \equiv 2 \pmod 3; \end{cases}$$

and

$$S_\sigma(v_{0,m}) = \begin{cases} \{2\} \cup \{5,8,11,\dots,3n+2\} \cup \{6,9,12,\dots,3n+3\} \text{ if } m \equiv 0 \pmod 3, \\ \{3\} \cup \{5,8,11,\dots,3n+2\} \cup \{6,9,12,\dots,3n+3\} \text{ if } m \equiv 1 \pmod 3, \\ \{1\} \cup \{5,8,11,\dots,3n+2\} \cup \{6,9,12,\dots,3n+3\} \text{ if } m \equiv 2 \pmod 3; \end{cases}$$

$$S_\sigma(v_{1,1}) = \{1,5,6,7,9,10,12\};$$

for $j \in \{2,3,4,\dots,m-1\}$,

$$S_\sigma(v_{1,j}) = \begin{cases} \{2,3,4,5,6,7,8,9,10,11,12\} \text{ if } j \equiv 0 \pmod 3, \\ \{1,3,4,5,6,7,8,9,10,11,12\} \text{ if } j \equiv 1 \pmod 3, \\ \{1,2,4,5,6,7,8,9,10,11,12\} \text{ if } j \equiv 2 \pmod 3; \end{cases}$$

and

$$S_\sigma(v_{1,m}) = \begin{cases} \{2,4,6,8,9,11,12\} \text{ if } m \equiv 0 \pmod 3, \\ \{3,4,6,8,9,11,12\} \text{ if } m \equiv 1 \pmod 3, \\ \{1,4,6,8,9,11,12\} \text{ if } m \equiv 2 \pmod 3; \end{cases}$$

for $i \in \{2,3,4,\dots,n-1\}$,

$$S_\sigma(v_{i,1}) = \{1, 3i+2, 3i+3, 3i+5, 3i+6, 3i+7, 3i+9\};$$

for $i \in \{2,3,4,\dots,n-1\}$ and $j \in \{2,3,4,\dots,m-1\}$,

$$S_\sigma(v_{i,j}) = \begin{cases} \{2,3,3i+1,3i+2,3i+3,3i+4,3i+5,3i+6,3i+7,3i+8,3i+9\} \text{ if } j \equiv 0 \pmod 3, \\ \{1,3,3i+1,3i+2,3i+3,3i+4,3i+5,3i+6,3i+7,3i+8,3i+9\} \text{ if } j \equiv 1 \pmod 3, \\ \{1,2,3i+1,3i+2,3i+3,3i+4,3i+5,3i+6,3i+7,3i+8,3i+9\} \text{ if } j \equiv 2 \pmod 3; \end{cases}$$

for $i \in \{2,3,4,\dots,n-2\}$,

$$S_\sigma(v_{i,m}) = \begin{cases} \{2,3i+1,3i+3,3i+4,3i+6,3i+8,3i+9\} \text{ if } m \equiv 0 \pmod 3, \\ \{3,3i+1,3i+3,3i+4,3i+6,3i+8,3i+9\} \text{ if } m \equiv 1 \pmod 3, \\ \{1,3i+1,3i+3,3i+4,3i+6,3i+8,3i+9\} \text{ if } m \equiv 2 \pmod 3; \end{cases}$$

and

$$S_\sigma(v_{n-1,1}) = \{1,4,6,3n-1,3n,3n+2,3n+3\};$$

for $j \in \{2,3,4,\dots,m-1\}$,

$$S_\sigma(v_{n-1,j}) = \begin{cases} \{2,3,4,5,6,3n-2,3n-1,3n,3n+1,3n+2,3n+3\} \text{ if } j \equiv 0 \pmod 3, \\ \{1,3,4,5,6,3n-2,3n-1,3n,3n+1,3n+2,3n+3\} \text{ if } j \equiv 1 \pmod 3, \\ \{1,2,4,5,6,3n-2,3n-1,3n,3n+1,3n+2,3n+3\} \text{ if } j \equiv 2 \pmod 3; \end{cases}$$

and

$$S_\sigma(v_{n-1,m}) = \begin{cases} \{2,5,6,3n-2,3n,3n+1,3n+3\} \text{ if } m \equiv 0 \pmod 3, \\ \{3,5,6,3n-2,3n,3n+1,3n+3\} \text{ if } m \equiv 1 \pmod 3, \\ \{1,5,6,3n-2,3n,3n+1,3n+3\} \text{ if } m \equiv 2 \pmod 3; \end{cases}$$

$$S_\sigma(v_{n,1}) = \{1,5,6,8,9,3n+1,3n+3\};$$

for $j \in \{2,3,4,\dots,m-1\}$,

$$S_\sigma(v_{n,j}) = \begin{cases} \{2,3,4,5,6,7,8,9,3n+1,3n+2,3n+3\} \text{ if } j \equiv 0 \pmod 3, \\ \{1,3,4,5,6,7,8,9,3n+1,3n+2,3n+3\} \text{ if } j \equiv 1 \pmod 3, \\ \{1,2,4,5,6,7,8,9,3n+1,3n+2,3n+3\} \text{ if } j \equiv 2 \pmod 3; \end{cases}$$

and

$$S_\sigma(v_{n,m}) = \begin{cases} \{2,4,6,7,9,3n+1,3n+3\} \text{ if } m \equiv 0 \pmod 3, \\ \{3,4,6,7,9,3n+1,3n+3\} \text{ if } m \equiv 1 \pmod 3, \\ \{1,4,6,7,9,3n+1,3n+3\} \text{ if } m \equiv 2 \pmod 3. \end{cases}$$

Observe that σ is an AVD proper edge-coloring of $P_m \boxtimes W_n$. Hence, $\chi'_{as}(P_m \boxtimes W_n) = 3(n+1)$. ■

Theorem 2.2.

(i) $\chi'_{as}(P_2 \boxtimes K_{1,3}) = 9$.

(ii) If $n \geq 4$, then $\chi'_{as}(P_2 \boxtimes K_{1,n}) = 2(n+1)$.

(iii) $\chi'_{as}(P_3 \boxtimes K_{1,3}) = 12$.

(iv) If $n \geq 4$, then $\chi'_{as}(P_3 \boxtimes K_{1,n}) = 3n+2$.

(v) If $m \geq 4$ and $n \geq 3$, then $\chi'_{as}(P_m \boxtimes K_{1,n}) = 3(n+1)$.

Proof:

Proof follows in the way similar to Theorem 2.1. ■

Theorem 2.3.

(i) If $n \geq 2$, then $\chi'_{as}(P_2 \boxtimes F_n) = 2(n+1)$.

(ii) If $n \geq 2$, then $\chi'_{as}(P_3 \boxtimes F_n) = 3n+2$.

(iii) If $m \geq 4$ and $n \geq 2$, then $\chi'_{as}(P_m \boxtimes F_n) = 3(n+1)$.

Proof:

Proof follows in the way similar to Theorem 2.1. ■

Theorem 2.4.

For every integers $r, s \geq 3$ and $r + s \geq 7$, then $\chi'_{as}(P_r \boxtimes P_s) = 9$.

Proof:

Since $\Delta(P_r \boxtimes P_s) = 8$, by Proposition 1.1, $\chi'_{as}(P_r \boxtimes P_s) \geq 9$. To show $\chi'_{as}(P_r \boxtimes P_s) \leq 9$, define $\sigma : E(P_r \boxtimes P_s) \to \{1, 2, 3, \ldots, 9\}$ as follows:

for $i \in \{1,2,3, \ldots s\}$ and $j \in \{1,2,3, \ldots, r\}$,

$$\sigma(v_{i,j}v_{i,j+1}) = \begin{cases} 3 \text{ if } i \equiv 0 \pmod 5 \text{ and } j \text{ is odd,} \\ 2 \text{ if } i \equiv 0 \pmod 5 \text{ and } j \text{ is even;} \end{cases}$$

$$\sigma(v_{i,j}v_{i,j+1}) = \begin{cases} 4 \text{ if } i \equiv 1 \pmod 5 \text{ and } j \text{ is odd,} \\ 3 \text{ if } i \equiv 1 \pmod 5 \text{ and } j \text{ is even;} \end{cases}$$

$$\sigma(v_{i,j}v_{i,j+1}) = \begin{cases} 5 \text{ if } i \equiv 2 \pmod 5 \text{ and } j \text{ is odd,} \\ 4 \text{ if } i \equiv 2 \pmod 5 \text{ and } j \text{ is even;} \end{cases}$$

$$\sigma(v_{i,j}v_{i,j+1}) = \begin{cases} 1 \text{ if } i \equiv 3 \pmod 5 \text{ and } j \text{ is odd,} \\ 5 \text{ if } i \equiv 3 \pmod 5 \text{ and } j \text{ is even;} \end{cases}$$

$$\sigma(v_{i,j}v_{i,j+1}) = \begin{cases} 2 \text{ if } i \equiv 4 \pmod 5 \text{ and } j \text{ is odd,} \\ 1 \text{ if } i \equiv 4 \pmod 5 \text{ and } j \text{ is even;} \end{cases}$$

for $i \in \{1,2,3, \ldots s\}$ and $j \in \{1,2,3, \ldots, r\}$,

$$\sigma(v_{i,j}v_{i+1,j}) = \begin{cases} 5 \text{ if } i \equiv 0 \pmod 5 \text{ and } j \text{ is odd,} \\ 1 \text{ if } i \equiv 0 \pmod 5 \text{ and } j \text{ is even;} \end{cases}$$

$$\sigma(v_{i,j}v_{i+1,j}) = \begin{cases} 1 \text{ if } i \equiv 1 \pmod 5 \text{ and } j \text{ is odd,} \\ 2 \text{ if } i \equiv 1 \pmod 5 \text{ and } j \text{ is even;} \end{cases}$$

$$\sigma(v_{i,j}v_{i+1,j}) = \begin{cases} 2 \text{ if } i \equiv 2 \pmod 5 \text{ and } j \text{ is odd,} \\ 3 \text{ if } i \equiv 2 \pmod 5 \text{ and } j \text{ is even;} \end{cases}$$

$$\sigma(v_{i,j}v_{i+1,j}) = \begin{cases} 3 \text{ if } i \equiv 3 \pmod 5 \text{ and } j \text{ is odd,} \\ 4 \text{ if } i \equiv 3 \pmod 5 \text{ and } j \text{ is even;} \end{cases}$$

$$\sigma(v_{i,j}v_{i+1,j}) = \begin{cases} 4 \text{ if } i \equiv 4 \pmod 5 \text{ and } j \text{ is odd,} \\ 5 \text{ if } i \equiv 4 \pmod 5 \text{ and } j \text{ is even;} \end{cases}$$

for $i \in \{1,2,3, \dots s\}$ and $j \in \{1,2,3, \dots, r-1\}$,

$$\sigma(v_{i,j}v_{i+1,j+1}) = \begin{cases} 6 \text{ if } i \text{ is odd,} \\ 8 \text{ if } i \text{ is even;} \end{cases}$$

for $i \in \{1,2,3, \dots s\}$ and $j \in \{2,3,4, \dots, r\}$,

$$\sigma(v_{i,j}v_{i+1,j-1}) = \begin{cases} 7 \text{ if } i \text{ is odd,} \\ 9 \text{ if } i \text{ is even.} \end{cases}$$

First, one can see that, by construction, σ is a proper edge-coloring. It remains to show that σ is an AVD proper edge-coloring. I compare the sets of colors of adjacent vertices of the same degree.

Case (i): degree *3* vertices.

Since there are only four vertices of degree 3 and since they form an independent set in $P_r \boxtimes P_s$ there is no problem with these vertices.

The induced vertex-color sets are:

$$S_\sigma(v_{1,1}) = \{1,4,6\};$$

$$S_\sigma(v_{1,r}) = \begin{cases} \{1,3,7\} \text{ if } r \text{ is odd and } r \notin 1, \\ \{2,4,7\} \text{ if } r \text{ is even.} \end{cases}$$

$$S_\sigma(v_{s,1}) = \begin{cases} \{3,4,7\} \text{ if } s \equiv 0 \pmod{10}, \\ \{4,5,9\} \text{ if } s \equiv 1 \pmod{10}, \\ \{1,5,7\} \text{ if } s \equiv 2 \pmod{10}, \\ \{1,2,9\} \text{ if } s \equiv 3 \pmod{10}, \\ \{2,3,7\} \text{ if } s \equiv 4 \pmod{10}, \\ \{3,4,9\} \text{ if } s \equiv 5 \pmod{10}, \\ \{4,5,7\} \text{ if } s \equiv 6 \pmod{10}, \\ \{1,5,9\} \text{ if } s \equiv 7 \pmod{10}, \\ \{1,2,7\} \text{ if } s \equiv 8 \pmod{10}, \\ \{2,3,9\} \text{ if } s \equiv 9 \pmod{10}. \end{cases}$$

If r is odd and $r \notin 1$,

$$S_\sigma(v_{s,r}) = \begin{cases} \{2,4,6\} \text{ if } s \equiv 0 \pmod{10}, \\ \{3,5,8\} \text{ if } s \equiv 1 \pmod{10}. \\ \{1,4,6\} \text{ if } s \equiv 2 \pmod{10}, \\ \{2,5,8\} \text{ if } s \equiv 3 \pmod{10}, \\ \{1,3,6\} \text{ if } s \equiv 4 \pmod{10}, \\ \{2,4,8\} \text{ if } s \equiv 5 \pmod{10}, \\ \{3,5,6\} \text{ if } s \equiv 6 \pmod{10}, \\ \{1,4,8\} \text{ if } s \equiv 7 \pmod{10}, \\ \{2,5,6\} \text{ if } s \equiv 8 \pmod{10}, \\ \{1,3,8\} \text{ if } s \equiv 9 \pmod{10}. \end{cases}$$

If r *is even*,

$$S_\sigma(v_{s,r}) = \begin{cases} \{3,5,6\} \text{ if } s \equiv 0 \pmod{10}, \\ \{1,4,8\} \text{ if } s \equiv 1 \pmod{10}, \\ \{2,5,6\} \text{ if } s \equiv 2 \pmod{10}, \\ \{1,3,8\} \text{ if } s \equiv 3 \pmod{10}, \\ \{2,4,6\} \text{ if } s \equiv 4 \pmod{10}, \\ \{3,5,8\} \text{ if } s \equiv 5 \pmod{10}, \\ \{1,4,6\} \text{ if } s \equiv 6 \pmod{10}, \\ \{2,5,8\} \text{ if } s \equiv 7 \pmod{10}, \\ \{1,3,6\} \text{ if } s \equiv 8 \pmod{10}, \\ \{2,4,8\} \text{ if } s \equiv 9 \pmod{10}. \end{cases}$$

Case (ii): degree 5 vertices.

The induced vertex-color sets are:

for $i \in \{2,3,4,\dots,s-1\}$

$$S_\sigma(v_{i,1}) = \begin{cases} \{3,4,5,7,8\} \text{ if } i \equiv 0 \pmod{10}, \\ \{1,4,5,6,9\} \text{ if } i \equiv 1 \pmod{10}, \\ \{1,2,5,7,8\} \text{ if } i \equiv 2 \pmod{10}, \\ \{1,2,3,6,9\} \text{ if } i \equiv 3 \pmod{10}, \\ \{2,3,4,7,8\} \text{ if } i \equiv 4 \pmod{10}, \\ \{3,4,5,6,9\} \text{ if } i \equiv 5 \pmod{10}, \\ \{1,4,5,7,8\} \text{ if } i \equiv 6 \pmod{10}, \\ \{1,2,5,6,9\} \text{ if } i \equiv 7 \pmod{10}, \\ \{1,2,3,7,8\} \text{ if } i \equiv 8 \pmod{10}, \\ \{2,3,4,6,9\} \text{ if } i \equiv 9 \pmod{10}; \end{cases}$$

for $i \in \{2,3,4,\dots,s-1\}$ and r ($r \notin 1$) is odd,

$$S_\sigma(v_{i,r}) = \begin{cases} \{2,4,5,6,9\} \text{ if } i \equiv 0 \pmod{10}, \\ \{1,3,5,7,8\} \text{ if } i \equiv 1 \pmod{10}, \\ \{1,2,4,6,9\} \text{ if } i \equiv 2 \pmod{10}, \\ \{2,3,5,7,8\} \text{ if } i \equiv 3 \pmod{10}, \\ \{1,3,4,6,9\} \text{ if } i \equiv 4 \pmod{10}, \\ \{2,4,5,7,8\} \text{ if } i \equiv 5 \pmod{10}, \\ \{1,3,5,6,9\} \text{ if } i \equiv 6 \pmod{10}, \\ \{1,2,4,7,8\} \text{ if } i \equiv 7 \pmod{10}, \\ \{2,3,5,6,9\} \text{ if } i \equiv 8 \pmod{10}, \\ \{1,3,4,7,8\} \text{ if } i \equiv 9 \pmod{10}; \end{cases}$$

for $i \in \{2,3,4,\dots,s-1\}$ and r is even,

$$S_\sigma(v_{i,r}) = \begin{cases} \{1,3,5,6,9\} \text{ if } i \equiv 0 \pmod{10}, \\ \{1,2,4,7,8\} \text{ if } i \equiv 1 \pmod{10}, \\ \{2,3,5,6,9\} \text{ if } i \equiv 2 \pmod{10}, \\ \{1,3,4,7,8\} \text{ if } i \equiv 3 \pmod{10}, \\ \{2,4,5,6,9\} \text{ if } i \equiv 4 \pmod{10}, \\ \{1,3,5,7,8\} \text{ if } i \equiv 5 \pmod{10}, \\ \{1,2,4,6,9\} \text{ if } i \equiv 6 \pmod{10}, \\ \{2,3,5,7,8\} \text{ if } i \equiv 7 \pmod{10}, \\ \{1,3,4,6,9\} \text{ if } i \equiv 8 \pmod{10}, \\ \{2,4,5,7,8\} \text{ if } i \equiv 9 \pmod{10}. \end{cases}$$

Observe that the sets of colors are different.

Case (iii): degree 8 vertices.

The induced vertex-color sets are:

for $i \in \{2,3,4,\dots,s-1\}$ and $j \in \{2,3,4,\dots,r-1\}$,

$$S_\sigma(v_{i,j}) = \begin{cases} \{1,2,3,\dots,9\}\backslash\{1\} \text{ for } i \equiv 0 \pmod 5 \text{ and } j \text{ is odd}, \\ \{1,2,3,\dots,9\}\backslash\{4\} \text{ for } i \equiv 0 \pmod 5 \text{ and } j \text{ is even}, \\ \{1,2,3,\dots,9\}\backslash\{2\} \text{ for } i \equiv 1 \pmod 5 \text{ and } j \text{ is odd}, \\ \{1,2,3,\dots,9\}\backslash\{5\} \text{ for } i \equiv 1 \pmod 5 \text{ and } j \text{ is even}, \\ \{1,2,3,\dots,9\}\backslash\{3\} \text{ for } i \equiv 2 \pmod 5 \text{ and } j \text{ is odd}, \\ \{1,2,3,\dots,9\}\backslash\{1\} \text{ for } i \equiv 2 \pmod 5 \text{ and } j \text{ is even}, \\ \{1,2,3,\dots,9\}\backslash\{4\} \text{ for } i \equiv 3 \pmod 5 \text{ and } j \text{ is odd}, \\ \{1,2,3,\dots,9\}\backslash\{2\} \text{ for } i \equiv 3 \pmod 5 \text{ and } j \text{ is even}, \\ \{1,2,3,\dots,9\}\backslash\{5\} \text{ for } i \equiv 4 \pmod 5 \text{ and } j \text{ is odd}, \\ \{1,2,3,\dots,9\}\backslash\{3\} \text{ for } i \equiv 4 \pmod 5 \text{ and } j \text{ is even}. \end{cases}$$

Note that for any i and j two adjacent vertices, $2 \le i \le s-1$ and $2 \le j \le r-1$,

$$S_\sigma(v_{i,j}) \ne S_\sigma(v_{i+1,j});\ S_\sigma(v_{i,j}) \ne S_\sigma(v_{i-1,j});\ S_\sigma(v_{i,j}) \ne S_\sigma(v_{i,j-1});$$
$$S_\sigma(v_{i,j}) \ne S_\sigma(v_{i,j+1});\ S_\sigma(v_{i,j}) \ne S_\sigma(v_{i-1,j-1});\ S_\sigma(v_{i,j}) \ne S_\sigma(v_{i-1,j+1});$$
$$S_\sigma(v_{i,j}) \ne S_\sigma(v_{i+1,j-1});\ S_\sigma(v_{i,j}) \ne S_\sigma(v_{i+1,j+1}).$$

Thus the sets of colors are different and σ is an AVD proper edge-coloring of $P_r \boxtimes P_s$. Hence, $\chi'_{as}(P_r \boxtimes P_s)=9$. ■

Theorem 2.5.

If $m \ge 1$ and $n \ge 3$, then $\chi'_{as}(C_{3m} \boxtimes W_n)=3(n+1)$.

Proof:

Let $m \geq 1$ and $n \geq 3$. Since $\Delta(C_{3m} \boxtimes W_n)=3n+2$, by Proposition 1.1, $\chi'_{as}(C_{3m} \boxtimes W_n) \geq 3n+3$. To show $\chi'_{as}(C_{3m} \boxtimes W_n) \leq 3n+3$, define $\sigma: E(C_{3m} \boxtimes W_n) \to \{1,2,3,\ldots,3n+3\}$ as follows:

for $i \in \{0,1,2,\ldots,n\}$ and $j \in \{1,2,3,\ldots,m\}$,

$$\sigma(v_{i,j}v_{i,j+1}) = \begin{cases} 3 \text{ if } j \equiv 0 \pmod 3, \\ 1 \text{ if } j \equiv 1 \pmod 3, \\ 2 \text{ if } j \equiv 2 \pmod 3; \end{cases}$$

for $i \in \{1,2,3,\ldots,n\}$ and $j \in \{2,3,4,\ldots,m\}$, $\sigma(v_{0,j}v_{i,j-1})=3i+2$;

for $i \in \{1,2,3,\ldots,n\}$ and $j \in \{1,2,3,\ldots,m\}$, $\sigma(v_{0,j}v_{i,j})=3i+3$;

for $i \in \{1,2,3,\ldots,n\}$ and $j \in \{1,2,3,\ldots,m-1\}$, $\sigma(v_{0,j}v_{i,j+1}) = 3i+1$;

for $i \in \{1,2,3,\ldots,n-2\}$ and $j \in \{2,3,4,\ldots,m\}$, $\sigma(v_{i,j}v_{i+1,j-1})=3i+8$;

for $i \in \{1,2,3,\ldots,n-2\}$ and $j \in \{1,2,3,\ldots,m\}$, $\sigma(v_{i,j}v_{i+1,j}) = 3i+9$;

for $i \in \{1,2,3,\ldots,n-2\}$ and $j \in \{1,2,3,\ldots,m-1\}$, $\sigma(v_{i,j}v_{i+1,j+1}) = 3i+7$;

for $j \in \{1,2,3,\ldots,m-1\}$, $\sigma(v_{n-1,j}v_{n,j+1})=4$; for $j \in \{2,3,4,\ldots,m\}$, $\sigma(v_{n-1,j}v_{n,j-1})=5$;

for $j \in \{1,2,3,\ldots,m\}$, $\sigma(v_{n-1,j}v_{n,j})=6$; for $j \in \{1,2,3,\ldots,m-1\}$, $\sigma(v_{1,j}v_{n,j+1})=7$;

for $j \in \{2,3,4,\ldots,m\}$, $\sigma(v_{1,j}v_{n,j-1})=8$; for $j \in \{1,2,3,\ldots,m\}$, $\sigma(v_{1,j}v_{n,j})=9$.

First, one can see that, by construction, σ is a proper edge-coloring. It remains to show that σ is an AVD proper edge-coloring. I compare the sets of colors of adjacent vertices of the same degree.

The induced vertex-color sets are:

for $j \in \{1,2,3,\ldots,m\}$,

$$S_\sigma(v_{0,j}) = \begin{cases} \{1,2,3,\ldots,3n+3\}\backslash\{1\} \text{ if } j \equiv 0 \pmod 3, \\ \{1,2,3,\ldots,3n+3\}\backslash\{2\} \text{ if } j \equiv 1 \pmod 3, \\ \{1,2,3,\ldots,3n+3\}\backslash\{3\} \text{ if } j \equiv 2 \pmod 3; \end{cases}$$

for $j \in \{1,2,3,\ldots,m\}$,

$$S_\sigma(v_{1,j}) = \begin{cases} \{1,2,3,\ldots,12\}\backslash\{1\} \text{ if } j \equiv 0 \pmod 3, \\ \{1,2,3,\ldots,12\}\backslash\{2\} \text{ if } j \equiv 1 \pmod 3, \\ \{1,2,3,\ldots,12\}\backslash\{3\} \text{ if } j \equiv 2 \pmod 3; \end{cases}$$

for $i \in \{2,3,4,\ldots,n-1\}$ and $j \in \{1,2,3,\ldots,m\}$,

$$S_\sigma(v_{i,j}) = \begin{cases} \{2,3,3i+1,3i+2,3i+3,3i+4,3i+5,3i+6,3i+7,3i+8,3i+9\} \text{ if } j \equiv 0 \pmod 3, \\ \{1,3,3i+1,3i+2,3i+3,3i+4,3i+5,3i+6,3i+7,3i+8,3i+9\} \text{ if } j \equiv 1 \pmod 3, \\ \{1,2,3i+1,3i+2,3i+3,3i+4,3i+5,3i+6,3i+7,3i+8,3i+9\} \text{ if } j \equiv 2 \pmod 3; \end{cases}$$

for $j \in \{1,2,3,\dots,m\}$,

$$S_\sigma(v_{n-1,j}) = \begin{cases} \{2,3,4,5,6,3n-2,3n-1,3n,3n+1,3n+2,3n+3\} \text{ if } j \equiv 0 \pmod 3, \\ \{1,3,4,5,6,3n-2,3n-1,3n,3n+1,3n+2,3n+3\} \text{ if } j \equiv 1 \pmod 3, \\ \{1,2,4,5,6,3n-2,3n-1,3n,3n+1,3n+2,3n+3\} \text{ if } j \equiv 2 \pmod 3; \end{cases}$$

for $j \in \{1,2,3,\dots,m\}$,

$$S_\sigma(v_{n,j}) = \begin{cases} \{2,3,4,5,6,7,8,9,3n+1,3n+2,3n+3\} \text{ if } j \equiv 0 \pmod 3, \\ \{1,3,4,5,6,7,8,9,3n+1,3n+2,3n+3\} \text{ if } j \equiv 1 \pmod 3, \\ \{1,2,4,5,6,7,8,9,3n+1,3n+2,3n+3\} \text{ if } j \equiv 2 \pmod 3. \end{cases}$$

Observe that σ is an AVD proper edge-coloring of $C_{3m} \boxtimes W_n$. Hence, $\chi'_{as}(C_{3m} \boxtimes W_n) = 3(n+1)$. ■

Theorem 2.6.

If $m \geq 1$ and $n \geq 3$, then $\chi'_{as}(C_{3m} \boxtimes K_{1,n}) = 3(n+1)$.

Proof:

Proof follows in the way similar to Theorem 2.5. ■

Theorem 2.7.

If $m \geq 1$ and $n \geq 2$, then $\chi'_{as}(C_{3m} \boxtimes F_n) = 3(n+1)$.

Proof:

Proof follows in the way similar to Theorem 2.5. ■

Theorem 2.8.

For every integer $s \geq 1$, we have $\chi'_{as}(P_3 \boxtimes C_{3s}) = 9$.

Proof:

Since $\Delta(P_3 \boxtimes C_{3s}) = 8$, by Proposition 1.1, $\chi'_{as}(P_3 \boxtimes C_{3s}) \geq 9$. To show $\chi'_{as}(P_3 \boxtimes C_{3s}) \leq 9$, define $\sigma : E(P_3 \boxtimes C_{3s}) \to \{1,2,3,\dots,9\}$ as follows:

Case (i): If s is odd

for $i \in \{1,2,3,\dots,3s\}$,

$$\sigma(v_{i,1}v_{i+1,1}) = \begin{cases} 3 \text{ if } i \equiv 0 \pmod 3, \\ 1 \text{ if } i \equiv 1 \pmod 3, \\ 2 \text{ if } i \equiv 2 \pmod 3; \end{cases}$$

$$\sigma(v_{i,2}v_{i+1,2}) = \begin{cases} 1 \text{ if } i \equiv 0 \pmod 3, \\ 2 \text{ if } i \equiv 1 \pmod 3, \\ 3 \text{ if } i \equiv 2 \pmod 3; \end{cases}$$

$$\sigma(v_{i,3}v_{i+1,3}) = \begin{cases} 2 \text{ if } i \equiv 0 \pmod 3, \\ 3 \text{ if } i \equiv 1 \pmod 3, \\ 1 \text{ if } i \equiv 2 \pmod 3; \end{cases}$$

for $i \in \{1,2,3, \dots ,3s\}$,

$$\sigma(v_{i,1}v_{i+1,2}) = \sigma(v_{i,2}v_{i+1,3}) = \begin{cases} 4 \text{ if } i \text{ is odd}, \\ 6 \text{ if } i \text{ is even}; \end{cases}$$
$$\sigma(v_{i,2}v_{i+1,1}) = \sigma(v_{i,3}v_{i+1,2}) = \begin{cases} 5 \text{ if } i \text{ is odd}, \\ 7 \text{ if } i \text{ is even}; \end{cases}$$

for $i \in \{1,2,3, \dots ,3s\}$

$$\sigma(v_{i,1}v_{i,2}) = 8; \ \ \sigma(v_{i,2}v_{i,3}) = 9;$$

And

$$\sigma(v_{1,1}v_{3s,2}) = 5; \ \sigma(v_{1,2}v_{3s,1}) = 6; \ \sigma(v_{1,2}v_{3s,3}) = 7; \ \sigma(v_{1,3}v_{3s,2}) = 4.$$

First, one can see that, by construction, σ is a proper edge-coloring. It remains to show that σ is an AVD proper edge-coloring. I compare the sets of colors of adjacent vertices of the same degree.

The induced vertex-color sets are:

$$S_\sigma(v_{1,1}) = \{1,3,4,5,8\};$$

for $i \in \{2,3,4, \dots ,3s-1\}$,

$$S_\sigma(v_{i,1}) = \begin{cases} \{2,3,5,6,8\} \text{ if } i \equiv 0 \pmod 6, \\ \{1,3,4,7,8\} \text{ if } i \equiv 1 \pmod 6, \\ \{1,2,5,6,8\} \text{ if } i \equiv 2 \pmod 6, \\ \{2,3,4,7,8\} \text{ if } i \equiv 3 \pmod 6, \\ \{1,3,5,6,8\} \text{ if } i \equiv 4 \pmod 6, \\ \{1,2,4,7,8\} \text{ if } i \equiv 5 \pmod 6; \end{cases}$$

$$S_\sigma(v_{3s,1}) = \{2,3,6,7,8\};$$

for $i \in \{1,2,3, \dots ,3s\}$,

$$S_\sigma(v_{i,2}) = \begin{cases} \{1,2,3, \dots ,9\} \backslash \{2\} \text{ if } i \equiv 0 \pmod 3, \\ \{1,2,3, \dots ,9\} \backslash \{3\} \text{ if } i \equiv 1 \pmod 3, \\ \{1,2,3, \dots ,9\} \backslash \{1\} \text{ if } i \equiv 2 \pmod 3; \end{cases}$$

$$S_\sigma(v_{1,3}) = \{2,3,4,5,9\};$$

for $i \in \{2,3,4, \dots ,3s-1\}$,

$$S_\sigma(v_{i,3}) = \begin{cases} \{1,2,4,7,9\} \text{ if } i \equiv 0 \pmod 6, \\ \{2,3,5,6,9\} \text{ if } i \equiv 1 \pmod 6, \\ \{1,3,4,7,9\} \text{ if } i \equiv 2 \pmod 6, \\ \{1,2,5,6,9\} \text{ if } i \equiv 3 \pmod 6, \\ \{2,3,4,7,9\} \text{ if } i \equiv 4 \pmod 6, \\ \{1,3,5,6,9\} \text{ if } i \equiv 5 \pmod 6; \end{cases}$$

$$S_\sigma(v_{3s,3}) = \{1,2,6,7,9\}.$$

Observe that σ is an AVD proper edge-coloring of $P_3 \boxtimes C_{3s}$.

Case (ii): If s is even

for $i \in \{1,2,3, \dots ,3s\}$

$$\sigma(v_{i,1}v_{i+1,1}) = \begin{cases} 3 \text{ if } i \equiv 0 \pmod 3, \\ 1 \text{ if } i \equiv 1 \pmod 3, \\ 2 \text{ if } i \equiv 2 \pmod 3; \end{cases}$$

$$\sigma(v_{i,2}v_{i+1,2}) = \begin{cases} 1 \text{ if } i \equiv 0 \pmod 3, \\ 2 \text{ if } i \equiv 1 \pmod 3, \\ 3 \text{ if } i \equiv 2 \pmod 3; \end{cases}$$

$$\sigma(v_{i,3}v_{i+1,3}) = \begin{cases} 2 \text{ if } i \equiv 0 \pmod 3, \\ 3 \text{ if } i \equiv 1 \pmod 3, \\ 1 \text{ if } i \equiv 2 \pmod 3; \end{cases}$$

for $i \in \{1,2,3, \dots ,3s\}$

$$\sigma(v_{i,1}v_{i+1,2}) = \sigma(v_{i,2}v_{i+1,3}) = \begin{cases} 4 \text{ if } i \text{ is odd}, \\ 6 \text{ if } i \text{ is even}; \end{cases}$$

$$\sigma(v_{i,2}v_{i+1,1}) = \sigma(v_{i,3}v_{i+1,2}) = \begin{cases} 5 \text{ if } i \text{ is odd}, \\ 7 \text{ if } i \text{ is even}; \end{cases}$$

for $i \in \{1,2,3, \dots ,3s\}$,

$$\sigma(v_{i,1}v_{i,2}) = 8; \ \sigma(v_{i,2}v_{i,3}) = 9;$$

and

$$\sigma(v_{1,1}v_{3s,2}) = \sigma(v_{1,2}v_{3s,3}) = 6; \ \sigma(v_{1,2}v_{3s,1}) = \sigma(v_{1,3}v_{3s,2}) = 7.$$

First, one can see that, by construction, σ is a proper edge-coloring. It remains to show that σ is an AVD proper edge-coloring. I compare the sets of colors of adjacent vertices of the same degree.

The induced vertex-color sets are:

$$S_\sigma(v_{1,1}) = \{1,3,4,6,8\};$$

for $i \in \{2,3,4, \dots ,3s-1\}$,

$$S_\sigma(v_{i,1})=\begin{cases}\{2,3,5,6,8\} \text{ if } i\equiv 0 \pmod 6,\\ \{1,3,4,7,8\} \text{ if } i\equiv 1 \pmod 6,\\ \{1,2,5,6,8\} \text{ if } i\equiv 2 \pmod 6,\\ \{2,3,4,7,8\} \text{ if } i\equiv 3 \pmod 6,\\ \{1,3,5,6,8\} \text{ if } i\equiv 4 \pmod 6,\\ \{1,2,4,7,8\} \text{ if } i\equiv 5 \pmod 6;\end{cases}$$

$$S_\sigma(v_{3s,1})=\{2,3,5,7,8\};$$

for $i\in\{1,2,3,\ldots,3s\}$,

$$S_\sigma(v_{i,2})=\begin{cases}\{1,2,3,\ldots,9\}\backslash\{2\} \text{ if } i\equiv 0 \pmod 3,\\ \{1,2,3,\ldots,9\}\backslash\{3\} \text{ if } i\equiv 1 \pmod 3,\\ \{1,2,3,\ldots,9\}\backslash\{1\} \text{ if } i\equiv 2 \pmod 3,\end{cases}$$

$$S_\sigma(v_{1,3})=\{2,3,5,7,9\};$$

for $i\in\{2,3,4,\ldots,3s-1\}$,

$$S_\sigma(v_{i,3})=\begin{cases}\{1,2,4,7,9\} \text{ if } i\equiv 0 \pmod 6,\\ \{2,3,5,6,9\} \text{ if } i\equiv 1 \pmod 6,\\ \{1,3,4,7,9\} \text{ if } i\equiv 2 \pmod 6,\\ \{1,2,5,6,9\} \text{ if } i\equiv 3 \pmod 6,\\ \{2,3,4,7,9\} \text{ if } i\equiv 4 \pmod 6,\\ \{1,3,5,6,9\} \text{ if } i\equiv 5 \pmod 6;\end{cases}$$

$$S_\sigma(v_{3s,3})=\{1,2,4,6,9\}.$$

Observe that σ is an AVD proper edge-coloring of $P_3\boxtimes C_{3s}$. Hence, $\chi'_{as}(P_3\boxtimes C_{3s})=9$.

3. Graph with $\chi'_{as}(G)\geq\Delta+2$

Theorem 3.1.

Let G be a k-regular graph of odd order at least $k+2$. If for any two nonadjacent vertices u and v, $N_G(u)\cup N_G(v)=V(G)\backslash\{u,v\}$, then $\chi'_{as}(G)\geq k+2$.

Proof:

Suppose $\chi'_{as}(G)=k+1$. Then there exists a AVD proper edge $(k+1)$-coloring $\sigma: E(G)\to N$. As G is a k-regular, for any vertex x, some color σ_i is not represented for the edges incident at x. Since there are $k+1$ colors and $|V(G)|>k+1$, by pigeonhole principle, some color, say, i is not represented at two vertices. Since S_σ is equal for these two vertices, they are non adjacent. Let the two non adjacent vertices be u and v. By hypothesis, i must be represented at all the vertices of $V(G)\backslash\{u,v\}$. This is clearly impossible, since $|V(G)\backslash\{u,v\}|$ is odd. Thus $\chi'_{as}(G)\geq k+2$.

In particular, consider $K_5-E(C_5)=C_5$, then $\chi'_{as}(K_5-E(C_5))=5$.

Problem 3.1.

If possible, to find an AVD proper edge-coloring with $2+\Delta$-color of $K_n - E(H)$ where H is a triangle free 2-factor of K_n.

4. Conclusion

In this paper, I compute AVD of strong product of simple graph. Calculation of AVD for simple graphs is important because the AVD of more complex graphs can be calculated by using AVD of its simple parts. Very good practical results can be achieved if the AVD is calculated for some real networks. This parameter is of particular interest because it is considered to be a reasonable measure for strong product of any graph.

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