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Summation Formulas for the Confluent Hypergeometric Function $\Phi_2^{(2r)}$ of Several Variables

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Abstract

In this paper, we establish a general summation formula for the confluent hypergeometric function $\Phi_2^{(2r)}$ of several variables by applying the generalized Kummer's summation theorem due to Lavoie et al. As an applications of our main result, we obtain certain new summation formulas for the confluent hypergeometric function $\Phi_2^{(4)}$. Also some summation and transformation formulas including a results obtained recently by Choi and Rathie have been obtained as special cases.

Keywords: Summation formulas; Kummer's summation theorem; Confluent hypergeometric function; Kampé de Fériet function

MSC 2010 No.: Primary 33C20, 33C70; Secondary 33B15, 33C05

1. Introduction

In 1992-1996, Lavoie et al. (1992), Lavoie et al. (1994) and Lavoie et al. (1996), established the generalizations of the well-known classical summation theorems of Watson, Dixon, Whipple and Kummer. Recently, many summation and transformation formulas for the different hypergeometric functions have been considered by applications of the above mentioned generalizations (see, for example, Ali (2013), Atash (2015), Choi and Rathie (2015a), Choi and Rathie (2015b), Kim and Rathie (2007), Kim and Rathie (2009), Mohsen et al. (2016), Srivastava et al. (2014)). Motivated from the above mentioned works, we establish further summation formula for the confluent hypergeometric function of several variables $\Phi_2^{(2r)}$.

2. Preliminaries

The following definitions are given in Srivastava and Manocha (1984).

Definition 2.1.

The generalized hypergeometric function ${}_pF_q$ with p numerator parameters and q denominator parameters is defined by

$$ {}_pF_q\left[\begin{array}{lc} a_1,\ldots,a_p\ ; & \\ & x \\ b_1,\ldots,b_q\ ; & \end{array}\right]=\sum_{n=0}^{\infty}\frac{(a_1)_n\,\ldots\,(a_p)_n}{(b_1)_n\,\ldots\,(b_q)_n}\,\frac{x^n}{n!}, \tag{1}$$

where $(a)_n$ denotes the Pochhammer's symbol defined by

$$(a)_n\;=\;\left\{\begin{array}{cl} 1 & ,\quad \text{if } n=0 \\ a(a+1)(a+2)\ldots(a+n-1) & ,\quad \text{if } n=1,2,3,\ldots \end{array}\right. \tag{2}$$

The special case of (1) when $p=2$ and $q=1$ is usually called Gauss's hypergeometric function.

Definition 2.2.

The Kampé de Fériet function of two variables $F^{p:q;k}_{l:m;n}[x,y]$ is defined by

$$F^{p:q;k}_{l:m;n}\left[\begin{array}{lc} (a_p)\ :\ (b_q)\ ;\ (c_k)\ ; & \\ & x\ ,\ y \\ (\alpha_l)\ :\ (\beta_m)\ ;\ (\gamma_n)\ ; & \end{array}\right]=\sum_{r,s=0}^{\infty}\frac{\prod\limits_{j=1}^{p}(a_j)_{r+s}\,\prod\limits_{j=1}^{q}(b_j)_r\,\prod\limits_{j=1}^{k}(c_j)_s}{\prod\limits_{j=1}^{l}(\alpha_j)_{r+s}\,\prod\limits_{j=1}^{m}(\beta_j)_r\,\prod\limits_{j=1}^{n}(\gamma_j)_s}\,\frac{x^r}{r!}\,\frac{y^s}{s!}. \tag{3}$$

Definition 2.3.

The confluent hypergeometric function of several variables $\Phi_2^{(n)}$ is defined by

$$\Phi_2^{(n)}(b_1,...,b_n;c;x_1,...,x_n)=\sum_{m_1,...,m_n=0}^{\infty}\frac{(b_1)_{m_1}...(b_n)_{m_n}}{(c)_{m_1+...+m_n}}\frac{x_1^{m_1}}{m_1!}...\frac{x_n^{m_n}}{m_n!}. \tag{4}$$

Clearly, for $r=2$ we have

$$\Phi_2^{(2)}=\Phi_2,$$

where Φ_2 is the Humbert's confluent hypergeometric function of two variable defined by

$$\Phi_2[a,b;c;x,y]=\sum_{m,n=0}^{\infty}\frac{(a)_m(b)_n}{(c)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!}. \tag{5}$$

The main aim of this research paper is to establish a general summation formula for the confluent hypergeometric function of $2r$ variables $\Phi_2^{(2r)}$ by using the following generalization of the classical Kummer's theorem due to Lavoie et al. (1996):

$$
{}_2F_1\left[\begin{matrix} a\ ,\ b & ; \\ 1+a-b+i & ; \end{matrix} -1\right] = \frac{\Gamma(\frac{1}{2})\Gamma(1+a-b+i)\Gamma(1-b)}{2^a\Gamma(1-b+\frac{1}{2}(i+|i|))}
$$
$$
\times\left\{\frac{A_i}{\Gamma(\frac{1}{2}a+\frac{1}{2}i+\frac{1}{2}-[\frac{1+i}{2}])\Gamma(1+\frac{1}{2}a-b+\frac{1}{2}i)}+\frac{B_i}{\Gamma(\frac{1}{2}a+\frac{1}{2}i-[\frac{i}{2}])\Gamma(\frac{1}{2}+\frac{1}{2}a-b+\frac{1}{2}i)}\right\}, \tag{6}
$$

where $[x]$ denotes the greatest integer less than or equal to x, $|x|$ denotes the usual absolute value of x and $(i=0,\pm1,\pm2,\pm3,\pm4,\pm5)$. The coefficients A_i and B_i are given in the following table:

Table 1

i	A_i	B_i
5	$-4(6+a-b)^2+2b(6+a-b)$ $+b^2+22(6+a-b)-13b-20$	$4(6+a-b)^2+2b(6+a-b)$ $-b^2-34(6+a-b)-b+62$
4	$2(a-b+3)(1+a-b)-(b-1)(b-4)$	$-4(a-b+2)$
3	$3b-2a-5$	$2a-b+1$
2	$1+a-b$	-2
1	-1	1
0	1	0
-1	1	1
-2	$a-b-1$	2
-3	$2a-3b-4$	$2a-b-2$
-4	$2(a-b-3)(a-b-1)-b(b+3)$	$4(a-b-2)$
-5	$4(a-b-4)^2-2b(a-b-4)$ $-b^2+8(a-b-4)-7b$	$4(a-b-4)^2+2b(a-b-4)$ $-b^2+16(a-b-4)-b+12$

3. Main Summation Formula

In this section, the following summation formula will be established:

$$\begin{aligned}
&\Phi_2^{(2r)}(b_1-i,b_1,b_2-i,b_2,...,b_r-i,b_r;c;x_1,-x_1,x_2,-x_2,...,x_r,-x_r)\\
&= \sum_{m_1,...,m_r=0}^{\infty} \frac{(b_1-i)_{2m_1}...(b_r-i)_{2m_r} x_1^{2m_1}...x_r^{2m_r}}{(c)_{2m_1+...+2m_r}(2m_1)!...(2m_r)!}\\
&\times (A_i^{(1)}E_1+B_i^{(1)}F_1)(A_i^{(2)}E_2+B_i^{(2)}F_2)...(A_i^{(r)}E_r+B_i^{(r)}F_r)\\
&+ \sum_{m_1,...,m_r=0}^{\infty} \frac{(b_1-i)_{2m_1+1}(b_2-i)_{2m_2}...(b_r-i)_{2m_r} x_1^{2m_1+1}x_2^{2m_2}...x_r^{2m_r}}{(c)_{2m_1+1+2m_2+...+2m_r}(2m_1+1)!(2m_2)!...(2m_r)!}\\
&\times (C_i^{(1)}G_1+D_i^{(1)}H_1)(A_i^{(2)}E_2+B_i^{(2)}F_2)...(A_i^{(r)}E_r+B_i^{(r)}F_r)+...
\end{aligned}$$

$$\begin{aligned}
&...+ \sum_{m_1,...,m_r=0}^{\infty} \frac{(b_1-i)_{2m_1}(b_2-i)_{2m_2+1}...(b_r-i)_{2m_r+1} x_1^{2m_1}x_2^{2m_2+1}...x_r^{2m_r+1}}{(c)_{2m_1+2m_2+1+...+2m_r+1}(2m_1)!(2m_2+1)!...(2m_r+1)!}\\
&\times (A_i^{(1)}E_1+B_i^{(1)}F_1)(C_i^{(2)}G_2+D_i^{(2)}H_2)...(C_i^{(r)}G_r+D_i^{(r)}H_r)\\
&+ \sum_{m_1,...,m_r=0}^{\infty} \frac{(b_1-i)_{2m_1+1}...(b_r-i)_{2m_r+1} x_1^{2m_1+1}...x_r^{2m_r+1}}{(c)_{2m_1+1+...+2m_r+1}(2m_1+1)!...(2m_r+1)!}\\
&\times (C_i^{(1)}G_1+D_i^{(1)}H_1)(C_i^{(2)}G_2+D_i^{(2)}H_2)...(C_i^{(r)}G_r+D_i^{(r)}H_r), \qquad (7)
\end{aligned}$$

where

$$E_r = \frac{2^{2m_r}\Gamma(\frac{1}{2})\Gamma(1-2m_r-b_r+i)\Gamma(1-b_r)}{\Gamma(1-b_r+\frac{1}{2}(i+|i|))\Gamma(-m_r+\frac{1}{2}i+\frac{1}{2}-[\frac{1+i}{2}])\Gamma(1-m_r-b_r+\frac{1}{2}i)},$$

$$F_r = \frac{2^{2m_r}\Gamma(\frac{1}{2})\Gamma(1-2m_r-b_r+i)\Gamma(1-b_r)}{\Gamma(1-b_r+\frac{1}{2}(i+|i|))\Gamma(-m_r+\frac{1}{2}i-[\frac{i}{2}])\Gamma(-m_r+\frac{1}{2}-b_r+\frac{1}{2}i)},$$

$$G_r = \frac{2^{2m_r+1}\Gamma(\frac{1}{2})\Gamma(-2m_r-b_r+i)\Gamma(1-b_r)}{\Gamma(1-b_r+\frac{1}{2}(i+|i|))\Gamma(-m_r+\frac{1}{2}i-[\frac{1+i}{2}])\Gamma(-m_r+\frac{1}{2}-b_r+\frac{1}{2}i)},$$

$$H_r = \frac{2^{2m_r+1}\Gamma(\frac{1}{2})\Gamma(-2m_r-b_r+i)\Gamma(1-b_r)}{\Gamma(1-b_r+\frac{1}{2}(i+|i|))\Gamma(-m_r-\frac{1}{2}+\frac{1}{2}i-[\frac{i}{2}])\Gamma(-m_r-b_r+\frac{1}{2}i)},$$

for $i=0,\pm1,\pm2,\pm3,\pm4,\pm5.$

The coefficients $A_i^{(1)},...,A_i^{(r)}$ and $B_i^{(1)},...,B_i^{(r)}$ can be obtained from the tables of A_i and B_i by replacing a and b by $-2m_j$ and b_j, $j=1,2,...,r$. The coefficients $C_i^{(1)},...,C_i^{(r)}$ and $D_i^{(1)},...,D_i^{(r)}$ can be obtained from the tables of A_i and B_i by replacing a and b by $-2m_j-1$ and b_j, $j=1,2,...,r$.

Proof:

Denoting the left hand side of (7) by $\Phi_2^{(2r)}$, then from the definition (4), we have

$$\Phi_2^{(2r)}=\sum_{n_1,p_1,\ldots,n_r,p_r=0}^{\infty}\frac{(b_1-i)_{n_1}(b_1)_{p_1}\ldots(b_r-i)_{n_r}(b_r)_{p_r}x_1^{n_1}(-x_1)^{p_1}\ldots x_r^{n_r}(-x_r)^{p_r}}{(c)_{n_1+p_1+\ldots+n_r+p_r}n_1!p_1!\ldots n_r!p_r!}. \tag{8}$$

Using the well-known result Srivastava and Manocha (1984)

$$(\alpha)_{m+n}=(\alpha)_m(\alpha+m)_n, \tag{9}$$

we have

$$\Phi_2^{(2r)}=\sum_{n_2,p_2,\ldots,n_r,p_r=0}^{\infty}\frac{(b_2-i)_{n_2}(b_2)_{p_2}\ldots(b_r-i)_{n_r}(b_r)_{p_r}x_2^{n_2}(-x_2)^{p_2}\ldots x_r^{n_r}(-x_r)^{p_r}}{(c)_{n_2+p_2+\ldots+n_r+p_r}n_2!p_2!\ldots n_r!p_r!}$$
$$\times\Phi_2[b_1-i,b_1;c+n_2+p_2+\ldots+n_r+p_r;x_1,-x_1]. \tag{10}$$

Now, applying the following result by Rathie (2013), (see also Choi and Rathie (2015a)):

$$\Phi_2(a,b;c;x,-x)=\sum_{m=0}^{\infty}\frac{(a)_m x^m}{(c)_m m!}\,{}_2F_1\left[\begin{matrix}-m\ ,\ b & ; & \\ & & -1\\ 1-m-a & ; & \end{matrix}\right], \tag{11}$$

we have

$$\Phi_2^{(2r)}=\sum_{m_1,n_2,p_2,\ldots,n_r,p_r=0}^{\infty}\frac{(b_1-i)_{m_!}(b_2-i)_{n_2}(b_2)_{p_2}\ldots(b_r-i)_{n_r}(b_r)_{p_r}x_1^{m_!}x_2^{n_2}(-x_2)^{p_2}\ldots x_r^{n_r}(-x_r)^{p_r}}{(c)_{m_1+n_2+p_2+\ldots+n_r+p_r}m_1!n_2!p_2!\ldots n_r!p_r!}$$
$$\times\,{}_2F_1\left[\begin{matrix}-m_1\ ,\ b_1 & ; & \\ & & -1\\ 1-m_1-b_1+i & ; & \end{matrix}\right]. \tag{12}$$

Then, by repeating the above steps r-times, we have

$$\Phi_2^{(2r)}=\sum_{m_1,\ldots,m_r=0}^{\infty}\frac{(b_1-i)_{m_1}\ldots(b_r-i)_{m_r}x_1^{m_1}\ldots x_r^{m_r}}{(c)_{m_1+\ldots+m_r}m_1!\ldots m_r!}f(b_1,i,m_1)\ldots f(b_r,i,m_r), \tag{13}$$

where

$$f(b_r,i,m_r)={}_2F_1\left[\begin{matrix}-m_r\ ,\ b_r & ; & \\ & & -1\\ 1-m_r-b_r+i & ; & \end{matrix}\right].$$

Separating (13) into even and odd powers of $(x_j, j=1,\ldots,r)$ by using the elementary identity

$$\sum_{n=0}^{\infty} A(n) = \sum_{n=0}^{\infty} A(2n) + \sum_{n=0}^{\infty} A(2n+1), \tag{14}$$

we have

$$\begin{aligned}
\Phi_2^{(2r)} = & \sum_{m_1,\ldots,m_r=0}^{\infty} \frac{(b_1-i)_{2m_1}\ldots(b_r-i)_{2m_r} x_1^{2m_1}\ldots x_r^{2m_r}}{(c)_{2m_1+\ldots+2m_r}(2m_1)!\ldots(2m_r)!} \\
& \times f(b_1,i,2m_1) f(b_2,i,2m_2)\ldots f(b_r,i,2m_r) \\
& + \sum_{m_1,\ldots,m_r=0}^{\infty} \frac{(b_1-i)_{2m_1+1}(b_2-i)_{2m_2}\ldots(b_r-i)_{2m_r} x_1^{2m_1+1} x_2^{2m_2}\ldots x_r^{2m_r}}{(c)_{2m_1+1+2m_2+\ldots+2m_r}(2m_1+1)!(2m_2)!\ldots(2m_r)!} \\
& \times f(b_1,i,2m_1+1) f(b_2,i,2m_2)\ldots f(b_r,i,2m_r) + \ldots
\end{aligned}$$

$$\begin{aligned}
& \ldots + \sum_{m_1,\ldots,m_r=0}^{\infty} \frac{(b_1-i)_{2m_1}(b_2-i)_{2m_2+1}\ldots(b_r-i)_{2m_r+1} x_1^{2m_1} x_2^{2m_2+1}\ldots x_r^{2m_r+1}}{(c)_{2m_1+2m_2+1+\ldots+2m_r+1}(2m_1)!(2m_2+1)!\ldots(2m_r+1)!} \\
& \times f(b_1,i,2m_1) f(b_2,i,2m_2+1)\ldots f(b_r,i,2m_r+1) \\
& + \sum_{m_1,\ldots,m_r=0}^{\infty} \frac{(b_1-i)_{2m_1+1}\ldots(b_r-i)_{2m_r+1} x_1^{2m_1+1}\ldots x_r^{2m_r+1}}{(c)_{2m_1+1+\ldots+2m_r+1}(2m_1+1)!\ldots(2m_r+1)!} \\
& \times f(b_1,i,2m_1+1) f(b_2,i,2m_2+1)\ldots f(b_r,i,2m_r+1).
\end{aligned} \tag{15}$$

Now, applying the generalized Kummer's theorem (6) on each ${}_2F_1(-1)$ in the right hand side of (15) and after some simplification, we get the right hand side of (7).This completes the proof of (7). ∎

4. Special Cases

In this section we will mention the following special cases of (7) and we will use the following results (see Srivastava and Manocha (1984)):

$$\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} = (\alpha)_n, \quad \frac{\Gamma(\alpha-n)}{\Gamma(\alpha)} = \frac{(-1)^n}{(1-\alpha)_n}, \tag{16}$$

$$(\alpha)_{2n} = 2^{2n}\left(\frac{1}{2}\alpha\right)_n \left(\frac{1}{2}\alpha+\frac{1}{2}\right)_n, \tag{17}$$

$$(2n)! = 2^{2n}\left(\frac{1}{2}\right)_n n!, \quad (2n+1)! = 2^{2n}\left(\frac{3}{2}\right)_n n!. \tag{18}$$

1. Taking $r=1$ in (7), we get

$$\Phi_2(b-i,b;c;x,-x)=\frac{\Gamma(\frac{1}{2})\Gamma(1-b)}{\Gamma(1-b+\frac{1}{2}(i+|i|))}\sum_{m=0}^{\infty}\frac{(b-i)_{2m}x^{2m}}{(c)_{2m}(2m)!}$$

$$\times\left\{\frac{A_i^{(1)}2^{2m}\Gamma(1-2m-b+i)}{\Gamma(-m+\frac{1}{2}i+\frac{1}{2}-[\frac{1+i}{2}])\Gamma(1-m-b+\frac{1}{2}i)}+\frac{B_i^{(1)}2^{2m}\Gamma(1-2m-b+i)}{\Gamma(-m+\frac{1}{2}i-[\frac{i}{2}])\Gamma(-m+\frac{1}{2}-b+\frac{1}{2}i)}\right\}$$

$$+\frac{\Gamma(\frac{1}{2})\Gamma(1-b)}{\Gamma(1-b+\frac{1}{2}(i+|i|))}\sum_{m=0}^{\infty}\frac{(b-i)_{2m+1}x^{2m+1}}{(c)_{2m+1}(2m+1)!}$$

$$\times\left\{\frac{C_i^{(1)}2^{2m+1}\Gamma(-2m-b+i)}{\Gamma(-m+\frac{1}{2}i-[\frac{1+i}{2}])\Gamma(-m+\frac{1}{2}-b+\frac{1}{2}i)}+\frac{D_i^{(1)}2^{2m+1}\Gamma(-2m-b+i)}{\Gamma(-m-\frac{1}{2}+\frac{1}{2}i-[\frac{i}{2}])\Gamma(-m-b+\frac{1}{2}i)}\right\}. \tag{19}$$

Further, replacing b by $a+i$ in (19) and using (14) we get a known result of Choi and Rathie (2015a) for $\Phi_2(a,a+i;c;x,-x)$, $i=0,\pm1,\pm2,\pm3,\pm4,\pm5$.

2. Taking $i=0$ in (7), we have

$$\Phi_2^{(2r)}(b_1,b_1,...,b_r,b_r;c;x_1,-x_1,...,x_r,-x_r)$$
$$=\sum_{m_1,...,m_r=0}^{\infty}\frac{(b_1)_{2m_1}...(b_r)_{2m_r}x_1^{2m_1}...x_r^{2m_r}}{(c)_{2m_1+...+2m_r}(2m_1)!...(2m_r)!}$$
$$\times\frac{2^{2m_1}\Gamma(\frac{1}{2})\Gamma(1-2m_1-b_1)}{\Gamma(1-m_1-b_1)\Gamma(\frac{1}{2}-m_1)}\times...\times\frac{2^{2m_r}\Gamma(\frac{1}{2})\Gamma(1-2m_r-b_r)}{\Gamma(1-m_r-b_r)\Gamma(\frac{1}{2}-m_r)}. \tag{20}$$

Now using the results (16)–(18) in (20), then after some simplification we obtain the following transformation formula:

$$\Phi_2^{(2r)}(b_1,b_1,...,b_r,b_r;c;x_1,-x_1,...,x_r,-x_r)$$
$$=F\,{}^{0:1;...;1}_{2:0;...;0}\left[\begin{matrix}- & : & b_1 & ; & ... & ; & b_r & ; \\ & & & & & & & \tfrac{1}{4}x_1^2,...,\tfrac{1}{4}x_r^2 \\ \frac{1}{2}c,\frac{1}{2}c+\frac{1}{2} & : & - & ; & ... & ; & - & ;\end{matrix}\right]. \tag{21}$$

Further, setting $x_1=...=x_r=x$ in (21) and using the result Srivastava and Karlsson (1985),

$$F\,{}^{p:1;...;1}_{q:0;...;0}\left[\begin{matrix}a_1,...,a_p & : & c_1 & ; & ... & ; & c_r & ; \\ & & & & & & & x,...,x \\ b_1,...,b_q & : & - & ; & ... & ; & - & ;\end{matrix}\right]={}_{p+1}F_q\left[\begin{matrix}a_1,...,a_p,c_1+...+c_r & ; \\ & x \\ b_1,...,b_q & ;\end{matrix}\right], \tag{22}$$

we get

$$\Phi_2^{(2r)}(b_1,b_1,...,b_r,b_r;c;x,-x,...,x,-x)={}_1F_2\left[\begin{matrix}b_1+...+b_r & ; \\ & \tfrac{1}{4}x^2 \\ \frac{1}{2}c,\frac{1}{2}c+\frac{1}{2} & ;\end{matrix}\right]. \tag{23}$$

3. Taking $r=2$, $i=\pm 1$ in (7) and using the results (16)–(18), we get

$$\begin{aligned}
&\Phi_2^{(4)}(b_1-1, b_1, b_2-1, b_2; c; x_1, -x_1, x_2, -x_2)\\
&= F\,{}^{0:1;1}_{2:0;0}\left[\begin{matrix} - : b_1 ; b_2 ; \\ \frac{1}{2}c, \frac{1}{2}c+\frac{1}{2} : - ; - ; \end{matrix}\ \frac{1}{4}x_1^2, \frac{1}{4}x_2^2\right]\\
&- \frac{x_1}{c} F\,{}^{0:1;1}_{2:0;0}\left[\begin{matrix} - : b_1 ; b_2 ; \\ \frac{1}{2}c+\frac{1}{2}, \frac{1}{2}c+1 : - ; - ; \end{matrix}\ \frac{1}{4}x_1^2, \frac{1}{4}x_2^2\right]\\
&- \frac{x_2}{c} F\,{}^{0:1;1}_{2:0;0}\left[\begin{matrix} - : b_1 ; b_2 ; \\ \frac{1}{2}c+\frac{1}{2}, \frac{1}{2}c+1 : - ; - ; \end{matrix}\ \frac{1}{4}x_1^2, \frac{1}{4}x_2^2\right]\\
&+ \frac{x_1 x_2}{c(c+1)} F\,{}^{0:1;1}_{2:0;0}\left[\begin{matrix} - : b_1 ; b_2 ; \\ \frac{1}{2}c+1, \frac{1}{2}c+\frac{3}{2} : - ; - ; \end{matrix}\ \frac{1}{4}x_1^2, \frac{1}{4}x_2^2\right]
\end{aligned} \tag{24}$$

and

$$\begin{aligned}
&\Phi_2^{(4)}(b_1+1, b_1, b_2+1, b_2; c; x_1, -x_1, x_2, -x_2)\\
&= F\,{}^{0:1;1}_{2:0;0}\left[\begin{matrix} - : b_1+1 ; b_2+1 ; \\ \frac{1}{2}c, \frac{1}{2}c+\frac{1}{2} : - ; - ; \end{matrix}\ \frac{1}{4}x_1^2, \frac{1}{4}x_2^2\right]\\
&+ \frac{x_1}{c} F\,{}^{0:1;1}_{2:0;0}\left[\begin{matrix} - : b_1+1 ; b_2+1 ; \\ \frac{1}{2}c+\frac{1}{2}, \frac{1}{2}c+1 : - ; - ; \end{matrix}\ \frac{1}{4}x_1^2, \frac{1}{4}x_2^2\right]\\
&+ \frac{x_2}{c} F\,{}^{0:1;1}_{2:0;0}\left[\begin{matrix} - : b_1+1 ; b_2+1 ; \\ \frac{1}{2}c+\frac{1}{2}, \frac{1}{2}c+1 : - ; - ; \end{matrix}\ \frac{1}{4}x_1^2, \frac{1}{4}x_2^2\right]\\
&+ \frac{x_1 x_2}{c(x+1)} F\,{}^{0:1;1}_{2:0;0}\left[\begin{matrix} - : b_1+1 ; b_2+1 ; \\ \frac{1}{2}c+1, \frac{1}{2}c+\frac{3}{2} : - ; - ; \end{matrix}\ \frac{1}{4}x_1^2, \frac{1}{4}x_2^2\right].
\end{aligned} \tag{25}$$

Further, setting $x_1 = x_2 = x$ in (24) and (25) and using the result (22), we get

$$\Phi_2^{(4)}(b_1-1, b_1, b_2-1, b_2; c; x, -x, x, -x)$$

$$= {}_1F_2\left[\begin{matrix} b_1+b_2 & ; \\ & & \tfrac{1}{4}x^2 \\ \tfrac{1}{2}c, \tfrac{1}{2}c+\tfrac{1}{2} & ; \end{matrix}\right] - \frac{x}{c}\,{}_1F_2\left[\begin{matrix} b_1+b_2 & ; \\ & & \tfrac{1}{4}x^2 \\ \tfrac{1}{2}c+\tfrac{1}{2}, \tfrac{1}{2}c+1 & ; \end{matrix}\right]$$

$$-\frac{x}{c}\,{}_1F_2\left[\begin{matrix} b_1+b_2 & ; \\ & & \tfrac{1}{4}x^2 \\ \tfrac{1}{2}c+\tfrac{1}{2}, \tfrac{1}{2}c+1 & ; \end{matrix}\right] + \frac{x^2}{c(c+1)}\,{}_1F_2\left[\begin{matrix} b_1+b_2 & ; \\ & & \tfrac{1}{4}x^2 \\ \tfrac{1}{2}c+1, \tfrac{1}{2}c+\tfrac{3}{2} & ; \end{matrix}\right], \tag{26}$$

and

$$\Phi_2^{(4)}(b_1+1, b_1, b_2+1, b_2; c; x, -x, x, -x)$$

$$= {}_1F_2\left[\begin{matrix} b_1+b_2+2 & ; \\ & & \tfrac{1}{4}x^2 \\ \tfrac{1}{2}c, \tfrac{1}{2}c+\tfrac{1}{2} & ; \end{matrix}\right] + \frac{x}{c}\,{}_1F_2\left[\begin{matrix} b_1+b_2+2 & ; \\ & & \tfrac{1}{4}x^2 \\ \tfrac{1}{2}c+\tfrac{1}{2}, \tfrac{1}{2}c+1 & ; \end{matrix}\right]$$

$$+\frac{x}{c}\,{}_1F_2\left[\begin{matrix} b_1+b_2+2 & ; \\ & & \tfrac{1}{4}x^2 \\ \tfrac{1}{2}c+\tfrac{1}{2}, \tfrac{1}{2}c+1 & ; \end{matrix}\right] + \frac{x^2}{c(c+1)}\,{}_1F_2\left[\begin{matrix} b_1+b_2+2 & ; \\ & & \tfrac{1}{4}x^2 \\ \tfrac{1}{2}c+1, \tfrac{1}{2}c+\tfrac{3}{2} & ; \end{matrix}\right]. \tag{27}$$

Remark 4.1.

I. On setting $x_2 = 0$ and replacing b_1 by $a+1$ in (24), we get the known results of Choi and Rathie (2015a) for $\Phi_2(a, a+1; c; x, -x)$.

II. On setting $x_2 = 0$ and replacing b_1 by $a-1$ in (25), we get the known results of Choi and Rathie (2015a) for $\Phi_2(a, a-1; c; x, -x)$.

4. Taking $r = 2$, $i = \pm 2$ in (7) and using the results (16)–(18), we get

$$\Phi_2^{(4)}(b_1-2, b_1, b_2-2, b_2; c; x_1, -x_1, x_2, -x_2)$$

$$= F\begin{matrix} 0:2;2 \\ 2:1;1 \end{matrix}\left[\begin{matrix} - & : & \tfrac{1}{2}b_1+\tfrac{1}{2}, b_1-1 & ; & \tfrac{1}{2}b_2+\tfrac{1}{2}, b_2-1 & ; \\ & & & & & & \tfrac{1}{4}x_1^2, \tfrac{1}{4}x_2^2 \\ \tfrac{1}{2}c, \tfrac{1}{2}c+\tfrac{1}{2} & : & \tfrac{1}{2}b_1-\tfrac{1}{2} & ; & \tfrac{1}{2}b_2-\tfrac{1}{2} & ; \end{matrix}\right]$$

$$-\frac{2x_1}{c} F\begin{matrix} 0:1;2 \\ 2:0;1 \end{matrix}\left[\begin{matrix} - & : & b_1 & ; & \tfrac{1}{2}b_2+\tfrac{1}{2}, b_2-1 & ; \\ & & & & & & \tfrac{1}{4}x_1^2, \tfrac{1}{4}x_2^2 \\ \tfrac{1}{2}c+\tfrac{1}{2}, \tfrac{1}{2}c+1 & : & - & ; & \tfrac{1}{2}b_2-\tfrac{1}{2} & ; \end{matrix}\right]$$

$$-\frac{2x_2}{c} F\begin{matrix} 0:2;1 \\ 2:1;0 \end{matrix}\left[\begin{matrix} - & : & \tfrac{1}{2}b_1+\tfrac{1}{2}, b_1-1 & ; & b_2 & ; \\ & & & & & & \tfrac{1}{4}x_1^2, \tfrac{1}{4}x_2^2 \\ \tfrac{1}{2}c+\tfrac{1}{2}, \tfrac{1}{2}c+1 & : & \tfrac{1}{2}b_1-\tfrac{1}{2} & ; & - & ; \end{matrix}\right]$$

$$+\frac{4x_1x_2}{c(c+1)}F\begin{matrix}0:1;1\\2:0;0\end{matrix}\left[\begin{matrix}- & : & b_1 & ; & b_2 & ; \\ \frac{1}{2}c+1,\frac{1}{2}c+\frac{3}{2} & : & - & ; & - & ;\end{matrix}\ \frac{1}{4}x_1^2,\frac{1}{4}x_2^2\right] \tag{28}$$

and

$$\Phi_2^{(4)}(a,b_1+2,b_1,b_2+2,b_2;c;x_1,-x_1,x_2,-x_2)$$
$$=F\begin{matrix}0:2;2\\2:1;1\end{matrix}\left[\begin{matrix}- & : & \frac{1}{2}b_1+\frac{3}{2},b_1+1 & ; & \frac{1}{2}b_2+\frac{3}{2},b_2+1 & ; \\ \frac{1}{2}c,\frac{1}{2}c+\frac{1}{2} & : & \frac{1}{2}b_1+\frac{1}{2} & ; & \frac{1}{2}b_2+\frac{1}{2} & ;\end{matrix}\ \frac{1}{4}x_1^2,\frac{1}{4}x_2^2\right]$$

$$+\frac{2x_1}{c}F\begin{matrix}0:1;2\\2:0;1\end{matrix}\left[\begin{matrix}- & : & b_1+2 & ; & \frac{1}{2}b_2+\frac{3}{2},b_2+1 & ; \\ \frac{1}{2}c+\frac{1}{2},\frac{1}{2}c+1 & : & - & ; & \frac{1}{2}b_2+\frac{1}{2} & ;\end{matrix}\ \frac{1}{4}x_1^2,\frac{1}{4}x_2^2\right]$$

$$+\frac{2x_2}{c}F\begin{matrix}0:2;1\\2:1;0\end{matrix}\left[\begin{matrix}- & : & \frac{1}{2}b_1+\frac{3}{2},b_1+1 & ; & b_2+2 & ; \\ \frac{1}{2}c+\frac{1}{2},\frac{1}{2}c+1 & : & \frac{1}{2}b_1+\frac{1}{2} & ; & - & ;\end{matrix}\ \frac{1}{4}x_1^2,\frac{1}{4}x_2^2\right]$$

$$+\frac{4x_1x_2}{c(c+1)}F\begin{matrix}0:1;1\\2:0;0\end{matrix}\left[\begin{matrix}- & : & b_1+2 & ; & b_2+2 & ; \\ \frac{1}{2}c+1,\frac{1}{2}c+\frac{3}{2} & : & - & ; & - & ;\end{matrix}\ \frac{1}{4}x_1^2,\frac{1}{4}x_2^2\right]. \tag{29}$$

Remark 4.2.

I. On setting $x_2=0$ and replacing b_1 by $a+2$ in (28), we get the known results of Choi and Rathie (2015a) for $\Phi_2(a,a+2;c;x,-x)$.

II. On setting $x_2=0$ and replacing b_1 by $a-2$ in (29), we get the known results of Choi and Rathie (2015a) for $\Phi_2(a,a-2;c;x,-x)$.

5. Conclusion

In the present paper, we have extended a known summation formula of Choi and Rathie (2015a) for the confluent hypergeometric function of 2-variables Φ_2 to general summation formula for the confluent hypergeometric function of $2r$- variables $\Phi_2^{(2r)}$. The results are derived by using the method of series manipulation with the help of generalized Kummer's summation theorem obtained earlier by Lavoie et al. (1996). Furthermore, some new and known transformation formulas for $\Phi_2^{(4)}$ and Φ_2 are also given as special cases of our main summation formula. The method used in this paper can be applied to extend other hypergeometric summation formulas given in the literature.

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