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Blow up of Solutions for a Coupled Kirchhoff-type Equations with Degenerate Damping Terms

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Abstract

In this paper, we investigate a system of coupled Kirchhoff-type equations with degenerate damping terms. We prove a nonexistence of global solutions with positive initial energy. Later, we give some estimates for lower bound of the blow up time.

Keywords: Nonexistence of solutions; Kirchhoff-type equation; Degenerate damping

MSC 2010 No.: 35B44, 35G61, 35L75

1. Introduction

In this paper, we study the following initial-boundary value problem for the coupled nonlinear Kirchhoff-type equations with degenerate damping and source terms

$$\begin{cases} u_{tt} - M(\|\nabla u\|^2 + \|\nabla v\|^2)\Delta u + \left(|u|^k + |v|^l\right)|u_t|^{p-1} u_t = f_1(u, v), & (x,t) \in \Omega \times (0,T), \\ v_{tt} - M(\|\nabla u\|^2 + \|\nabla v\|^2)\Delta v + \left(|v|^\theta + |u|^\varrho\right)|v_t|^{q-1} v_t = f_2(u, v), & (x,t) \in \Omega \times (0,T), \\ u(x,t) = v(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & x \in \Omega, \\ v(x,0) = v_0(x),\ v_t(x,0) = v_1(x), & x \in \Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain with smooth boundary $\partial\Omega$ in R^n $(n = 1, 2, 3)$; $p, q \geq 1$, $k, l, \theta, \varrho \geq 0$; $f_i(.\,,.) : R^2 \longrightarrow R$ are given functions to be specified later. $M(s)$ is a locally Lipschitz function.

In the case of $M(s) \equiv 1$, Rammaha and Sakuntasathien (2010) considered the following system

$$\begin{cases} u_{tt} - \triangle u + \left(|u|^k + |v|^l\right)|u_t|^{p-1} u_t = f_1(u,v), \\ v_{tt} - \triangle v + \left(|v|^\theta + |u|^\varrho\right)|v_t|^{q-1} v_t = f_2(u,v). \end{cases} \tag{2}$$

They studied the global well posedness of the solution of the problem (2). Benaissa et al. (2012) and Zennir (2013) considered the same problem treated in Rammaha and Sakuntasathien (2010), and he studied the blow up and growth properties. Also, some authors studied the system with degenerate damping terms (see Pişkin (2015a), Zennir (2014) and Wu (2013)).

Ye (2016) considered the following system

$$\begin{cases} u_{tt} - M(\|\nabla u\|^2 + \|\nabla v\|^2)\Delta u + |u_t|^{p-1} u_t = f_1(u,v), \\ v_{tt} - M(\|\nabla u\|^2 + \|\nabla v\|^2)\Delta v + |v_t|^{q-1} v_t = f_2(u,v), \end{cases} \tag{3}$$

with initial-boundary conditions. He proved the global existence and the energy decay results. Narasimha (1968) introduced the model (3) for studying the nonlinear vibrations of an elasting string.

Motivated by the above studies, in this paper we proved a blow up of solutions for (1). However, when both Kirchhoff-type terms ($M(s)$) and degenerate damping terms are present, then the analysis of their interaction is not easy.

This paper is organized as follows. In Section 2, we give some lemmas, assumptions and the local existence theorem. In Section 3, we state and prove a blow up of solutions. In Section 4, some estimates for lower bound of the blow up time is given.

2. Preliminaries

In this section, we shall give some lemmas and assumptions which will be used throughout this paper. Let $\|.\|$ and $\|.\|_p$ denote the usual $L^2(\Omega)$ norm and $L^p(\Omega)$ norm, respectively.

To state and prove our main result, let's assume that

$(\mathbf{A1})$ $M(s)$ is a nonnegative C^1 function for $s \geq 0$ satisfying

$$M(s) = 1 + s^\gamma,\ \gamma > 1.$$

$(\mathbf{A2})$ For the nonlinear term, we assume

$$\begin{cases} p, q \geq 1, & \text{if } n = 1, 2, \\ 1 \leq p, q \leq 5, & \text{if } n = 3. \end{cases}$$

The nonlinear source terms $f_1(u,v)$ and $f_2(u,v)$ satisfy

$$f_1(u,v) = a|u+v|^{2(r+1)}(u+v) + b|u|^r u|v|^{r+2},$$
$$f_2(u,v) = a|u+v|^{2(r+1)}(u+v) + b|v|^r v|u|^{r+2},$$

where $a, b > 0$ are constants and r satisfies

$$\begin{cases} -1 < r, & \text{if } n = 1, 2, \\ -1 < r \leq 1, & \text{if } n = 3. \end{cases} \tag{4}$$

One can easily verify that

$$uf_1(u,v) + vf_2(u,v) = 2(r+2) F(u,v), \ \forall (u,v) \in R^2, \tag{5}$$

where

$$F(u,v) = \frac{1}{2(r+2)} \left[a|u+v|^{2(r+2)} + 2b|uv|^{r+2} \right]. \tag{6}$$

For the sake of simplicity, we take $a = b = 1$ throughout this paper. We define the energy function as follows

$$E(t) = \frac{1}{2}\left(\|u_t\|^2 + \|v_t\|^2\right) + \frac{1}{2}(\|\nabla u\|^2 + \|\nabla v\|^2) + \frac{1}{2(\gamma+1)}(\|\nabla u\|^2 + \|\nabla v\|^2)^{\gamma+1} - \int_\Omega F(u,v)\,dx. \tag{7}$$

Lemma 2.1. (Messaoudi and Houari (2010))

Let c_0 and c_1 positive constants. Then, we have following inequality

$$c_0\left(|u|^{2(r+2)} + |v|^{2(r+2)}\right) \leq 2(r+2) F(u,v) \leq c_1\left(|u|^{2(r+2)} + |v|^{2(r+2)}\right). \tag{8}$$

Lemma 2.2. (Sobolev-Poincare inequality) (Adams and Fournier (2003))

Let $2 \leq q < \infty$ $(n = 1,2)$ or $2 \leq q \leq 2n/(n-2)$ $(n \geq 3)$. Then, there exists a constant $C_* = C_*(\Omega, q)$ such that

$$\|u\|_q \leq C_* \|\nabla u\| \text{ for } u \in H_0^1(\Omega).$$

Lemma 2.3. (Messaoudi (2001))

Assume that

$$p \leq 2\frac{n-1}{n-2}, \quad n \geq 3,$$

holds. Then, there exists a positive constant $C > 1$ depending on Ω only such that

$$\|u\|_p^s \leq C\left(\|\nabla u\|^2 + \|u\|_p^p\right),$$

for any, $u \in H_0^1(\Omega), \ 2 \leq s \leq p$.

Lemma 2.4.

Let $E(t)$ be a energy functional of problem (1). Then we have

$$\frac{d}{dt}E(t) = -\int_\Omega \left(|u|^k + |v|^l\right)|u_t|^{p+1}dx - \int_\Omega \left(|v|^\theta + |u|^\varrho\right)|v_t|^{q+1}dx. \tag{9}$$

Proof:

Multiplying the first equation in (1) by u_t and the second one by v_t, integrating over Ω. Then, we obtain

$$\int_0^t E'(\tau)d\tau = -\int_0^t \int_\Omega (\left(|u|^k + |v|^l\right)|u_\tau|^{p+1} + \left(|v|^\theta + |u|^\varrho\right)|v_\tau|^{q+1})dxd\tau,$$

$$E(t) - E(0) = -\int_0^t \int_\Omega \left(\left(|u|^k + |v|^l\right)|u_\tau|^{p+1} + \left(|v|^\theta + |u|^\varrho\right)|v_\tau|^{q+1}\right)dxd\tau \text{ for } t \geq 0. \quad (10)$$

■

The local existence and uniqueness of solutions for the problem (1) which can be established by combining arguments of Georgiev and Todorova (1994), Ono (1997), Pişkin (2015b), Rammaha and Sakuntasathien (2010).

Theorem 2.5. (Local existence)

Suppose that (A1), (A2) and (4) hold. Let $u_0, v_0 \in H_0^1(\Omega) \cap L^{r+1}(\Omega)$ and $u_1, v_1 \in L^2(\Omega)$ are given. Then, problem (1) has a unique solution satisfying

$$u, v \in C\left([0,T); H_0^1(\Omega) \cap L^{r+1}(\Omega)\right),$$

$$u_t \in C\left([0,T); L^2(\Omega) \cap L^{p+1}(\Omega \times [0,T))\right),$$

$$v_t \in C\left([0,T); L^2(\Omega) \cap L^{q+1}(\Omega \times [0,T))\right),$$

for some $T > 0$.

3. Blow up of solutions

In this section, we state and prove the blow up results. Firstly, we give the following two lemmas.

Lemma 3.1. (Houari (2010))

Assume that (4) holds. Then, there exists $\eta > 0$ such that for any $(u,v) \in H_0^1(\Omega) \times H_0^1(\Omega)$ the inequality

$$\|u+v\|_{2(r+2)}^{2(r+2)} + 2\|uv\|_{r+2}^{r+2} \leq \eta\left(\|\nabla u\|^2 + \|\nabla v\|^2\right)^{r+2}, \quad (11)$$

holds.

We introduce the following:

$$B = \eta^{\frac{1}{2(r+2)}},\ \alpha_1 = B^{-\frac{r+2}{r+1}},\ E_1 = \left(\frac{1}{2} - \frac{1}{2(r+2)}\right)\alpha_1^2, \quad (12)$$

where η is the optimal constant in (11).

Lemma 3.2.

Suppose that assumptions (A1), (A2) and (4) hold. Let (u, v) be a solution of (1). Moreover, assume that $E(0) < E_1$ and

$$\left(\|\nabla u_0\|^2 + \|\nabla v_0\|^2\right)^{\frac{1}{2}} > \alpha_1. \tag{13}$$

Then there exists a constant $\alpha_2 > \alpha_1$ such that

$$\left(\|\nabla u\|^2 + \|\nabla v\|^2 + \frac{1}{\gamma+1}(\|\nabla u\|^2 + \|\nabla v\|^2)^{\gamma+1}\right)^{\frac{1}{2}} > \alpha_2, \text{ for } t > 0, \tag{14}$$

$$\left(\|u+v\|_{2(r+2)}^{2(r+2)} + 2\|uv\|_{r+2}^{r+2}\right)^{\frac{1}{2(r+2)}} \geq B\alpha_2, \text{ for } t > 0, \tag{15}$$

for all $t \in [0, T)$.

Proof:

Our techniques of proof follows carefully the steps in Vitillaro (1999), with necessary modifications imposed by the nature of our problem. We first note that by (7), (11) and the definition of B, we have

$$\begin{aligned}
E(t) &\geq \frac{1}{2}(\|\nabla u\|^2 + \|\nabla v\|^2) + \frac{1}{2(\gamma+1)}(\|\nabla u\|^2 + \|\nabla v\|^2)^{\gamma+1} - \int_\Omega F(u,v)\,dx \\
&= \frac{1}{2}(\|\nabla u\|^2 + \|\nabla v\|^2) + \frac{1}{2(\gamma+1)}(\|\nabla u\|^2 + \|\nabla v\|^2)^{\gamma+1} \\
&\quad - \frac{1}{2(r+2)}(\|u+v\|_{2(r+2)}^{2(r+2)} + 2\|uv\|_{r+2}^{r+2}) \\
&\geq \frac{1}{2}(\|\nabla u\|^2 + \|\nabla v\|^2) + \frac{1}{2(\gamma+1)}(\|\nabla u\|^2 + \|\nabla v\|^2)^{\gamma+1} \\
&\quad - \frac{1}{2(r+2)}\eta(\|\nabla u\|^2 + \|\nabla v\|^2)^{r+2} \\
&\geq \frac{1}{2}(\|\nabla u\|^2 + \|\nabla v\|^2) + \frac{1}{2(\gamma+1)}(\|\nabla u\|^2 + \|\nabla v\|^2)^{\gamma+1} \\
&\quad - \frac{B^{2(r+2)}}{2(r+2)}(\|\nabla u\|^2 + \|\nabla v\|^2 + \frac{1}{(\gamma+1)}(\|\nabla u\|^2 + \|\nabla v\|^2)^{\gamma+1})^{r+2} \\
&= \frac{1}{2}\alpha^2 - \frac{B^{2(r+2)}}{2(r+2)}\alpha^{2(r+2)} = G(\alpha),
\end{aligned} \tag{16}$$

where

$$\alpha = (\|\nabla u\|^2 + \|\nabla v\|^2 + \frac{1}{(\gamma+1)}(\|\nabla u\|^2 + \|\nabla v\|^2)^{\gamma+1})^{1/2}.$$

It is not difficult to verify that G is increasing for $0 < \alpha < \alpha_1$, decreasing for $\alpha > \alpha_1$, $G(\alpha) \to -\infty$ as $\alpha \to \infty$, and

$$G(\alpha_1) = \frac{1}{2}\alpha_1^2 - \frac{B^{2(r+2)}}{2(r+2)}\alpha_1^{2(r+2)} = E_1, \tag{17}$$

where α_1 is given in (12). Since $E(0) < E_1$, there exists $\alpha_2 > \alpha_1$ such that $G(\alpha_2) = E(0)$.

Set $\alpha_0 = (\|\nabla u_0\|^2 + \|\nabla v_0\|^2 + \frac{1}{(\gamma+1)}(\|\nabla u_0\|^2 + \|\nabla v_0\|^2)^{\gamma+1})^{1/2}$. Then, by (16) we get $G(\alpha_0) \leq E(0) = G(\alpha_2)$, which implies that $\alpha_0 \geq \alpha_2$.

Now, to established (14), we suppose by contradiction that

$$(\|\nabla u(t_0)\|^2 + \|\nabla v(t_0)\|^2 + \frac{1}{(\gamma+1)}(\|\nabla u(t_0)\|^2 + \|\nabla v(t_0)\|^2)^{\gamma+1})^{1/2} < \alpha_2,$$

for some $t_0 > 0$. By the continuity of

$$(\|\nabla u(t_0)\|^2 + \|\nabla v(t_0)\|^2 + \frac{1}{(\gamma+1)}(\|\nabla u(t_0)\|^2 + \|\nabla v(t_0)\|^2)^{\gamma+1})^{1/2},$$

we can obtain that,

$$(\|\nabla u(t_0)\|^2 + \|\nabla v(t_0)\|^2 + \frac{1}{(\gamma+1)}(\|\nabla u(t_0)\|^2 + \|\nabla v(t_0)\|^2)^{\gamma+1})^{1/2} > \alpha_1.$$

Again, the use of (16) leads to

$$\begin{aligned} E(t_0) &\geq G(\|\nabla u(t_0)\|^2 + \|\nabla v(t_0)\|^2 + \frac{1}{(\gamma+1)}(\|\nabla u(t_0)\|^2 + \|\nabla v(t_0)\|^2)^{\gamma+1}) \\ &> G(\alpha_2) \\ &= E(0). \end{aligned}$$

This is imposible since $E(t) \leq E(0)$ for all $t \in [0, T)$. Hence, (14) is established.

To prove (15), we make use of (7) to get

$$\begin{aligned} &\frac{1}{2}(\|\nabla u\|^2 + \|\nabla v\|^2) + \frac{1}{2(\gamma+1)}(\|\nabla u\|^2 + \|\nabla v\|^2)^{\gamma+1} \\ &\qquad \leq E(0) + \frac{1}{2(r+2)}(\|u+v\|_{2(r+2)}^{2(r+2)} + 2\|uv\|_{r+2}^{r+2}). \end{aligned}$$

Consequently, (14) yields

$$\begin{aligned} \frac{1}{2(r+2)} &(\|u+v\|_{2(r+2)}^{2(r+2)} + 2\|uv\|_{r+2}^{r+2}) \\ &\geq \frac{1}{2}(\|\nabla u\|^2 + \|\nabla v\|^2 + \frac{1}{\gamma+1}(\|\nabla u\|^2 + \|\nabla v\|^2)^{\gamma+1}) - E(0) \\ &\geq \frac{1}{2}\alpha_2^2 - E(0) \\ &\geq \frac{1}{2}\alpha_2^2 - G(\alpha_2) \\ &= \frac{B^{2(r+2)}}{2(r+2)}\alpha_2^{2(r+2)}. \end{aligned} \tag{18}$$

Therefore, (18) and (12) yield the desired result. This completes the proof. ∎

Theorem 3.3.

Suppose that (A1), (A2), (4) hold and

$$2(r+2) > \max\{2\gamma+2,\ k+p+1,\ l+p+1,\ \theta+q+1,\ \varrho+q+1\}.$$

Assume further that

$$\left(\|\nabla u_0\|^2 + \|\nabla v_0\|^2\right)^{\frac{1}{2}} > \alpha_1,\ E\left(0\right) < E_1.$$

Then, any the solution of (1) can not exist for all time.

Proof:

We set

$$H\left(t\right) = E_1 - E\left(t\right). \tag{19}$$

By using (7) and (19), we have

$$\begin{aligned} 0 < H\left(0\right) \leq H\left(t\right) = E_1 - \frac{1}{2}\left(\|u_t\|^2 + \|v_t\|^2\right) - \frac{1}{2}(\|\nabla u\|^2 + \|\nabla v\|^2) \\ -\frac{1}{2(\gamma+1)}(\|\nabla u\|^2 + \|\nabla v\|^2)^{\gamma+1} + \int_\Omega F\left(u,v\right)dx. \end{aligned} \tag{20}$$

From (18) and (8), we have

$$\begin{aligned} & E_1 - \frac{1}{2}\left(\|u_t\|^2 + \|v_t\|^2\right) - \frac{1}{2}(\|\nabla u\|^2 + \|\nabla v\|^2) \\ & \quad -\frac{1}{2(\gamma+1)}(\|\nabla u\|^2 + \|\nabla v\|^2)^{\gamma+1} + \int_\Omega F\left(u,v\right)dx \\ & \leq E_1 - \frac{1}{2}\alpha_1^2 + \frac{c_1}{2\left(r+2\right)}\left(\|u\|_{2(r+2)}^{2(r+2)} + \|v\|_{2(r+2)}^{2(r+2)}\right) \\ & \leq -\frac{1}{2\left(r+2\right)}\alpha_1^2 + \frac{c_1}{2\left(r+2\right)}\left(\|u\|_{2(r+2)}^{2(r+2)} + \|v\|_{2(r+2)}^{2(r+2)}\right) \\ & \leq \frac{c_1}{2\left(r+2\right)}\left(\|u\|_{2(r+2)}^{2(r+2)} + \|v\|_{2(r+2)}^{2(r+2)}\right). \end{aligned} \tag{21}$$

By combining (20) and (21), we have

$$0 < H\left(0\right) \leq H\left(t\right) \leq \frac{c_1}{2\left(r+2\right)}\left(\|u\|_{2(r+2)}^{2(r+2)} + \|v\|_{2(r+2)}^{2(r+2)}\right). \tag{22}$$

We then define

$$\Psi\left(t\right) = H^{1-\sigma}\left(t\right) + \varepsilon\left(\int_\Omega u_t u dx + \int_\Omega v_t v dx\right), \tag{23}$$

where ε small to be chosen later and

$$\begin{aligned} 0 < \sigma \leq \min\left\{\frac{r+1}{2\left(r+2\right)}, \frac{2r+3-(k+p)}{2p\left(r+2\right)}, \frac{2r+3-(l+p)}{2p\left(r+2\right)},\right. \\ \left.\frac{2r+3-(\varrho+q)}{2q\left(r+2\right)}, \frac{2r+3-(\theta+q)}{2q\left(r+2\right)}\right\}. \end{aligned} \tag{24}$$

A direct differentiation of (23) gives

$$\begin{aligned}\Psi'(t) &= (1-\sigma)\,H^{-\sigma}(t)\,H'(t) + \varepsilon\left(\int_\Omega |u_t|^2\,dx + \int_\Omega |v_t|^2\,dx\right)\\ &\quad +\varepsilon\left(\int_\Omega u_{tt}u dx + \int_\Omega v_{tt}v dx\right)\\ &= (1-\sigma)\,H^{-\sigma}(t)\,H'(t) + \varepsilon(\|u_t\|^2 + \|v_t\|^2)\\ &\quad -\varepsilon(\|\nabla u\|^2 + \|\nabla v\|^2) - \varepsilon(\|\nabla u\|^2 + \|\nabla v\|^2)^{\gamma+1} + 2\varepsilon(r+2)\int_\Omega F(u,v)dx\\ &\quad -\varepsilon\left(\int_\Omega u\left(|u|^k + |v|^l\right)u_t\,|u_t|^{p-1}\,dx + \int_\Omega v\left(|v|^\theta + |u|^\varrho\right)v_t\,|v_t|^{q-1}\,dx\right). \qquad (25)\end{aligned}$$

From the definition of $H(t)$, we obtain

$$\begin{aligned}&-(\|\nabla u\|^2 + \|\nabla v\|^2)^{\gamma+1}\\ &= 2(\gamma+1)\,H(t) - 2(\gamma+1)\,E_1 + (\gamma+1)\left(\|u_t\|^2 + \|v_t\|^2\right)\\ &\quad + (\gamma+1)\left(\|\nabla u\|^2 + \|\nabla v\|^2\right) - 2(\gamma+1)\int_\Omega F(u,v)\,dx. \qquad (26)\end{aligned}$$

Inserting (26) into (25), we get

$$\begin{aligned}\Psi'(t) &= (1-\sigma)\,H^{-\sigma}(t)\,H'(t) + \varepsilon(\gamma+2)\left(\|u_t\|^2 + \|v_t\|^2\right)\\ &\quad +\varepsilon\gamma\left(\|\nabla u\|^2 + \|\nabla v\|^2\right) + 2\varepsilon(\gamma+1)\,H(t) - 2\varepsilon(\gamma+1)\,E_1\\ &\quad +\varepsilon\Big(1 - \frac{\gamma+1}{r+2}\Big)(\|u+v\|_{2(r+2)}^{2(r+2)} + 2\,\|uv\|_{r+2}^{r+2})\\ &\quad -\varepsilon\left(\int_\Omega u\left(|u|^k + |v|^l\right)u_t\,|u_t|^{p-1}\,dx + \int_\Omega v\left(|v|^\theta + |u|^\varrho\right)v_t\,|v_t|^{q-1}\,dx\right).\end{aligned}$$

Then, using (15), we have

$$\begin{aligned}\Psi'(t) &\geq (1-\sigma)\,H^{-\sigma}(t)\,H'(t) + \varepsilon(\gamma+2)\left(\|u_t\|^2 + \|v_t\|^2\right)\\ &\quad +\varepsilon\gamma\left(\|\nabla u\|^2 + \|\nabla v\|^2\right)\\ &\quad +2(\gamma+1)\,\varepsilon H(t) + \varepsilon c'(\|u+v\|_{2(r+2)}^{2(r+2)} + 2\,\|uv\|_{r+2}^{r+2})\\ &\quad -\varepsilon\left(\int_\Omega u\left(|u|^k + |v|^l\right)u_t\,|u_t|^{p-1}\,dx + \int_\Omega v\left(|v|^\theta + |u|^\varrho\right)v_t\,|v_t|^{q-1}\,dx\right), \qquad (27)\end{aligned}$$

where $c' = 1 - \frac{\gamma+1}{r+2} - 2(\gamma+1)\,E_1(B\alpha_2)^{-2(r+2)} > 0$, since $\alpha_2 > B^{-\frac{r+2}{r+1}}$. In order to estimate the last two terms in (27).

Thanks to the following Young's inequality,

$$XY \leq \frac{\delta^k X^k}{k} + \frac{\delta^{-l} Y^l}{l},$$

where $X,Y \geq 0,\ \delta > 0,\ k,l \in R^{+}$ such that $\frac{1}{k}+\frac{1}{l}=1$. Consequently, applying the above inequality we find

$$\int_{\Omega} uu_t \left|u_t\right|^{p-1} dx \leq \frac{\delta_1^{p+1}}{p+1}\left\|u\right\|_{p+1}^{p+1}+\frac{p\delta_1^{-\frac{p+1}{p}}}{p+1}\left\|u_t\right\|_{p+1}^{p+1},$$

and therefore,

$$\int_{\Omega}\left(|u|^{k}+|v|^{l}\right)uu_t\left|u_t\right|^{p-1}dx \leq \frac{\delta_1^{p+1}}{p+1}\int_{\Omega}\left(|u|^{k}+|v|^{l}\right)|u|^{p+1}dx + \frac{p\delta_1^{-\frac{p+1}{p}}}{p+1}\int_{\Omega}\left(|u|^{k}+|v|^{l}\right)\left|u_t\right|^{p+1}dx.$$

In the same way, we conclude that

$$\int_{\Omega} vv_t \left|v_t\right|^{q-1} dx \leq \frac{\delta_2^{q+1}}{q+1}\left\|v\right\|_{q+1}^{q+1}+\frac{q\delta_2^{-\frac{q+1}{q}}}{q+1}\left\|v_t\right\|_{q+1}^{q+1},$$

and therefore,

$$\int_{\Omega} v\left(|v|^{\theta}+|u|^{\varrho}\right)v_t\left|v_t\right|^{q-1}dx \leq \frac{\delta_2^{q+1}}{q+1}\int_{\Omega}\left(|v|^{\theta}+|u|^{\varrho}\right)|v|^{q+1}dx + \frac{q\delta_2^{-\frac{q+1}{q}}}{q+1}\int_{\Omega}\left(|v|^{\theta}+|u|^{\varrho}\right)\left|v_t\right|^{q+1}dx,$$

where $\delta_1,\ \delta_2$ are constants depending on the time t and specified later. Therefore, (27) becomes

$$\begin{aligned}\Psi'(t) \geq{}& (1-\sigma)H^{-\sigma}(t)H'(t)+\varepsilon(\gamma+2)\left(\left\|u_t\right\|^2+\left\|v_t\right\|^2\right)\\ &+\varepsilon\gamma\left(\left\|\nabla u\right\|^2+\left\|\nabla v\right\|^2\right)+2\varepsilon(\gamma+1)H(t)+\varepsilon c'\left(\left\|u+v\right\|_{2(r+2)}^{2(r+2)}+2\left\|uv\right\|_{r+2}^{r+2}\right)\\ &-\varepsilon\frac{\delta_1^{p+1}}{p+1}\int_{\Omega}\left(|u|^{k}+|v|^{l}\right)|u|^{p+1}dx-\varepsilon\frac{p\delta_1^{-\frac{p+1}{p}}}{p+1}\int_{\Omega}\left(|u|^{k}+|v|^{l}\right)\left|u_t\right|^{p+1}dx\\ &-\varepsilon\frac{\delta_2^{q+1}}{q+1}\int_{\Omega}\left(|v|^{\theta}+|u|^{\varrho}\right)|v|^{q+1}dx-\varepsilon\frac{q\delta_2^{-\frac{q+1}{q}}}{q+1}\int_{\Omega}\left(|v|^{\theta}+|u|^{\varrho}\right)\left|v_t\right|^{q+1}dx. \qquad (28)\end{aligned}$$

Therefore, by taking δ_1 and δ_2 so that

$$\delta_1^{-\frac{p+1}{p}} = k_1H^{-\sigma}(t),$$
$$\delta_2^{-\frac{q+1}{q}} = k_2H^{-\sigma}(t),$$

where $k_1,\ k_2>0$ are specified later, we get

$$\begin{aligned}\Psi'(t) \geq{}& ((1-\sigma)-K\varepsilon)H^{-\sigma}(t)H'(t)+\varepsilon(\gamma+2)\left(\left\|u_t\right\|^2+\left\|v_t\right\|^2\right)\\ &+\varepsilon\gamma\left(\left\|\nabla u\right\|^2+\left\|\nabla v\right\|^2\right)+2\varepsilon(\gamma+1)H(t)+\varepsilon c'\left(\left\|u\right\|_{2(r+2)}^{2(r+2)}+\left\|v\right\|_{2(r+2)}^{2(r+2)}\right)\\ &-\varepsilon\frac{k_1^{-p}H^{\sigma p}(t)}{p+1}\int_{\Omega}\left(|u|^{k}+|v|^{l}\right)|u|^{p+1}dx\\ &-\varepsilon\frac{k_2^{-q}H^{\sigma q}(t)}{q+1}\int_{\Omega}\left(|v|^{\theta}+|u|^{\varrho}\right)|v|^{q+1}dx, \qquad (29)\end{aligned}$$

where $K = \frac{k_1 p}{p+1} + \frac{k_2 q}{q+1}$.

Thanks to Young's inequality, we obtain

$$\begin{aligned}\int_\Omega \left(|u|^k + |v|^l\right)|u|^{p+1}\,dx &\le \int_\Omega |u|^{k+p+1}\,dx + \int_\Omega |v|^l\,|u|^{p+1}\,dx \\ &\le \int_\Omega |u|^{k+p+1}\,dx + \frac{l}{l+p+1}\delta_1^{\frac{l+p+1}{l}}\int_\Omega |v|^{l+p+1}\,dx \\ &\quad + \frac{p+1}{l+p+1}\delta_1^{-\frac{l+p+1}{p+1}}\int_\Omega |u|^{l+p+1}\,dx \\ &= \|u\|_{k+p+1}^{k+p+1} + \frac{l}{l+p+1}\delta_1^{\frac{l+p+1}{l}}\|v\|_{l+p+1}^{l+p+1} \\ &\quad + \frac{p+1}{l+p+1}\delta_1^{-\frac{l+p+1}{p+1}}\|u\|_{l+p+1}^{l+p+1}. \end{aligned} \tag{30}$$

Similarly

$$\begin{aligned}\int_\Omega \left(|v|^\theta + |u|^\varrho\right)|v|^{q+1}\,dx &\le \|v\|_{\theta+q+1}^{\theta+q+1} + \frac{\varrho}{\varrho+q+1}\delta_2^{\frac{\varrho+q+1}{\varrho}}\|u\|_{\varrho+q+1}^{\varrho+q+1} \\ &\quad + \frac{q+1}{\varrho+q+1}\delta_2^{-\frac{\varrho+q+1}{q+1}}\|v\|_{\varrho+q+1}^{\varrho+q+1}. \end{aligned} \tag{31}$$

Inserting (31) and (30) into (29), we conclude that

$$\begin{aligned}\Psi'(t) &\ge ((1-\sigma) - K\varepsilon)H^{-\sigma}(t)\,H'(t) + \varepsilon(\gamma+2)\left(\|u_t\|^2 + \|v_t\|^2\right) \\ &\quad + \varepsilon\gamma\left(\|\nabla u\|^2 + \|\nabla v\|^2\right) + 2\varepsilon(\gamma+1)\,H(t) + \varepsilon c'\left(\|u\|_{2(r+2)}^{2(r+2)} + \|v\|_{2(r+2)}^{2(r+2)}\right) \\ &\quad - \varepsilon\frac{k_1^{-p}H^{\sigma p}(t)}{p+1}\left(\|u\|_{k+p+1}^{k+p+1} + \frac{l}{l+p+1}\delta_1^{\frac{l+p+1}{l}}\|v\|_{l+p+1}^{l+p+1}\right. \\ &\quad \left. + \frac{p+1}{l+p+1}\delta_1^{-\frac{l+p+1}{p+1}}\|u\|_{l+p+1}^{l+p+1}\right) \\ &\quad - \varepsilon\frac{k_2^{-q}H^{\sigma q}(t)}{q+1}\left(\|v\|_{\theta+q+1}^{\theta+q+1} + \frac{\varrho}{\varrho+q+1}\delta_2^{\frac{\varrho+q+1}{\varrho}}\|u\|_{\varrho+q+1}^{\varrho+q+1}\right. \\ &\quad \left. + \frac{q+1}{\varrho+q+1}\delta_2^{-\frac{\varrho+q+1}{q+1}}\|v\|_{\varrho+q+1}^{\varrho+q+1}\right). \end{aligned} \tag{32}$$

Since

$$2(r+2) > \max\{2(\gamma+1), k+p+1, l+p+1, \theta+q+1, \varrho+q+1\},$$

we have

$$H^{\sigma p}(t)\,\|u\|_{k+p+1}^{k+p+1} \le C\left(\|u\|_{2(r+2)}^{2\sigma p(r+2)+k+p+1} + \|v\|_{2(r+2)}^{2\sigma p(r+2)}\,\|u\|_{k+p+1}^{k+p+1}\right), \tag{33}$$

$$H^{\sigma q}(t)\,\|v\|_{\theta+q+1}^{\theta+q+1} \le C\left(\|v\|_{2(r+2)}^{2\sigma q(r+2)+\theta+q+1} + \|u\|_{2(r+2)}^{2\sigma q(r+2)}\,\|v\|_{\theta+q+1}^{\theta+q+1}\right), \tag{34}$$

$$\begin{aligned}\frac{l}{l+p+1}\delta_1^{\frac{l+p+1}{l}}H^{\sigma p}(t)\|v\|_{l+p+1}^{l+p+1}\\ \leq C\frac{l}{l+p+1}\delta_1^{\frac{l+p+1}{l}}\left(\|v\|_{2(r+2)}^{2\sigma p(r+2)+l+p+1}+\|u\|_{2(r+2)}^{2\sigma p(r+2)}\|v\|_{l+p+1}^{l+p+1}\right),\end{aligned}\tag{35}$$

and

$$\begin{aligned}\frac{\varrho}{\varrho+q+1}\delta_2^{\frac{\varrho+q+1}{\varrho}}H^{\sigma q}(t)\|u\|_{\varrho+q+1}^{\varrho+q+1}\\ \leq C\frac{\varrho}{\varrho+q+1}\delta_2^{\frac{\varrho+q+1}{\varrho}}\left(\|u\|_{2(r+2)}^{2\sigma q(r+2)+\varrho+q+1}+\|v\|_{2(r+2)}^{2\sigma q(r+2)}\|u\|_{\varrho+q+1}^{\varrho+q+1}\right).\end{aligned}\tag{36}$$

By using (24) and the following algebraic inequality

$$z^{\upsilon}\leq z+1\leq\left(1+\frac{1}{a}\right)(z+a),\quad\forall z\geq 0,\ 0<\upsilon\leq 1,\ a\geq 0,\tag{37}$$

we have, for all $t\geq 0$,

$$\begin{aligned}\|u\|_{2(r+2)}^{2\sigma p(r+2)+k+p+1}&\leq d\left(\|u\|_{2(r+2)}^{2(r+2)}+H(0)\right)\\&\leq d\left(\|u\|_{2(r+2)}^{2(r+2)}+H(t)\right),\end{aligned}\tag{38}$$

$$\|v\|_{2(r+2)}^{2\sigma q(r+2)+\theta+q+1}\leq d\left(\|v\|_{2(r+2)}^{2(r+2)}+H(t)\right),\tag{39}$$

where $d=1+\frac{1}{H(0)}$. Similarly

$$\|u\|_{2(r+2)}^{2\sigma q(r+2)+\varrho+q+1}\leq d\left(\|u\|_{2(r+2)}^{2(r+2)}+H(t)\right),\tag{40}$$

$$\|v\|_{2(r+2)}^{2\sigma p(r+2)+l+p+1}\leq d\left(\|v\|_{2(r+2)}^{2(r+2)}+H(t)\right).\tag{41}$$

Also, since

$$(a+b)^{\lambda}\leq C\left(a^{\lambda}+b^{\lambda}\right),\ \ a,b>0,$$

by Young's inequality and using (24) and (37), we have

$$\begin{aligned}\|v\|_{2(r+2)}^{2\sigma p(r+2)}\|u\|_{k+p+1}^{k+p+1}&\leq|\Omega|^{\frac{2(r+2)-(k+p+1)}{2(r+2)}}(\|v\|_{2(r+2)}^{2\sigma p(r+2)}\|u\|_{2(r+2)}^{k+p+1})\\&=|\Omega|^{\frac{2(r+2)-(k+p+1)}{2(r+2)}}\left(\|v\|_{2(r+2)}^{\sigma p}\|u\|_{2(r+2)}^{\frac{k+p+1}{2(r+2)}}\right)^{2(r+2)}\\&\leq|\Omega|^{\frac{2(r+2)-(k+p+1)}{2(r+2)}}(c'\|v\|_{2(r+2)}^{\frac{2\sigma p(r+2)+k+p+1}{2(r+2)}}+c''\|u\|_{2(r+2)}^{\frac{2\sigma p(r+2)+k+p+1}{2(r+2)}})^{2(r+2)}\\&\leq C\left(\|v\|_{2(r+2)}^{2(r+2)}+\|u\|_{2(r+2)}^{2(r+2)}\right).\end{aligned}\tag{42}$$

Similarly

$$\|u\|_{2(r+2)}^{2\sigma q(r+2)}\|v\|_{\theta+q+1}^{\theta+q+1}\leq C\left(\|u\|_{2(r+2)}^{2(r+2)}+\|v\|_{2(r+2)}^{2(r+2)}\right),\tag{43}$$

$$\|u\|_{2(r+2)}^{2\sigma p(r+2)}\|v\|_{l+p+1}^{l+p+1}\leq C\left(\|u\|_{2(r+2)}^{2(r+2)}+\|v\|_{2(r+2)}^{2(r+2)}\right),\tag{44}$$

and

$$\|v\|_{2(r+2)}^{2\sigma q(r+2)} \|u\|_{\varrho+q+1}^{\varrho+q+1} \leq C \left(\|v\|_{2(r+2)}^{2(r+2)} + \|u\|_{2(r+2)}^{2(r+2)} \right). \tag{45}$$

Inserting (33)-(36) and (38)-(45) into (32), we have

$$\begin{aligned}
\Psi'(t) \geq & \left((1-\sigma) - K\varepsilon\right) H^{-\sigma}(t) H'(t) + \varepsilon(\gamma+2)\left(\|u_t\|^2 + \|v_t\|^2\right) \\
& + \varepsilon\gamma\left(\|\nabla u\|^2 + \|\nabla v\|^2\right) \\
& + \varepsilon\left[2(\gamma+1) - Ck_1^{-p}\left(1 + \frac{l}{l+p+1}\delta_1^{\frac{l+p+1}{l}} + \frac{p+1}{l+p+1}\delta_1^{-\frac{l+p+1}{p+1}}\right)\right. \\
& \left. - Ck_2^{-q}\left(1 + \frac{\varrho}{\varrho+q+1}\delta_2^{\frac{\varrho+q+1}{\varrho}} + \frac{q+1}{\varrho+q+1}\delta_2^{-\frac{\varrho+q+1}{q+1}}\right)\right] H(t) \\
& + \varepsilon\left[c' - Ck_1^{-p}\left(1 + \frac{l}{l+p+1}\delta_1^{\frac{l+p+1}{l}} + \frac{p+1}{l+p+1}\delta_1^{-\frac{l+p+1}{p+1}}\right)\right. \\
& \left. - Ck_2^{-q}\left(1 + \frac{\varrho}{\varrho+q+1}\delta_2^{\frac{\varrho+q+1}{\varrho}} + \frac{q+1}{\varrho+q+1}\delta_2^{-\frac{\varrho+q+1}{q+1}}\right)\right] \left(\|u\|_{2(r+2)}^{2(r+2)} + \|v\|_{2(r+2)}^{2(r+2)}\right)
\end{aligned} \tag{46}$$

At this point, and for large values of k_1 and k_2, we can find positive constats K_1 and K_2 such that (46) becomes

$$\begin{aligned}
\Psi'(t) \geq & \left((1-\sigma) - K\varepsilon\right) H^{-\sigma}(t) H'(t) + \varepsilon(\gamma+2)\left(\|u_t\|^2 + \|v_t\|^2\right) \\
& + \varepsilon\gamma\left(\|\nabla u\|^2 + \|\nabla v\|^2\right) + \varepsilon K_1 H(t) + \varepsilon K_2 \left(\|u\|_{2(r+2)}^{2(r+2)} + \|v\|_{2(r+2)}^{2(r+2)}\right) \\
\geq & \beta\left(\|u_t\|^2 + \|v_t\|^2 + H(t) + \|\nabla u\|^2 + \|\nabla v\|^2 + \|u\|_{2(r+2)}^{2(r+2)} + \|v\|_{2(r+2)}^{2(r+2)}\right),
\end{aligned} \tag{47}$$

where $\beta = \min\{\varepsilon(\gamma+2), \varepsilon\gamma, \varepsilon K_1, \varepsilon K_2\}$ and we pick ε small enough so that $(1-\sigma) - K\varepsilon \geq 0$. Consequently we have

$$\Psi(t) \geq \Psi(0) > 0, \ \forall t \geq 0. \tag{48}$$

We now estimate $\Psi(t)^{\frac{1}{1-\sigma}}$. Applying Hölder inequality, we obtain

$$\begin{aligned}
\left| \int_\Omega u_t u dx + \int_\Omega v_t v dx \right|^{\frac{1}{1-\sigma}} & \leq \|u_t\|^{\frac{1}{1-\sigma}} \|u\|^{\frac{1}{1-\sigma}} + \|v_t\|^{\frac{1}{1-\sigma}} \|v\|^{\frac{1}{1-\sigma}} \\
& \leq C\left(\|u_t\|^{\frac{1}{1-\sigma}} \|u\|_{2(r+2)}^{\frac{1}{1-\sigma}} + \|v_t\|^{\frac{1}{1-\sigma}} \|v\|_{2(r+2)}^{\frac{1}{1-\sigma}}\right).
\end{aligned} \tag{49}$$

Young's inequality gives

$$\left| \int_\Omega u_t u dx + \int_\Omega v_t v dx \right|^{\frac{1}{1-\sigma}} \leq C(\|u_t\|^{\frac{\mu}{1-\sigma}} + \|u\|_{2(r+2)}^{\frac{\theta}{1-\sigma}} + \|v_t\|^{\frac{\mu}{1-\sigma}} + \|v\|_{2(r+2)}^{\frac{\theta}{1-\sigma}}), \tag{50}$$

where $\frac{1}{\mu} + \frac{1}{\theta} = 1$. We take $\mu = 2(1-\sigma)$ to get $\theta = \frac{2(1-\sigma)}{1-2\sigma}$ by (24). Therefore, (50) becomes

$$\left| \int_\Omega u_t u dx + \int_\Omega v_t v dx \right|^{\frac{1}{1-\sigma}} \leq C(\|u_t\|^2 + \|u\|_{2(r+2)}^{\frac{2}{1-2\sigma}} + \|v_t\|^2 + \|v\|_{2(r+2)}^{\frac{2}{1-2\sigma}}). \tag{51}$$

By using Lemma 2.3, we obtain

$$\left|\int_\Omega u_t u dx + \int_\Omega v_t v dx\right|^{\frac{1}{1-\sigma}} \leq C(\|u_t\|^2 + \|v_t\|^2 + \|u\|_{2(r+2)}^{2(r+2)} + \|v\|_{2(r+2)}^{2(r+2)} + \|\nabla u\|^2 + \|\nabla v\|^2). \quad (52)$$

Thus,

$$\begin{aligned}\Psi^{\frac{1}{1-\sigma}}(t) &= \left[H^{1-\sigma}(t) + \varepsilon\left(\int_\Omega u_t u dx + \int_\Omega v_t v dx\right)\right]^{\frac{1}{1-\sigma}} \\ &\leq C\left(\|u_t\|^2 + \|v_t\|^2 + H(t) + \|u\|_{2(r+2)}^{2(r+2)} + \|v\|_{2(r+2)}^{2(r+2)} + \|\nabla u\|^2 + \|\nabla v\|^2\right). \quad (53)\end{aligned}$$

Combining (47) and (53) we arrive at

$$\Psi'(t) \geq \xi \Psi^{\frac{1}{1-\sigma}}(t), \quad (54)$$

where ξ is a positive constant.

A simple integration of (54) over $(0,t)$ yields

$$\Psi^{\frac{\sigma}{1-\sigma}}(t) \geq \frac{1}{\Psi^{-\frac{\sigma}{1-\sigma}}(0) - \frac{\xi\sigma t}{1-\sigma}},$$

which implies that the solution blows up in a finite time T^*, with

$$T^* \leq \frac{1-\sigma}{\xi\sigma\Psi^{\frac{\sigma}{1-\sigma}}(0)}.$$

■

4. Lower bounds for the blow up time

In this section, we discuss the lower bounds for the blow up time. Firstly, we give the following lemma (see Peyravi (2017) and Pişkin (2017)).

Lemma 4.1.

There exist two positive c_1 and c_2 such that

$$\begin{aligned}\int_\Omega |f_1(u,v)|^2 dx &\leq c_1(\|\nabla u\|^2 + \|\nabla v\|^2)^{2r+3}, \\ \int_\Omega |f_2(u,v)|^2 dx &\leq c_2(\|\nabla u\|^2 + \|\nabla v\|^2)^{2r+3}, \quad (55)\end{aligned}$$

are satisfied.

Theorem 4.2.

Suppose that (A1), (4) hold and $(u_0,u_1),(v_0,v_1) \in (H_0^1(\Omega)\cap L^{r+1}(\Omega)) \times L^2(\Omega)$. Assume further that $1 < p, q < 2r+1$. Then, the finite blow-up time T^* satisfies the following estimate

$$\int_{\phi(0)}^{\phi(t)} \frac{d\tau}{(E(0)+\tau) + 2^{4(r+1)}(c_1+c_2)((E(0))^{2r+3} + \tau^{2r+3})} \leq T^*,$$

where $\phi(0) = \int_\Omega F(u(0), v(0))dx$ and the positive constants c_1 and c_2 are specified in (55).

Proof:

Define

$$\phi(t) = \int_\Omega F(u, v)dx.$$

By differentiating $\phi(t)$ and using Young's inequality, we get

$$\begin{aligned}\phi'(t) &= \int_\Omega u_t F_u + v_t F_v dx \\ &\leq \frac{1}{2}\int_\Omega (u_t^2 + v_t^2)dx + \frac{1}{2}\int_\Omega (F_u^2 + F_v^2)dx. \end{aligned} \tag{56}$$

By the Lemma 4.1, we obtain

$$\phi'(t) \leq \frac{1}{2}\int_\Omega (u_t^2 + v_t^2)dx + (\frac{c_1 + c_2}{2})(\|\nabla u\|^2 + \|\nabla v\|^2)^{2r+3}. \tag{57}$$

Therefore, from (7) and Lemma 2.4, we have

$$\begin{aligned}\int_\Omega (u_t^2 + v_t^2)dx + (\|\nabla u\|^2 + \|\nabla v\|^2) &\leq 2E(t) + 2\int_\Omega F(u, v)dx \\ &\leq 2E(0) + 2\int_\Omega F(u, v)dx. \end{aligned} \tag{58}$$

Combining (57)-(58), we get

$$\begin{aligned}\phi'(t) &\leq \phi(t) + E(0) + 2^{2r+2}(c_1 + c_2)\left[\phi(t) + E(0)\right]^{2r+3} \\ &\leq \phi(t) + E(0) + 2^{4(r+1)}(c_1 + c_2)\left[(\phi(t))^{2r+3} + (E(0))^{2r+3}\right]. \end{aligned} \tag{59}$$

Integrating (59) from 0 to t, we obtain

$$\int_{\phi(0)}^{\phi(t)} \frac{d\tau}{(E(0) + \tau) + 2^{4(r+1)}(c_1 + c_2)((E(0))^{2r+3} + \tau^{2r+3})} \leq T^*.$$

Thus, we obtain the desired result. ■

5. Conclusion

In this paper, we obtained a blow up and a lower bounds for the blow up time for a coupled Kirchhoff-type equations with degenerate damping terms. This improves and extends many results in the literature.

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