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New Notions From (r, s)-Generalized Fuzzy e-open Sets

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Abstract

The present article discuss (r, s)-generalized fuzzy e-border, (r, s)-generalized fuzzy e-exterior and (r, s)-generalized fuzzy e-frontier in double fuzzy topologies. Furthermore, some characterizations of generalized double fuzzy e-continuous, generalized double fuzzy e-open, generalized double fuzzy e-closed and generalized double fuzzy e-closure-irresolute functions are studied and investigated. Moreover, the interrelations among the new concepts are discussed with some necessary examples.

Keywords: (r, s)-generalized fuzzy e-border; (r, s)-generalized fuzzy e-exterior; (r, s)- generalized fuzzy e-frontier; Generalized double fuzzy e-open; Generalized double fuzzy e-closed; Generalized double fuzzy e-continuous; Generalized double fuzzy e-closure-irresolute functions

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1. Introduction and Preliminaries

Fuzzy theory (Zadeh (1965)) was initiated in 1965. Later on, Chang (1968) utilized it to define fuzzy topological spaces. Intuitionistic fuzzy topologies introduced by Coker (1996) and Coker (1997), are based on intuitionistic fuzzy sets. In Samanta and Mondal (2002), the authors succeeded to intuitionism the structure of topology. The name “intuitionistic" is no longer used in mathematics because of some suspicions about the convenience of this term. This suspicions were refuted in Garcia and Rodabaugh (2005), when the authors substituted the term “intuitionistic" by “double" and renamed its related topologies. The concept of intuitionistic gradation of openness has been renamed to “double fuzzy topological spaces:.

There are a variety of applications for topology. For instance, ophthalmologist have found that when the sight is restored to a blind person, the person will have a topological vision for some time. During this short time, the person who has recovered from the blindness cannot differentiate between a circle and a square and any closed curves. He has to exercise for some time to precisely characterize various closed curves. Based on this idea, Zeeman has constructed a topological model of the brain and the visual perception (Zeeman (1962)).

Topological psychology (Leeper (1930); Lewin (1930)) is a dialectical topic in which mathematicians have different opinions on it. The German scientist Lewin studied topology and applied the topological notions in his theories in psychology. He tried hard to formalize his theories into an evident form in order to avoid the rigidity and obstinacy of the results. Lewin presented concepts incidentally and progressively developed them through experimental and observation methods.

The topological relaionships between spatial objects are essential information used in Geographic information system (GIS), along with positional and attribute information. Information on topological relationships can be used for spatial queries, spatial analyses, data quality control, and others. Topological relationships may be fuzzy or crisp based on the certainty or uncertainty of spatial objects and the nature of their relationships (Shi et al. (2007)). The initiations of e-open sets, e-continuity and e-compactness in topological spaces are due to Ekici (2007), Ekici (2008a), Ekici (2008b), Ekici (2008c), and Ekici (2009).

In this paper $X \neq \phi,\ I_0 = (0, 1],\ I_1 = [0, 1)$ and $I - [0, 1]$. I^X is the family of fuzzy sets defined on the universal set X. $P_t(X)$ refers to the collection of all fuzzy points defined on X. $\underline{0}$ and $\underline{1}$ refer to the smallest and the largest elements in I^X, respectively. The complement of $A \in I^X$ is denoted by $\underline{1} - A$. For any crisp map $f : X \to Y$, the direct image $f(A)$ and inverse image $f^{-1}(A)$ of f are given by $f(A)(y) = \bigvee_{f(x)=y} A(x)$ and $f^{-1}(B)(x) = B(f(x))$ for all $A \in I^X,\ B \in I^Y$ and $x \in X$, respectively.

The notion of double fuzzy topology (briefly, dfts), (r, s)-fuzzy open (briefly, (r, s)-fo) and its complement are introduced and studied in Samanta and Mondal (1997). The closure and interior operators in dfts are introduced and studied in Lee and Im (2001). The concept of (r, s)-generalized fuzzy closed (briefly, (r, s)-gfc) and its complement, the closure and interior operators of the same are studied in Abbas and El-Sanousy (2012). The class of new sets namely, (r, s)-fuzzy regular open (briefly, (r, s)-fro), (r, s)-fuzzy δ semiopen (briefly, (r, s)-$f\delta so$), (r, s)-fuzzy δ pre open (briefly, (r, s)-$f\delta po$), (r, s)-fuzzy β open (briefly, (r, s)-$f\beta o$), (r, s)-fuzzy e-open (briefly, (r, s)-feo), (r, s)-generalized fuzzy δ semiopen (briefly, (r, s)-gfδso), (r, s)-generalized fuzzy δ pre-open (briefly, (r, s)-gfδpo), (r, s)-generalized fuzzy β-open (briefly, (r, s)-gfβo),(r, s)-generalized fuzzy e-open (briefly, (r, s)-gf$e$o) and its complement with their closure and interior operators are introduced and studied in Periyasamy et al. (2019). Also (r, s)-generalized fuzzy border, (r, s)-generalized fuzzy frontier and (r, s)-generalized fuzzy exterior are introduced in Fatimah et al. (2017).

This paper takes some investigations on (r, s)-generalized fuzzy e-border, (r, s)-generalized fuzzy e-exterior and (r, s)-gneralized fuzzy e-frontier in double fuzzy topological spaces. Moreover, some characteristic properties of generalized double fuzzy e-continuous, generalized double fuzzy e-open, generalized double fuzzy e-closed and generalized double fuzzy e-closure irresolute function are studied and investigated. Also, we present the relationships between the new notions and already known functions. Our new results are a generalization to the corresponding notions in topology and fuzzy topology, and therefore, we believe that it will be more useful and applicable for GIS modeling and for other scientific fields.

2. Properties of (r, s)-generalized fuzzy e-open sets

In this section, we study (r, s)-generalized fuzzy e-border, (r, s)-generalized fuzzy e-exterior and (r, s)-generalized fuzzy e-frontier. Some of its interesting properties and characterizations are examined.

Definition 2.1.

For any dfts $(X, T, T^\star),\ A \in I^X,\ \alpha \in I_0$ and $\beta \in I_1$, we have

(1) $geC_{T,T^\star}(A, r, s) = \bigwedge\{B \in I^X | A \leq B$ and B is (r, s)-$gfec\}$, where $geC_{T,T^\star}(A, r, s)$ is the (r, s)-generalized fuzzy e-closure of A.
(2) $geI_{T,T^\star}(A, r, s) = \bigvee\{B \in I^X | B \leq A$ and B is (r, s)-$gfeo\}$, where $geI_{T,T^\star}(A, r, s)$ is the (r, s)-generalized fuzzy e-interior of A.

Proposition 2.2.

For any dfts $(X, T, T^\star),\ A \in I^X,\ \alpha \in I_0$ and $\beta \in I_1$, we have

(1) $geI_{T,T^\star}(A, r, s)$ is the largest (r, s)-gfeo set such that $geI_{T,T^\star}(A, r, s) \leq A$.
(2) $A = geI_{T,T^\star}(A, r, s)$, if A is an (r, s)-gfeo set.
(3) $geI_{T,T^\star}(geI_{T,T^\star}(A, r, s), r, s) = geI_{T,T^\star}(A, r, s)$, if A is an (r, s)-gfeo set.
(4) $\underline{1} - geI_{T,T^\star}(A, r, s) = geC_{T,T^\star}(\underline{1} - A, r, s)$.
(5) $\underline{1} - geC_{T,T^\star}(A, r, s) = geI_{T,T^\star}(\underline{1} - A, r, s)$.
(6) If $A \leq B$, then $geI_{T,T^\star}(A, r, s) \leq geI_{T,T^\star}(B, r, s)$.
(7) If $A \leq B$, then $geC_{T,T^\star}(A, r, s) \leq geC_{T,T^\star}(B, r, s)$.

(8) $geI_{T,T^\star}(A,r,s) \wedge geI_{T,T^\star}(B,r,s) = geI_{T,T^\star}(A \wedge B,r,s)$.
(9) $geI_{T,T^\star}(A,r,s) \vee geI_{T,T^\star}(B,r,s) = geI_{T,T^\star}(A \vee B,r,s)$.

Proof:

(i) and (ii) follow from the definitions and (iii) follows from (ii).

(iv) $geC_{T,T^\star}(\underline{1}-A,r,s) = \bigwedge\{B|B \text{ is } (r,s)\text{-}gfec \text{ set}, \ B \geq \underline{1}-A\}.$
$= \underline{1} - \bigvee\{\underline{1}-B|\underline{1}-B \text{ is } (r,s) \ gfeo \text{ set}, \ \underline{1}-B \leq A\}.$
$= \underline{1} - geI_{T,T^\star}(A,r,s).$

(v) It is similar to (iv).

(vi) It is clear that $geI_{T,T^\star}(A,r,s) = \bigvee\{D|D \text{ is } (r,s)\text{-}gfeo \text{ and}, \ D \leq A\}.$
$= \bigvee\{D|D \leq B \ D \text{ is an } (r,s)\text{-}gfeo\}$
$= geI_{T,T^\star}(B,r,s).$

(vii) If $A \leq B$, it is similar to (vi).

(viii)$geI_{T,T^\star}(A \wedge B,r,s) = \bigvee\{D|D \text{ is } (r,s)\text{-}gfeo \text{ and}, \ D \leq (A \wedge B)\}.$
$= (\bigvee\{D|D \text{ is } (r,s)\text{-}gfeo \text{ and } D \leq A\})$
$\wedge (\bigvee\{D|D \text{ is } (r,s)\text{-}gfeo \text{ and } D \leq B\})$
$= (geI_{T,T^\star}(A,r,s)) \wedge (geI_{T,T^\star}(B,r,s)).$

(ix) It is similar to (viii). ■

Definition 2.3.

For any dfts $(X,T,T^\star), \ A \in I^X, \ r \in I_0$ and $s \in I_1$, we have

(1) $eB_{T,T^\star}(A,r,s) = A - eI_{T,T^\star}(A,r,s)$, where $eB_{T,T^\star}(A,r,s)$ is the (r,s)-fuzzy e-border of A.
(2) $geB_{T,T^\star}(A,r,s) = A - geI_{T,T^\star}(A,r,s)$, where $geB_{T,T^\star}(A,r,s)$ is the (r,s)-generalized fuzzy e-border of A.

Proposition 2.4.

For any dfts $(X,T,T^\star), \ A \in I^X, \ r \in I_0$ and $s \in I_1$, we have

(1) $geB_{T,T^\star}(A,r,s) \leq eB_{T,T^\star}(A,r,s)$.
(2) If A is an (r,s)-gfeo, then $geB_{T,T^\star}(A,r,s) = \underline{0}$.
(3) $geB_{T,T^\star}(A,r,s) \leq geC_{T,T^\star}(\underline{1}-A,r,s)$.
(4) $geI_{T,T^\star}(geB_{T,T^\star}(A,r,s),r,s) \leq A$.
(5) $geB_{T,T^\star}(A \vee B,r,s) \leq (geB_{T,T^\star}(A,r,s)) \vee (GeB_{T,T^\star}(B,r,s))$.
(6) $geB_{T,T^\star}(A \wedge B,r,s) \geq (geB_{T,T^\star}(A,r,s)) \wedge (GeB_{T,T^\star}(B,r,s))$.

Proof:

(i)For any $A \in I^X$, since $eI_{T,T^\star}(A,r,s) \leq geI_{T,T^\star}(A,r,s)$, then $A - geI_{T,T^\star}(A,r,s) \leq A - eI_{T,T^\star}(A,r,s)$. Therefore $geB_{T,T^\star}(A,r,s) \leq eB_{T,T^\star}(A,r,s)$.

(ii) For any an (r,s)-gfeo set $A \in I^X$, we have $A = geI_{T,T^\star}(A,r,s)$. Thus $geB_{T,T^\star}(A,r,s) = \underline{0}$.

(iii)
$$\begin{aligned} geB_{T,T^\star}(A,r,s) &= A - geI_{T,T^\star}(A,r,s) \\ &= A - (\underline{1} - geC_{T,T^\star}(\underline{1} - A)) \\ &\leq \underline{1} - \underline{1} + geC_{T.T^\star}(\underline{1} - A, r, s) \\ &= geC_{T,T^\star}(\underline{1} - A). \end{aligned}$$

(iv) $geI_{T,T^\star}(geB_{T,T^\star}(A,r,s),r,s) = geI_{T,T^\star}(A - geI_{T,T^\star}(A,r,s),r,s) \leq A - geI_{T,T^\star}(A,r,s) \leq A$,

by (i) of Proposition 2.2. Therefore $geI_{T,T^\star}(geB_{T,T^\star}(A,r,s),r,s) \leq A$.

(v)
$$\begin{aligned} geB_{T,T^\star}(A \vee B, r, s) &= (A \vee B) - geI_{T,T^\star}(A \vee B) \\ &= (A \vee B) - (geI_{T,T^\star}(A,r,s)) \vee (geI_{T,T^\star}(B,r,s)) \\ &\leq (A - geI_{T,T^\star}(A,r,s)) \vee (B - geI_{T,T^\star}(B,r,s)) \\ &= (geB_{T,T^\star}(A,r,s)) \vee (geB_{T,T^\star}(B,r,s)). \end{aligned}$$

Therefore, $geB_{T,T^\star}(A \vee B, r, s) \leq (geB_{T,T^\star}(A,r,s)) \vee (geB_{T,T^\star}(B,r,s))$.

(vi) It is similar to (v). ■

Definition 2.5.

For any dfts $(X, T, T^\star)$, $A \in I^X$, $\alpha \in I_0$ and $\beta \in I_1$, we have

(1) $eF_{T,T^\star}(A,r,s) = eC_{T,T^\star}(A,r,s) - eI_{T,T^\star}(A,r,s)$, where $eF_{T,T^\star}(A,r,s)$ is the (r,s)-fuzzy e-frontier of A.
(2) $geF_{T,T^\star}(A,r,s) = geC_{T,T^\star}(A,r,s) - geI_{T,T^\star}(A,r,s)$, where $geF_{T,T^\star}(A,r,s)$ is the (r,s)-generalized fuzzy e-frontier of A.

Proposition 2.6.

For any dfts $(X, T, T^\star)$, $A \in I^X$, $r \in I_0$ and $s \in I_1$, we have

(1) $geF_{T,T^\star}(A,r,s) \leq eF_{T,T^\star}(A,r,s)$.
(2) $geB_{T,T^\star}(A,r,s) \leq geF_{T,T^\star}(A,r,s)$.
(3) $geF_{T,T^\star}(\underline{1} - A, r, s) = geF_{T,T^\star}(A,r,s)$.
(4) $geF_{T,T^\star}(geI_{T,T^\star}(A,r,s),r,s) \leq geF_{T,T^\star}(A,r,s)$.
(5) $geF_{T,T^\star}(geC_{T,T^\star}(A,r,s),r,s) \leq geF_{T,T^\star}(A,r,s)$.
(6) $A - geF_{T,T^\star}(A,r,s) \leq geI_{T,T^\star}(A,r,s)$.
(7) $geF_{T,T^\star}(A \vee B, r, s) \leq (geF_{T,T^\star}(A,r,s)) \vee (GeF_{T,T^\star}(B,r,s))$.
(8) $geF_{T,T^\star}(A \wedge B, r, s) \geq (geF_{T,T^\star}(A,r,s)) \wedge (GeF_{T,T^\star}(B,r,s))$.

Proof:

(i) $$\begin{aligned} geF_{T,T^\star}(A,r,s) &= geC_{T,T^\star}(A,r,s) - geI_{T,T^\star}(A,r,s) \\ &\leq eC_{T,T^\star}(A,r,s) - eI_{T,T^\star}(A,r,s) \\ &= eF_{T,T^\star}(A,r,s). \end{aligned}$$

(ii) $$\begin{aligned} geB_{T,T^\star}(A,r,s) &= A - geI_{T,T^\star}(A,r,s) \\ &\leq geC_{T,T^\star}(A,r,s) - geI_{T,T^\star}(A,r,s), \end{aligned}$$

since $A \leq geC_{T,T^\star}(A,r,s) = geF_{T,T^\star}(A,r,s)$. Therefore, $geB_{T,T^\star}(A,r,s) \leq geF_{T,T^\star}(A,r,s)$.

(iii) $$\begin{aligned} geF_{T,T^\star}(A,r,s) &= geC_{T,T^\star}(A,r,s) - geI_{T,T^\star}(A,r,s) \\ &= geC_{T,T^\star}(A,r,s) - (\underline{1} - geC_{T,T^\star}(\underline{1} - A,r,s)) \\ &= geC_{T,T^\star}(A,r,s) - \underline{1} + geC_{T,T^\star}(\underline{1} - A,r,s) \\ &= -geI_{T,T^\star}(\underline{1} - A,r,s) + geC_{T,T^\star}(\underline{1} - A,r,s) \\ &= geF_{T,T^\star}(\underline{1} - A,r,s). \end{aligned}$$

(iv) $$\begin{aligned} geF_{T,T^\star}(geI_{T,T^\star}(A,r,s),r,s) &= geC_{T,T^\star}(geC_{T,T^\star}(A,r,s),r,s) \\ &\quad - geI_{T,T^\star}(geI_{T,T^\star}(A,r,s),r,s) \\ &\leq geC_{T,T^\star}(A,r,s) - geI_{T,T^\star}(A,r,s) \\ &= geF_{T,T^\star}(A,r,s). \end{aligned}$$

(v) $$\begin{aligned} geF_{T,T^\star}(geC_{T,T^\star}(A,r,s),r,s) &= geC_{T,T^\star}(geC_{T,T^\star}(A,r,s),r,s) \\ &\quad - geI_{T,T^\star}(geC_{T,T^\star}(A,r,s),r,s) \\ &= geC_{T,T^\star}(A,r,s) - geI_{T,T^\star}(geC_{T,T^\star}(A,r,s),r,s) \\ &\geq geC_{T,T^\star}(A,r,s) - geI_{T,T^\star}(A,r,s) \\ &= geF_{T,T^\star}(A,r,s). \end{aligned}$$

(vi) $$\begin{aligned} A - geF_{T,T^\star}(A,r,s) &= A - (geC_{T,T^\star}(A,r,s) - geI_{T,T^\star}(A,r,s)) \\ &\leq geC_{T,T^\star}(A,r,s) - geC_{T,T^\star}(A,r,s) + geI_{T,T^\star}(A,r,s) \\ &= geI_{T,T^\star}(A,r,s). \end{aligned}$$

(vii) $$\begin{aligned} geF_{T,T^\star}(A \vee B,r,s) &= geC_{T,T^\star}(A \vee B,r,s) - geI_{T,T^\star}(A \vee B,r,s) \\ &= geC_{T,T^\star}(A \vee B,r,s) - (geI_{T,T^\star}(A,r,s) \vee geI_{T,T^\star}(B,r,s)) \\ &= (geC_{T,T^\star}(A,r,s) \vee geC_{T,T^\star}(B,r,s)) - (geI_{T,T^\star}(A,r,s) \vee geI_{T,T^\star}(B,r,s)) \\ &\leq (geC_{T,T^\star}(A,r,s) - geI_{T,T^\star}(A,r,s)) \vee (geC_{T,T^\star}(A,r,s) - geI_{T,T^\star}(B,r,s)) \\ &= geF_{T,T^\star}(A,r,s) \vee geF_{T,T^\star}(B,r,s). \end{aligned}$$

(viii) It is similar to (vii). ■

Definition 2.7.

For any dfts $(X, T, T^\star)$, for each $A \in I^X$, $r \in I_0$ and $s \in I_1$, we have

(1) $eE_{T,T^\star}(A,r,s) = eI_{T,T^\star}(\underline{1} - A,r,s)$ where $eE_{T,T^\star}(A,r,s)$ is the (r,s)-fuzzy e-exterior of A.
(2) $geE_{T,T^\star}(A,r,s) = geI_{T,T^\star}(\underline{1} - A,r,s)$ where $geE_{T,T^\star}(A,r,s)$ is the (r,s)-generalized fuzzy e-exterior of A.

Proposition 2.8.

For any dfts $(X, T, T^{\star})$, for every $A \in I^X,\ r \in I_0$ and $s \in I_1$, we have

(1) $eE_{T,T^{\star}}(A, r, s) \leq geE_{T,T^{\star}}(A, r, s)$.
(2) $geE_{T,T^{\star}}(A, r, s) = \underline{1} - geC_{T,T^{\star}}(A, r, s)$.
(3) $geE_{T,T^{\star}}(geE_{T,T^{\star}}(A, r, s), r, s) = geI_{T,T^{\star}}(geC_{T,T^{\star}}(A, r, s), r, s)$.
(4) If $A \leq B$, then $geE_{T,T^{\star}}(A, r, s) \geq geE_{T,T^{\star}}(B, r, s)$.
(5) $geE_{T,T^{\star}}(\underline{1}, r, s) = \underline{0}$.
(6) $geE_{T,T^{\star}}(\underline{0}, r, s) = \underline{1}$.
(7) $geI_{T,T^{\star}}(A, r, s) \leq geE_{T,T^{\star}}(geE_{T,T^{\star}}(A, r, s), r, s)$.
(8) $geE_{T,T^{\star}}(A \vee B, r, s) = geE_{T,T^{\star}}(A, r, s) \wedge geE_{T,T^{\star}}(B, r, s)$.
(9) $geE_{T,T^{\star}}(A \wedge B, r, s) = geE_{T,T^{\star}}(A, r, s) \vee geE_{T,T^{\star}}(B, r, s)$.

Proof:

(i) Since

$$\begin{aligned} geC_{T,T^{\star}}(A, r, s) &\leq eC_{T,T^{\star}}(A, r, s), \\ \underline{1} - geC_{T,T^{\star}}(A, r, s) &\geq \underline{1} - eC_{T,T^{\star}}(A, r, s) \\ \Rightarrow geI_{T,T^{\star}}(\underline{1} - A, r, s) &\geq eI_{T,T^{\star}}(\underline{1} - A, r, s), \end{aligned}$$

therefore, by definition, $geE_{T,T^{\star}}(A, r, s) \geq eE_{T,T^{\star}}(A, r, s)$.

(ii) It follows from the definitions.

(iii) $geE_{T,T^{\star}}(geE_{T,T^{\star}}(A, r, s), r, s) = geI_{T,T^{\star}}(\underline{1} - geE_{T,T^{\star}}(A, r, s), r, s) = geI_{T,T^{\star}}(\underline{1} - geI_{T,T^{\star}}(A, r, s), r, s)$. Therefore $geE_{T,T^{\star}}(geE_{T,T^{\star}}(A, r, s), r, s) = geI_{T,T^{\star}}(geC_{T,T^{\star}}(A, r, s), r, s)$.

(iv) Let $A \leq B$. By using Proposition 2.2(i), $geC_{T,T^{\star}}(A, r, s) \leq geC_{T,T^{\star}}(B, r, s)$. Therefore $\underline{1} - geC_{T,T^{\star}}(A, r, s) \geq \underline{1} - geC_{T,T^{\star}}(B, r, s)$. But $geI_{T,T^{\star}}(\underline{1} - A, r, s) \geq geI_{T,T^{\star}}(\underline{1} - B, r, s)$. Hence, $geE_{T,T^{\star}}(A, r, s) \geq geE_{T,T^{\star}}(B, r, s)$. (v) By (ii) $geE_{T,T^{\star}}(\underline{1}, r, s) = \underline{1} - geC_{T,T^{\star}}(\underline{1}, r, s)$ $= \underline{1} - \underline{1} = \underline{0}$.

(vi) It is similar to (v).

(vii) Since $A \leq geC_{T,T^{\star}}(A, r, s)$, and $geI_{T,T^{\star}}(A, r, s) \leq geI_{T,T^{\star}}(geC_{T,T^{\star}}\ (A, r, s), r, s)$. Then by (iii), $geI_{T,T^{\star}}(A, r, s) \leq geE_{T,T^{\star}}(geE_{T,T^{\star}}(A, r, s), r, s)$.

(viii) $geE_{T,T^{\star}}(A \vee B, r, s) = geI_{T,T^{\star}}(\underline{1} - (A \vee B), r, s) = geI_{T,T^{\star}}(\underline{1} - A) \wedge (\underline{1} - B), r, s) = (geI_{T,T^{\star}}((\underline{1} - A), r, s) \wedge (geI_{T,T^{\star}}((\underline{1} - B), r, s)) = geE_{T,T^{\star}}(A, r, s) \wedge geE_{T,T^{\star}}(B, r, s)$

(ix) Obvious. ∎

3. Characterizations of generalized double fuzzy e-closure irresolute and generalized double fuzzy e-continuous functions

In this section, some characterizations of generalized double fuzzy e-continuous,generalized double fuzzy e-open, generalized double fuzzy e-closed and generalized double fuzzy e-closure irresolute functions are studied.

Definition 3.1.

A map $f:(X,T,T^\star)\to(Y,S,S^\star)$ between any two dfts's $(X,T,T^\star)$ and $(Y,S,S^\star)$ is called:

(1) generalized double fuzzy e-open (briefly, gdfe-open) if for every (r,s)-gfeo set $A\in I^X,\ r\in I_0$ and $s\in I_1,\ f(A)$ is an (r,s)-gfeo in I^Y.
(2) generalized double fuzzy e-closed (briefly, gdfe-closed) if for every (r,s)-gfec set $A\in I^X,\ r\in I_0$ and $s\in I_1,\ f(A)$ is an (r,s)-gfec in I^Y.
(3) generalized double fuzzy e-continuous (briefly, gdfe-continuous) if for every (r,s)-fo set $A\in I^Y,\ r\in I_0$ and $s\in I_1,\ f^{-1}(A)$ is an (r,s)-gfeo in I^X.
(4) generalized double fuzzy e-closure irresolute (briefly, gdfec-Irr) if $f^{-1}(geC_{T,T^\star}(f(A),r,s)$ is an (r,s)-gfec set for every (r,s)-gfec set $A\in I^Y,\ r\in I_0,\ s\in I_1$.

Theorem 3.2.

For any two dfts's $(X,T,T^\star)$ and $(Y,S,S^\star)$ and any bijective map $f:(X,T,T^\star)\to(Y,S,S^\star)$, we have

(1) f is gdfec-Irr function.
(2) For every fuzzy set A in $I^X,\ f(geC_{T,T^\star}(A,r,s))\le geC_{S,S^\star}(f(A),r,s)$.
(3) For every fuzzy set B in $I^Y,\ geC_{T,T^\star}(f^{-1}(B),r,s)\le f^{-1}(geC_{S,S^\star}(B,r,s)$.

Proof:

(i) $\Rightarrow$ (ii) Suppose $A\in I^X$ and $geC_{T,T^\star}(f(A),r,s)\in I^Y$ is an (r,s)-gfec, then by (i), we have have $f^{-1}(geC_{S,S^\star}(f(A),r,s)\in I^X$ is an (r,s)-gfec set, $r\in I_0$ and $s\in I_1$. Therefore, $geC_{S,S^\star}(f^{-1}(geC_{S,S^\star}(f(A),r,s),r,s)=f^{-1}(geC_{S,S^\star}(f(A),r,s))$. Since $A\le f^{-1}(f(A))$ and $geC_{T,T^\star}(A,r,s)\le geC_{S,S^\star}(f^{-1}(f(A),r,s))$. Also, $f(A)\le geC_{S,S^\star}(f(A),r,s)$. Then $geC_{T,T^\star}(A,r,s)\le geC_{S,S^\star}(f^{-1}(geC_{S,S^\star}(f(A),r,s),r,s))$
$=f^{-1}(geC_{S,S^\star}(f(A),r,s))$.

(ii) $\Rightarrow$ (iii) Suppose $B\in I^Y$, by (ii)

$f(geC_{S,S^\star}(f^{-1}(B),r,s)\le geC_{S,S^\star}(f(f^{-1}(B)),r,s)\le geC_{S,S^\star}(B,r,s)$.

That is, $f(geC_{T,T^\star}(f^{-1}(B),r,s))\le geC_{S,S^\star}(B,r,s)$.

Therefore, $f^{-1}(f(geC_{S,S^\star}(f^{-1}(B),r,s)))\le f^{-1}(geC_{S,S^\star}(B,r,s))$.

Hence, $geC_{T,T^\star}(f^{-1}(B), r, s) \leq f^{-1}(geC_{S,S^\star}(B, r, s))$

(iii) $\Rightarrow$ (i) Suppose $B \in I^Y$ is an (r, s)-gfec set. Then, $geC_{S,S^\star}(B, r, s) = B$. By (iii) $geC_{T,T^\star}(f^{-1}(B), r, s) \leq f^{-1}(geC_{S,S^\star}(B, r, s)) = f^{-1}(B)$.

But $f^{-1}(B) \leq geC_{T,T^\star}(f^{-1}(B), r, s)$.

Therefore, $f^{-1}(B) = geC_{S,S^\star}(B, r, s)$. That is, $f^{-1}(B) \in I^X$ is (r, s)-gfec. Thus, f is gdfec-Irr map. ∎

Proposition 3.3.

The map $f : (X, T, T^\star) \to (Y, S, S^\star)$ between any two dfts's $(X, T, T^\star)$ and $(Y, S, S^\star)$ is a gdfe-closed iff for each $A \in I^X,\ geC_{T,T^\star}(f(A), r, s) \leq f(geC_{T,T^\star}(A, r, s))$.

Proof:

Suppose that f is a gdfe-closed function and A is any fuzzy set in X. Then $f(geC_{T,T^\star}(A, r, s))$ is an (r, s)-gfec in I^Y. Therefore, $geC_{T,T^\star}(f(geC_{T,T^\star}(A, r, s), r, s) = f(geC_{T,T^\star}(A, r, s))$. Since $A \leq geC_{T,T^\star}(A, r, s) \Rightarrow f(A) \leq f(geC_{T,T^\star}(A, r, s))$. Therefore, $geC_{T,T^\star}(f(A), r, s) \leq geC_{T,T^\star}(f(geC_{T,T^\star}(A, r, s)), r, s) = f(geC_{T,T^\star}(A, r, s))$. Hence, for every fuzzy set $A \in I^X,\ geC_{T,T^\star}(f(A), r, s) \leq f(geC_{T,T^\star}(A, r, s)$. Conversely, suppose that for every fuzzy set $A \in I^X,\ geC_{T,T^\star}(f(A), r, s) \leq f(geC_{T,T^\star}(A, r, s))$. Since A is an (r, s)-gfec set, we have $geC_{T,T^\star}(A, r, s) = A$. Therefore, $f(geC_{T,T^\star}(A, r, s)) = f(A) \leq geC_{S,S^\star}(A, r, s)$. Hence, $f(A) = f(geC_{T,T^\star}(A, r, s)) = geC_{T,T^\star}(f(A), r, s)$, which implies that $f(A) \in I^Y$ is an (r, s)-gfec set, that is f is gdfe-closed function. ∎

Proposition 3.4.

For any two dfts's $(X, T, T^\star)$ and $(Y, S, S^\star)$ and any gdfec-Irr function $f : (X, T, T^\star) \to (Y, S, S^\star)$, $geB_{T,T^\star}(f^{-1}(A), r, s)$ is zero for every (r, s)-gfeo set $A \in I^Y$.

Proof:

Let A be an (r, s)-gfeo set in $I^Y,\ f^{-1}(A) \in I^X$ is (r, s)-gfeo. Therefore, $geI_{S,S^\star}(f^{-1}(A), r, s) = f^{-1}(A)$. By definition, $geB_{S,S^\star}(f^{-1}(A), r, s) = f^{-1}(A) - geI_{S,S^\star}(f^{-1}(A), r, s)$. Hence, $geB_{T,T^\star}(f^{-1}(A), r, s) = f^{-1}(A) - f^{-1}(A) = \underline{0}$. ∎

Definition 3.5.

A dfts $(X, T, T^\star)$ is said to be a double fuzzy e-$(T, T^\star)_{\frac{1}{2}}$ space (briefly, dfe-$(T, T^\star)_{\frac{1}{2}}$), if each (r, s)-fec set is (r, s)-fc set in X.

Proposition 3.6.

For any dfts $(X, T, T^\star)$ and dfe-$(T, T^\star)_{\frac{1}{2}}$ space $(Y, S, S^\star)$, if the map $f : (X, T, T^\star) \to (Y, S, S^\star)$ is a bijective, the next statements will be equivalent:

(1) f and f^{-1} are gdfec-Irr.
(2) f is gdfe-continuous and gdfe-open.
(3) f is gdfe-continuous and gdfe-closed.
(4) $f(geC_{T,T^\star}(A,r,s)) = geC_{S,S^\star}(f(A),r,s)$, for every $A \in I^X$.

Proof:

(i) $\Rightarrow$ (ii) Suppose B is an (r,s)-gfeo set in X. Since f^{-1} is gdfec-Irr, $(f^{-1})^{-1}(B) \in I^Y$ is (r,s)-gfeo set, so f is gdfe-open. Now, let $C \in I^Y$ be an (r,s)-feo set, then it is an (r,s)-gfeo. But by hypothesis, f^{-1} are gdfec-Irr, then $f^{-1}(D) \in I^X$ is an (r,s)-gfeo, that is f is gdfe-continuous.

(ii) $\Rightarrow$ (iii) Let $A \in I^X$ is an (r,s)-gfec set, then $\underline{1} - A \in I^X$ is an (r,s)-gfeo set. By (ii), $\underline{1} - f(A) = f(\underline{1} - A)$ is an (r,s)-gfeo set in Y, which implies that $f(A)$ is an (r,s)-gfec set. Hence f is a gdfe-closed function.

(iii) $\Rightarrow$ (iv) Let $A \in I^X$, we have $A \leq f^{-1}(f(A))$ and $f(A) \leq geC_{S,S^\star}(f(A),r,s) \Rightarrow A \leq f^{-1}(geC_{S,S^\star}(f(A),r,s)$. Now, $geC_{S,S^\star}(f(A),r,s) \in I^Y$ is an (r,s)-gfec set. But $(Y,S,S^\star)$ is a dfe-$(T,T^\star)_{\frac{1}{2}}$ space, and $geC_{S,S^\star}(f(A),r,s)$ is an (r,s)-fc set, then $geC_{S,S^\star}(f(A),r,s) \in I^Y$ is an (r,s)-fec set. Since f is gdfe-continuous, $f^{-1}(geC_{S,S^\star}(f(A),r,s))$ is an (r,s)-gfec set, which implies $geC_{S,S^\star}(f^{-1}(geC_{S,S^\star}(f(A),r,s),r,s) = f^{-1}(geC_{S,S^\star}(f(A),r,s))$. But $geC_{S,S^\star}(f(A),r,s) \leq geC_{S,S^\star}(f^{-1}(geC_{S,S^\star}(f(A),r,s)),r,s)$, and $geC_{S,S^\star}(f(A),r,s) \leq f^{-1}(geC_{S,S^\star}(f(A),r,s))$. Therefore, $f(geC_{T,T^\star}(A,r,s)) \leq geC_{S,S^\star}(f(A),r,s)$. Also, $geC_{S,S^\star}(f(A),r,s) \leq f(geC_{T,T^\star}(A,r,s))$
$\Rightarrow f(geC_{T,T^\star}(A,r,s) = geC_{S,S^\star}(f(A),r,s)$.

(iv) $\Rightarrow$ (i) Let $A \in I^X$, by hypothesis of (iv), $f(geC_{T,T^\star}(A,r,s) = geC_{S,S^\star}(f(A),r,s)$. Therefore, $f(geC_{T,T^\star}(A,r,s) \leq geC_{S,S^\star}(f(A),r,s)$. Then, by Proposition 3.3, f is a gdfec-Irr function. Now, suppose $B \in I^Y$ is (r,s)-gfec. Then, $geC_{S,S^\star}(B,r,s) = B \Rightarrow f(geC_{S,S^\star}(B,r,s)) = f(B)$. But by (iv), $geC_{T,T^\star}(f(B),r,s) = f(geC_{S,S^\star}(B,r,s))$. Therefore, $geC_{T,T^\star}(f(B),r,s) = f(B)$. Then, $f(B) \in I^Y$ is an (r,s)-gfec set. Therefore, f^{-1} is gdfec-Irr. ■

4. Interrelations

In this section, we present the relationship among the concepts introduced in Sections 3 and 4.

Proposition 4.1.

If A is an (r,s)-gfec set in a dfts $(X,T,T^\star)$ then:

(1) $geB_{T,T^\star}(A,r,s) = geF_{T,T^\star}(A,r,s)$.
(2) $geE_{T,T^\star}(A,r,s) = \underline{1} - A$.

Proof:

(i) Let $A \in I^X$ be an (r,s)-gfec, we have $geC_{T,T^\star}(A,r,s) = A$. But by definiton

$geB_{T,T^\star}(A,r,s) = A - geI_{T,T^\star}(A,r,s) = geC_{T,T^\star}(A,r,s) - geI_{T,T^\star}(A,r,s) = geF_{T,T^\star}(A,r,s)$. Therefore $geB_{T,T^\star}(A,r,s) = geF_{T,T^\star}(A,r,s)$.

(ii) Let A be an (r,s)-gfec set, we get $geC_{T,T^\star}(A,r,s) = A \Rightarrow geI_{T,T^\star}(\underline{1} - A,r,s) = \underline{1} - A$. Therefore by definition, $geE_{T,T^\star}(A,r,s) = \underline{1} - A$. ■

Proposition 4.2.

For any two dfts's $(X,T,T^\star)$ and $(Y,S,S^\star)$ and any map $f : (X,T,T^\star) \to (Y,S,S^\star)$

(1) If f is any function, then $geE_{T,T^\star}(f^{-1}(A),r,s) \leq geC_{T,T^\star}(\underline{1} - f^{-1}(A),r,s)$, for each fuzzy set $A \in I^Y$.
(2) If f is a gdfe-continuous function, then for every (r,s)-fec set $A \in I^Y$, we have $geB_{T,T^\star}(f^{-1}(A),r,s) = geF_{T,T^\star}(f^{-1}(A),r,s)$.

Proof:

(i) Let $A \in I^Y$. Then, by definition $geE_{T,T^\star}(f^{-1}(A),r,s) = geI_{T,T^\star}(\underline{1} - f^{-1}(A),r,s) \leq \underline{1} - f^{-1}(A)$.

Also, $geE_{T,T^\star}(f^{-1}(A),r,s) \leq \underline{1} - geI_{T,T^\star}(f^{-1}(A),r,s) = geC_{T,T^\star}(\underline{1} - f^{-1}(A),r,s)$.

Therefore, $geE_{T,T^\star}(f^{-1}(A),r,s) \leq geC_{T,T^\star}(\underline{1} - f^{-1}(A),r,s)$.

(ii) Let A be an (r,s)-fec set in Y. Then, $f^{-1}(A)$ is an (r,s)-gfec set in X. Therefore, $geC_{T,T^\star}(f^{-1}(A),r,s) = f^{-1}(A)$. Hence, $geF_{T,T^\star}(f^{-1}(A),r,s) = geC_{T,T^\star}(f^{-1}(A),r,s) - geI_{T,T^\star}(f^{-1}(A),r,s) = f^{-1}(A) - geI_{T,T^\star}(f^{-1}(A),r,s) = geB_{T,T^\star}(f^{-1}(A),r,s)$. Therefore, $geF_{T,T^\star}(f^{-1}(A),r,s) = geB_{T,T^\star}(f^{-1}(A),r,s)$. ■

Definition 4.3.

A dfts $(X,T,T^\star)$ is said to be a $g^\star$ double fuzzy e-$(T,T^\star)_{\frac{1}{2}}$ space (briefly, $g^\star dfe$-$(T,T^\star)_{\frac{1}{2}}$ if each (r,s)-gfec set in X is an (r,s)-gfc set.

Proposition 4.4.

Let $(X,T,T^\star)$ be a $g^\star dfe$-$(T,T^\star)_{\frac{1}{2}}$ space and A be an (r,s)-gfec set in X. Then the following statements hold:

(1) $gB_{T,T^\star}(A,r,s) = gF_{T,T^\star}(A,r,s)$.
(2) $gE_{T,T^\star}(A,r,s) = \underline{1} - A$.

Proof:

Let $A \in I^X$ be an (r,s)-gfec set. Then A is an (r,s)-gfc set in X, which implies $gC_{T,T^\star}(A,r,s) = A$. But by definition, $gB_{T,T^\star}(A,r,s) = A - gI_{T,T^\star}(A,r,s) = gC_{T,T^\star}(A,r,s) - gI_{T,T^\star}(A,r,s) = gF_{T,T^\star}(A,r,s)$.

(ii) By definition, $gE_{T,T^\star}(A,r,s) = gI_{T,T^\star}(A,r,s) = \underline{1} - A$. ■

Proposition 4.5.

Let $(X,T,T^\star)$ and $(Y,S,S^\star)$ be dfts's and let $f : (X,T,T^\star) \to (Y,S,S^\star)$ be gdfec-Irr function such that $(X,T,T^\star)$ is $g^\star dfe$-$(T,T^\star)_{\frac{1}{2}}$. Then for every an (r,s)-gfec set $A \in I^X$, the following statements hold:

(1) $gB_{T,T^\star}(f^{-1}(A),r,s) = gF_{T,T^\star}(f^{-1}(A),r,s)$.
(2) $gE_{T,T^\star}(f^{-1}(A),r,s) = \underline{1} - f^{-1}(A)$.

Proof:

(i) Suppose $A \in I^Y$ is an (r,s)-gfec set. Then, $f^{-1}(A)$ is an (r,s)-gfec set in X. Since by hypothesis, $(X,T,T^\star)$ is $g^\star dfe$-$(T,T^\star)_{\frac{1}{2}}$ space, $f^{-1}(A)$ is an (r,s)-gfc set in X. Therefore, $gC_{T,T^\star}(f^{-1}(A),r,s) = f^{-1}(A)$. Also $gB_{T,T^\star}(f^{-1}(A),r,s) = f^{-1}(A) - gI_{T,T^\star}(f^{-1}(A),r,s) = gC_{T,T^\star}(f^{-1}(A),r,s) - gI_{T,T^\star}(f^{-1}(A),r,s) = gF_{T,T^\star}(f^{-1}(A),r,s)$. Therefore $gB_{T,T^\star}(f^{-1}(A),r,s) = gF_{T,T^\star}(f^{-1}(A),r,s)$.

(ii) By definition $gE_{T,T^\star}(f^{-1}(A),r,s) = gI_{T,T^\star}(\underline{1} - f^{-1}(A),r,s) = \underline{1} - f^{-1}(A)$. ■

The above is not true if $(X,T,T^\star)$ is not $g^\star dfe$-$(T,T^\star)_{\frac{1}{2}}$ space, as in the following example.

Example 4.6.

Let $X = \{a,b\}$ and $f : (X,T,T^\star) \to (Y,S,S^\star)$ be the identity map. Define A_1, A_2, A_3, B_1 and B_2 as follows: $A_1(a) = 0.67, A_1(b) = 0.64, A_2(a) = 0.67, A_2(b) = 0.35, A_3(a) = 0.33, A_3(b) = 0.34, B_1(a) = 0.20, B_1(b) = 0.30, B_2(a) = 0.67, B_2(b) = 0.49$, and define $(T,T^\star)$ and $(S,S^\star)$ as follows:

$$T(A) = \begin{cases} 1, & \text{if } A \in \{\underline{0},\underline{1}\}, \\ \frac{1}{4}, & \text{if } A = A_1, \\ \frac{1}{2}, & \text{if } A = A_2, \\ \frac{3}{4}, & \text{if } A = A_3, \\ 0, & \text{otherwise.} \end{cases} \qquad T^\star(A) = \begin{cases} 0, & \text{if } A \in \{\underline{0},\underline{1}\}, \\ \frac{3}{4}, & \text{if } A = A_1, \\ \frac{1}{2}, & \text{if } A = A_2, \\ \frac{1}{4}, & \text{if } A = A_3, \\ 1, & \text{otherwise.} \end{cases}$$

$$S(A) = \begin{cases} 1, & \text{if } A \in \{\underline{0},\underline{1}\}, \\ \frac{1}{4}, & \text{if } A = B_1, \\ \frac{1}{8}, & \text{if } A = B_2, \\ 0, & \text{otherwise.} \end{cases} \qquad S^\star(A) = \begin{cases} 0, & \text{if } A \in \{\underline{0},\underline{1}\}, \\ \frac{1}{8}, & \text{if } A = B_1, \\ \frac{1}{4}, & \text{if } A = B_2, \\ 1, & \text{otherwise.} \end{cases}$$

Then, A is an (r,s)-gfec set in $(X,S,S^\star)$ and $f^{-1}(A) = A$ is an (r,s)-gfec set which is not

(r,s)-gfc set in $(X,T,T^\star)$. Therefore, f is gdfec-Irr function, but $(X,T,T^\star)$ is not $g^\star$dfe-$(T,T^\star)_{\frac{1}{2}}$ space.

Proposition 4.7.

Let $f:(X,T,T^\star)\to(Y,S,S^\star)$ be gdfe-closed map such that $(Y,S,S^\star)$ is $g^\star dfe$-$(T,T^\star)_{\frac{1}{2}}$ space. Then for each (r,s)-gfec set A in I^X, the following statements hold:

(1) $gB_{S,S^\star}(f(A),r,s)=gF_{S,S^\star}(f(A),r,s)$.
(2) $gE_{S,S^\star}(f(A),r,s)=\underline{1}-f(A)$.

Proof:

Similar to the proof of Proposition 4.5. ■

The above statement is not true if $(Y,S,S^\star)$ is not $g^\star dfe$-$(T,T^\star)_{\frac{1}{2}}$ space, as shown in the following example.

Example 4.8.

Let $X=\{a,b\}$ and $f:(X,T,T^\star)\to(Y,S,S^\star)$ be the identity map. Define A_1,A_2,A_3,B_1 and B_2 as follows: $A_1(a)=0.67, A_1(b)=0.64, A_2(a)=0.67, A_2(b)=0.35, A_3(a)=0.33, A_3(b)=0.34, B_1(a)=0.20, B_1(b)=0.30, B_2(a)=0.67, B_2(b)=0.49$, and define $(T,T^\star)$ and $(S,S^\star)$ as follows:

$$T(A)=\begin{cases}1, & \text{if } A\in\{\underline{0},\underline{1}\},\\ \frac{1}{4}, & \text{if } A=B_1,\\ \frac{1}{8}, & \text{if } A=B_2,\\ 0, & \text{otherwise.}\end{cases}\qquad T^\star(A)=\begin{cases}0, & \text{if } A\in\{\underline{0},\underline{1}\},\\ \frac{1}{8}, & \text{if } A=B_1,\\ \frac{1}{4}, & \text{if } A=B_2,\\ 1, & \text{otherwise.}\end{cases}$$

$$S(A)=\begin{cases}1, & \text{if } A\in\{\underline{0},\underline{1}\},\\ \frac{1}{4}, & \text{if } A=A_1,\\ \frac{1}{2}, & \text{if } A=A_2,\\ \frac{3}{4}, & \text{if } A=A_3,\\ 0, & \text{otherwise.}\end{cases}\qquad S^\star(A)=\begin{cases}0, & \text{if } A\in\{\underline{0},\underline{1}\},\\ \frac{3}{4}, & \text{if } A=A_1,\\ \frac{1}{2}, & \text{if } A=A_2,\\ \frac{1}{4}, & \text{if } A=A_3,\\ 1, & \text{otherwise.}\end{cases}$$

Then, A is an (r,s)-gdfe closed set in $(X,T,T^\star)$ Therefore, f is gdfe closed function. but $f(A)=A$ is not (r,s)-gfc set in $(X,S,S^\star)$. Hence $(X,S,S^\star)$ is not $g^\star$dfe-$(T,T^\star)_{\frac{1}{2}}$ space.

Proposition 4.9.

Let $(X,T,T^\star)$, $(Y,S,S^\star)$ and $(Z,U,U^\star)$ be dfts's $f:(X,T,T^\star)\to(Y,S,S^\star)$ and g :

$(Y, S, S^\star) \to (Z, U, U^\star)$ be gdfec-Irr functions such that $(X, T, T^\star)$ is $g^\star dfe$-$(T, T^\star)_{\frac{1}{2}}$ space. Then for each (r, s)-gfec set in Z, the following statements hold:

(1) $gB_{T,T^\star}((g \circ f)^{-1}(A), r, s) = gF_{T,T^\star}((g \circ f)^{-1}(A), r, s)$.
(2) $gE_{T,T^\star}((g \circ f)^{-1}(A), r, s) = \underline{1} - (g \circ f)^{-1}(A)$.

Proof:

(i) Let A be a (r, s)-gfec set in Z. Then by hypothesis of g is gdfec-Irr, $g^{-1}(A) \in I^Y$ is a (r, s)-gfec set. Also, f is gdfec-Irr, so $f^{-1}(g^{-1}(A)) \in I^X$ is (r, s)-gfec set. Thus $(g \circ f)^{-1}(A) \in I^X$ is a (r, s)-gfec. Since $(X, T, T^\star)$ is $g^\star dfe$-$(T, T^\star)_{\frac{1}{2}}$, $(g \circ f)^{-1}(A)$ is a (r, s)-gfc in X. So by definition, we have $gB_{T,T^\star}((g \circ f)^{-1}(A), r, s) = (g \circ f)^{-1}(A) - gI_{T,T^\star}((g \circ f)^{-1}(A), r, s) = gC_{T,T^\star}((g \circ f)^{-1}(A), r, s) - gI_{T,T^\star}((g \circ f)^{-1}(A), r, s) = gF_{T,T^\star}((g \circ f)^{-1}(A), r, s)$.

(ii) By definition, $gE_{T,T^\star}((g \circ f)^{-1}(A), r, s) = gI_{T,T^\star}(\underline{1} - (g \circ f)^{-1}(A), r, s)\ \underline{1} - gC_{T,T^\star}((g \circ f)^{-1}(A), r, s) = \underline{1} - (g \circ f)^{-1}(A)$, since $(g \circ f)^{-1}(A)$ is a (r, s)-gfc set. Therefore, $gE_{T,T^\star}((g \circ f)^{-1}(A), r, s) = \underline{1} - (g \circ f)^{-1}(A)$.

The above statement is not true if $(X, T, T^\star)$ is not a $g^\star dfe$-$(T, T^\star)_{\frac{1}{2}}$ space, as shown in the following example. ■

Example 4.10.

Let $X = \{a, b\}$ and $f : (X, T, T^\star) \to (Y, S, S^\star)$ and $g : (Y, S, S^\star) \to (Z, U, U^\star)$be the identity map. Define A_1, A_2, A_3, B_1, B_2 and C_1 as follows: $A_1(a) = 0.67, A_1(b) = 0.64, A_2(a) = 0.67, A_2(b) = 0.35, A_3(a) = 0.33, A_3(b) = 0.34, B_1(a) = 0.20, B_1(b) = 0.30, B_2(a) = 0.67, B_2(b) = 0.49$, $C_1(a) = 0.50, C_1(b) = 0.40$, and define $(T, T^\star)$, $(S, S^\star)$ and $(U, U^\star)$ as follows:

$$T(A) = \begin{cases} 1, & \text{if } A \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{4}, & \text{if } A = A_1, \\ \frac{1}{2}, & \text{if } A = A_2, \\ \frac{3}{4}, & \text{if } A = A_3, \\ 0, & \text{otherwise.} \end{cases} \qquad T^\star(A) = \begin{cases} 0, & \text{if } A \in \{\underline{0}, \underline{1}\}, \\ \frac{3}{4}, & \text{if } A = A_1, \\ \frac{1}{2}, & \text{if } A = A_2, \\ \frac{1}{4}, & \text{if } A = A_3, \\ 1, & \text{otherwise.} \end{cases}$$

$$S(A) = \begin{cases} 1, & \text{if } A \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{4}, & \text{if } A = B_1, \\ \frac{1}{8}, & \text{if } A = B_2, \\ 0, & \text{otherwise.} \end{cases} \qquad S^\star(A) = \begin{cases} 0, & \text{if } A \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{8}, & \text{if } A = B_1, \\ \frac{1}{4}, & \text{if } A = B_2, \\ 1, & \text{otherwise.} \end{cases}$$

$$U(A)=\begin{cases}1, & \text{if } A\in\{\underline{0},\underline{1}\},\\ \frac{1}{4}, & \text{if } A=C_1,\\ 0, & \text{otherwise.}\end{cases}\qquad U^\star(A)=\begin{cases}0, & \text{if } A\in\{\underline{0},\underline{1}\},\\ \frac{1}{8}, & \text{if } A=C_1,\\ 1, & \text{otherwise.}\end{cases}$$

Then, f and g are gdfec-Irr function, but $(X,T,T^\star)$ is not $g^\star$dfp-$(T,T^\star)_{\frac{1}{2}}$ space as $(g\circ f)^{-1}(A)$ is a (r,s)-gfec set but not an (r,s)-gfc set.

5. Conclusion

In this paper (r,s)-generalized fuzzy e-border, (r,s)-generalized fuzzy e-exterior and (r,s)-generalized fuzzy e-frontier in double fuzzy topologies. Furthermore, some characterizations of generalized double fuzzy e-continuous, generalized double fuzzy e-open, generalized double fuzzy e-closed and generalized double fuzzy e-closure-irresolute functions are studied and investigated. Moreover, the interrelations among the new concepts are discussed with some necessary examples.

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