Applications and Applied Mathematics: An International Journal (AAM)

Volume 14 | Issue 1 Article 25

6-2019

Chlodowsky Szász-Kantorovich operators via Dunkl analogue

Nadeem Rao
Jamia Millia Islamia

Abdul Wafi
Jamia Millia Islamia

Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466
Vol. 14, Issue 1 (June 2019), pp. 370–381

Applications and Applied Mathematics:
An International Journal
(AAM)

Chlodowsky Szász-Kantorovich operators via Dunkl analogue

[1]Nadeem Rao and [2]Abdul Wafi

[1,2]Department of Mathematics
Jamia Millia Islamia
New Delhi, India-110025
[1]nadeemrao1990@gmail.com, [2]abdulwafi2k2@gmail.com

Received: June 10, 2018; Accepted: November 12, 2018

Abstract

The main objective of the present article is to design a sequence of Chlodowsky Szász-Kantorovich operators based on Dunkl analogue for the purpose to achieve faster rate of convergence in terms of two positive and unbounded real number sequences and the basic results are estimated. Further, the uniform approximation by means of Korovkin theorem using test functions $e_i(t) = t^i, i = 0, 1, 2$ is investigated. Moreover, the local and global approximation results are discussed for these sequences of linear positive operators.

Keywords: Szász operators; modulus of continuity; rate of convergence; Dunkl analogue

MSC 2010 No.: 41A10, 41A25, 41A28, 41A35, 41A36

1. Introduction

Recently, Sucu[2014] constructed Szász type operators as follows:

$$S_n(g;u) := \frac{1}{e_\mu(nu)} \sum_{k=0}^{\infty} \frac{(nu)^k}{\gamma_\mu(k)} g\left(\frac{k+2\mu\theta_k}{n}\right), \; n \in \mathbb{N} = \{1, 2, 3...\}, \tag{1}$$

using generalized exponential function (see Rosenblum (1994)) given by

$$e_\mu(t) = \sum_{k=0}^{\infty} \frac{t^k}{\gamma_\mu(k)}, t \in [0, \infty), \tag{2}$$

where the coefficient $\gamma_\mu(\nu)$ are defined as follows for $\nu \in \mathbb{N}_0 = \{0\} \bigcup \mathbb{N}$ and $\mu > -1/2$

$$\gamma_\mu(2\nu) = \frac{2^{2\nu}\nu!\Gamma(\nu+\mu+1/2)}{\Gamma(\mu+1/2)}, \gamma_\mu(2\nu+1) = \frac{2^{2\nu+1}\nu!\Gamma(\nu+\mu+3/2)}{\Gamma(\mu+1/2)},$$

and the recursive relation for γ_μ is defined as

$$\gamma_\mu(\nu+1) = (\nu+1+2\mu\theta_{\nu+1})\gamma_\mu(\nu),\ \nu \in \mathbb{N}_0, \tag{3}$$

with θ_ν is defined to be 0 if $\nu \in 2\mathbb{N}$ and 1 if $\nu \in 2\mathbb{N}+1$. Many research papers about the generalizations of the operators defined by (1) have been published to discuss the better approximation results in various functional spaces by several mathematicians such as Wafi and Rao [(2018), (2019),(2018)], Karaisa et al.[2016] and Icoz et al.[2015] Motivated by the above, the purpose of this manuscript is to construct the Chlodowsky Sz*á*sz Kantorovich type operators to approximate the Lebesgue integrable functions as follows:

$$K_n^*(f;x) := \frac{b_n}{e_\mu(a_n x)} \sum_{k=0}^{\infty} \frac{(a_n x)^k}{\gamma_\mu(k)} \int_{\frac{k+2\mu\theta_k}{b_n}}^{\frac{k+2\mu\theta_k+1}{b_n}} f(t)dt, \tag{4}$$

where a_n and b_n are unbounded and positive real numbers increasing sequences defined as

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \infty \text{ and } \frac{a_n}{b_n} = 1 + \frac{1}{b_n}.$$

In subsequence sections, we investigate the convergence of the operators defined in (4) with the aid of Kantorovich theorem and an asymptotic approximation result is studied to approximate the second order differentiable functions. The order of approximation is deduced with the help of second order modulus of continuity and Peetre's K-functional for the operators (4). In the last, weighted and statistical approximation results are obtained.

2. Preliminaries

Lemma 2.1.

Let $e_i(t) = t^i, i = 0, 1, 2$ be the test functions. Then, for the operators K_n^*, we have

$$\begin{aligned}
K_n^*(e_0;x) &= 1,\\
K_n^*(e_1;x) &= \frac{a_n}{b_n}x + \frac{1}{b_n},\\
K_n^*(e_2;x) &= \frac{a_n^2}{b_n^2}x^2 + \frac{a_n}{b_n^2}\left(2 + 2\mu\frac{e_\mu(-a_n x)}{e_\mu(a_n x)}\right)x + \frac{1}{3b_n^2},\\
K_n^*(e_3;x) &= \frac{a_n^3}{b_n^3}x^3 + \left(\frac{9}{2} - 2\mu\frac{e_\mu(-a_n x)}{e_\mu(a_n x)}\right)\frac{a_n^2}{b_n^3}x^2\\
&\quad + \left(\frac{7}{2} + 4\mu^2 + 5\mu\frac{e_\mu(-a_n x)}{e_\mu(a_n x)}\right)\frac{a_n}{b_n^3}x + \frac{1}{b_n^3},\\
K_n^*(e_4;x) &= \frac{a_n^4}{b_n^4}x^4 + \left(7 + 4\mu\frac{e_\mu(-a_n x)}{e_\mu(a_n x)}\right)\frac{a_n^3}{b_n^4}x^3\\
&\quad + \left(12 + 4\mu^2 - 10\mu\frac{e_\mu(-a_n x)}{e_\mu(a_n x)}\right)\frac{a_n^2}{b_n^4}x^2\\
&\quad + \left(6 + 16\mu^2 + 12\mu\frac{e_\mu(-a_n x)}{e_\mu(a_n x)} + 8\mu^3\frac{e_\mu(-a_n x)}{e_\mu(a_n x)}\right)\frac{a_n}{b_n^3}x + \frac{1}{5b_n^4}.
\end{aligned}$$

Proof:

For $i=0$,

$$K_n^*(e_0;x)=K_n^*(f;x):=\frac{b_n}{e_\mu(a_nx)}\sum_{k=0}^{\infty}\frac{(a_nx)^k}{\gamma_\mu(k)}\int_{\frac{k+2\mu\theta_k}{b_n}}^{\frac{k+2\mu\theta_k+1}{b_n}}dt,$$
$$=K_n^*(f;x):=\frac{b_n}{e_\mu(a_nx)}\sum_{k=0}^{\infty}\frac{(a_nx)^k}{\gamma_\mu(k)}\left(\frac{1}{b_n}\right),$$
$$=1.$$

For $i=1$,

$$K_n^*(e_1;x)=\frac{b_n}{e_\mu(a_nx)}\sum_{k=0}^{\infty}\frac{(a_nx)^k}{\gamma_\mu(k)}\int_{\frac{k+2\mu\theta_k}{b_n}}^{\frac{k+2\mu\theta_k+1}{b_n}}tdt,$$
$$=\frac{b_n}{2e_\mu(a_nx)}\sum_{k=0}^{\infty}\frac{(a_nx)^k}{\gamma_\mu(k)}\left(\frac{(k+2\mu\theta_k+1)^2}{b_n^2}-\frac{(k+2\mu\theta_k)^2}{b_n^2}\right),$$
$$=\frac{b_n}{2e_\mu(a_nx)}\sum_{k=0}^{\infty}\frac{(a_nx)^k}{\gamma_\mu(k)}\left(\frac{1+2(k+2\mu\theta_k)}{b_n^2}\right)$$
$$=\frac{b_n}{e_\mu(a_nx)}\sum_{k=0}^{\infty}\frac{(a_nx)^k}{\gamma_\mu(k)}\left(\frac{k+2\mu\theta_k}{b_n^2}\right)+\frac{1}{2b_n}$$
$$=\frac{a_n}{b_n}x+\frac{1}{2b_n}.$$

For $i=2$,

$$K_n^*(e_2;x)=\frac{b_n}{e_\mu(a_nx)}\sum_{k=0}^{\infty}\frac{(a_nx)^k}{\gamma_\mu(k)}\int_{\frac{k+2\mu\theta_k}{b_n}}^{\frac{k+2\mu\theta_k+1}{b_n}}t^2dt,$$
$$=\frac{b_n}{3e_\mu(a_nx)}\sum_{k=0}^{\infty}\frac{(a_nx)^k}{\gamma_\mu(k)}\left(\frac{(k+2\mu\theta_k+1)^3}{b_n^3}-\frac{(k+2\mu\theta_k)^3}{b_n^3}\right),$$
$$=\frac{b_n}{3e_\mu(a_nx)}\sum_{k=0}^{\infty}\frac{(a_nx)^k}{\gamma_\mu(k)}\left(\frac{3(k+2\mu\theta_k)^2+3(k+2\mu\theta_k)+1}{b_n^3}\right),$$
$$=\frac{b_n}{e_\mu(a_nx)}\sum_{k=0}^{\infty}\frac{(a_nx)^k}{\gamma_\mu(k)}\frac{(k+2\mu\theta_k)^2}{b_n^3}$$
$$+\frac{b_n}{e_\mu(a_nx)}\sum_{k=0}^{\infty}\frac{(a_nx)^k}{\gamma_\mu(k)}\frac{(k+2\mu\theta_k)}{b_n^3}+\frac{1}{3b_n^2}$$
$$=\frac{a_n^2}{b_n^2}x^2+\frac{a_n}{b_n^2}\left(2+2\mu\frac{e_\mu(-a_nx)}{e_\mu(a_nx)}\right)x+\frac{1}{3b_n^2}.$$

In the same manner, the rest part of this Lemma 2.1 can be proved easily. ■

Lemma 2.2.

Let $\psi_x^i(t) = (t-x)^i, i = 0,1,2$ be the central moments. Then, for the operators defined by (4), we have

$$\begin{aligned}
&K_n^*(\psi_x^0;x) = 1,\\
&K_n^*(\psi_x^1;x) = \left(\frac{a_n}{b_n} - 1\right)x + \frac{1}{b_n},\\
&K_n^*(\psi_x^2;x) = \left(\frac{a_n^2}{b_n^2} - 2\frac{a_n}{b_n} + 1\right)x^2 + \left(\left(2 + 2\mu\frac{e_{\mu} - a_n x}{e_{\mu}(a_n x)}\right)\frac{a_n}{b_n} - 2\right)\frac{x}{b_n},\\
&K_n^*(\psi_x^4;x) = o\left(\frac{1}{b_n}\right)\left(x^3 + x^2 + x + 1\right).
\end{aligned}$$

Proof:

We can prove Lemma 2.2 using Lemma 2.1 and linearity property as

$$\begin{aligned}
&K_n^*(\psi_x;x) = K_n^*(t;x) - xK_n^*(1;x)\\
&K_n^*(\psi_x^2;x) = K_n^*(t^2;x) - 2xK_n^*(t;x) + x^2K_n^*(1;x)\\
&K_n^*(\psi_x^4;x) = K_n^*(t^4;x) - 4xK_n^*(t^3;x) + 6x^2K_n^*(t^2;x) - 4x^3K_n^*(t;x) + x^4K_n^*(1;x).
\end{aligned}$$

■

3. Rate of convergence

Definition 3.1.

For $f \in C[0,\infty)$, where $C[0,\infty)$ is the set of all continuous functions on $[0,\infty)$, the modulus of continuity for a uniformly continuous function f is

$$\omega(f;\delta) = \sup_{|t-y|\leq\delta} |f(t) - f(y)|, \qquad t,y \in [0,\infty).$$

Let $f \in C[0,\infty)$ be a uniformly continuous function and $\delta > 0$. Then, one has

$$|f(t) - f(y)| \leq \left(1 + \frac{(t-y)^2}{\delta^2}\right)\omega(f;\delta). \tag{5}$$

Theorem 3.2.

For the operators K_n^* given in (4) and for each $f \in C[0,\infty) \bigcap E$, $K_n^* \rightrightarrows f$ on each compact subset of $[0,\infty)$ where $E := \left\{ f : x \geq 0, \frac{f(x)}{1+x^2} \text{ is convergent as } x \to \infty \right\}$ and $\rightrightarrows$ denotes the uniform convergence.

Proof:

From the Korovkin-type property (vi) of Theorem 4.1.4 in Altomare and Campiti[1994], it is sufficient to show that

$$K_n^*(e_i;x) \to e_i(x), \text{ for } i = 0,1,2.$$

Using Lemma 2.1, it is obvious $K_n^*(e_0;x) \to e_0(x)$ as $n \to \infty$ and for $i = 1$

$$\lim_{n\to\infty} K_n^*(e_1;x) = \lim_{n\to\infty}\left(\left(\frac{a_n}{b_n}-1\right)x+\frac{1}{b_n}\right) = e_1(x).$$

Similarly, we can prove for $i = 2$, $K_n^*(e_2;x) \to e_2$ which proves Theorem 3.2. ∎

4. Local approximation results

Let $C_B[0,\infty)$ be the space of real valued continuous and bounded functions endowed with the norm $\|f\| = \sup\limits_{0\leq x<\infty} |f(x)|$. Then, for any $f \in C_B[0,\infty)$ and $\delta > 0$, we have Peetre's K-functional is defined as

$$K_2(f,\delta) = inf\{\|f-g\| + \delta\|g''\| : g \in C_B^2[0,\infty)\},$$

where $C_B^2[0,\infty) = \{g \in C_B[0,\infty) : g', g'' \in C_B[0,\infty)\}$. By DeVore and Lorentz[1993], there exits an absolute constant $C > 0$ such that

$$K_2(f;\delta) \leq C\omega_2(f;\sqrt{\delta}), \tag{6}$$

where $\omega_2(f;\delta)$ is the second order modulus of continuity is defined as

$$\omega_2(f;\sqrt{\delta}) = \sup_{0<h\leq\sqrt{\delta}} \sup_{x\in[0,\infty)} |f(x+2h) - 2f(x+h) + f(x)|.$$

Now, for $f \in C_B[0,\infty)$, $x \geq 0$ and $n > 1$, we consider the auxiliary operator $\widehat{K}_n^*$ as follows

$$\widehat{K}_n^*(f;x) = K_n^*(f;x) + f(x) - f\left(\frac{a_n}{b_n}x + \frac{1}{b_n}\right). \tag{7}$$

Lemma 4.1.

For $g \in C_B^2[0,\infty)$ and, for all $x \geq 0$, we have

$$|\widehat{K}_n^*(g;x) - g(x)| \leq \xi_n(x)\|g''\|,$$

where

$$\xi_n(x) = K_n^*(\psi_x^2;x) + \left(K_n^*(\psi_x^1;x)\right)^2.$$

Proof:

In view of the definition of auxiliary operators defined by (7), we have

$$\widehat{K}_n^*(1;x)=1,\ \widehat{K}_n^*(\psi_x;x)=0 \text{ and } |\widehat{K}_n^*(f;x)|\leq 3\|f\|. \tag{8}$$

From Taylor's series expansion for $g\in C_B^2[0,\infty)$, we get

$$g(t)=g(x)+(t-x)g'(x)+\int_x^t (t-v)g''(v)dv. \tag{9}$$

Operating $\widehat{T}_n$ on both the sides, one has

$$\widehat{K}_n(g;x)-g(x)=g'(x)\widehat{K}_n^*(t-x;x)+\widehat{K}_n^*\Big(\int_x^t (t-v)g''(v)dv;x\Big).$$

From (7) and (8), we have

$$\begin{aligned}\widehat{K}_n*(g;x)-g(x)&=\widehat{K}_n^*\Big(\int_x^t (t-v)g''(v)dv;x\Big)\\&=K_n^*\Big(\int_x^t (t-v)g''(v)dv;x\Big)-\int_x^{\frac{a_n}{b_n}x+\frac{1}{b_n}}\left(\frac{a_n}{b_n}x+\frac{1}{b_n}-v\right)g''(v)dv.\end{aligned}$$

$$|\widehat{K}_n^*(g;x)-g(x)|\leq\left|K_n^*\Big(\int_x^t (t-v)g''(v)dv;x\Big)\right|+\left|\int_x^{\frac{a_n}{b_n}x+\frac{1}{b_n}}\left(\frac{a_n}{b_n}x+\frac{1}{b_n}-v\right)g''(v)dv\right|. \tag{10}$$

Since,

$$\left|\int_x^t (t-v)g''(v)dv\right|\leq(t-x)^2\parallel g''\parallel. \tag{11}$$

Then,

$$\left|\int_x^{\frac{a_n}{b_n}x+\frac{1}{b_n}}\left(\frac{a_n}{b_n}x+\frac{1}{b_n}-v\right)g''(v)dv\right|\leq\left(\frac{a_n}{b_n}x+\frac{1}{b_n}-x\right)^2\parallel g''\parallel. \tag{12}$$

Using (11) and (12) in (10), we deduce

$$|\widehat{K}_n^*(g;x)-g(x)| \leq \left\{K_n^*((t-x)^2;x)+\left(\frac{a_n}{b_n}x+\frac{1}{b_n}-x\right)^2\right\}\|g''\| \\ =\xi_n(x)\|g''\|.$$

Hence, Lemma 4.1 is proved. ■

Theorem 4.2.

Let $f\in C_B^2[0,\infty)$. Then, there exists a constant $C>0$ such that

$$|\ K_n^*(f;x)-f(x)\ |\leq C\omega_2\big(f;\sqrt{\xi_n(x)}\big)+\omega(f;K_n^*(\psi_x;x)),$$

where $\xi_n(x)$ is defined in Lemma 4.1.

Proof:

For $g\in C_B^2[0,\infty)$ and $f\in C_B[0,\infty)$ and by the definition of $\widehat{K}_n$, we have

$$|K_n^*(f;x)-f(x)|\leq|\widehat{K}_n(f-g;x)|+|(f-g)(x)|+|\widehat{K}_n(g;x)-g(x)| \\ +\left|f\Big(\frac{a_n}{b_n}x+\frac{1}{b_n}\Big)-f(x)\right|.$$

In view of Lemma 4.1 and relations in (8), one get

$$|K_n^*(f;x)-f(x)|\leq 4\|f-g\|+|\widehat{K}_n(g;x)-g(x)|+\left|f\Big(\frac{a_n}{b_n}x+\frac{1}{b_n}\Big)-f(x)\right| \\ \leq 4\|f-g\|+\xi_n(x)\|g''\|+\omega\Big(f;K_n^*(\psi_x;x)\Big).$$

By the definition of Peetre's K-functional, we have

$$|K_n^*(f;x)-f(x)|\leq C\omega_2\big(f;\sqrt{\xi_n(x)}\big)+\omega(f;K_n^*(\psi_x;x)),$$

which is a required result. ■

For two fixed real values $\beta_1,\beta_2>0$, we consider the Lipschitz type space as

$$Lip_M^{\beta_1,\beta_2}(\gamma):=\Big\{f\in C_B[0,\infty):|f(t)-f(x)|\leq M\frac{|t-x|^\gamma}{(t+\beta_1x+\beta_2x^2)^{\frac{\gamma}{2}}}:x,t\in(0,\infty)\Big\},$$

where M is a positive constant and $0<\gamma\leq 1$.

Theorem 4.3.

Let $f \in Lip_M^{\beta_1,\beta_2}(\gamma)$ and $x \in (0,\infty)$. Then, for the operators defined by (4), we have

$$|K_n^*(f;x) - f(x)| \leq M\left(\frac{\eta_n(x)}{\beta_1 x + \beta_2 x^2}\right)^{\frac{\gamma}{2}}, \tag{13}$$

where $\gamma \in (0,1]$ and $\eta_n(x) = K_n^*(\psi_x^2;x)$.

Proof:

For $\gamma = 1$ and $x \in (0,\infty)$, we have

$$\begin{aligned}|K_n^*(f;x) - f(x)| &\leq K_n^*(|f(t) - f(x)|;x)\\ &\leq MK_n^*\left(\frac{|t-x|}{(t+\beta_1 x + \beta_2 x^2)^{\frac{1}{2}}};x\right).\end{aligned}$$

It is obvious that $\frac{1}{t+\beta_1 x+\beta_2 x^2} < \frac{1}{\beta_1 x+\beta_2 x^2}$ for all $x \in (0,\infty)$, one has

$$\begin{aligned}|K_n^*(f;x) - f(x)| &\leq \frac{M}{(\beta_1 x + \beta_2 x^2)^{\frac{1}{2}}}(K_n^*((t-x)^2;x))^{\frac{1}{2}}\\ &\leq M\left(\frac{\eta_n(x)}{\beta_1 x + \beta_2 x^2}\right)^{\frac{1}{2}}.\end{aligned}$$

Thus, the Theorem 4.3 holds for $\gamma = 1$. Now, for $\gamma \in (0,\infty)$ and in the light of Hölder's inequality with $p = \frac{2}{\gamma}$ and $q = \frac{2}{2-\gamma}$, we have

$$\begin{aligned}|K_n^*(f;x) - f(x)| &\leq \left(K_n^*(|f(t) - f(x)|^{\frac{2}{\gamma}};x)\right)^{\frac{\gamma}{2}}\\ &\leq M\left(K_n^*\left(\frac{|t-x|^2}{(t+\beta_1 x + \beta_2 x^2)};x\right)\right)^{\frac{\gamma}{2}}.\end{aligned}$$

Since, $\frac{1}{t+\beta_1 x+\beta_2 x^2} < \frac{1}{\beta_1 x+\beta_2 x^2}$ for all $x \in (0,\infty)$, we have

$$\begin{aligned}|K_n^*(f;x) - f(x)| &\leq M\left(\frac{K_n^*(|t-x|^2;x)}{\beta_1 x + \beta_2 x^2}\right)^{\frac{\gamma}{2}}\\ &\leq M\Big(\frac{\eta_n(x)}{\beta_1 x + \beta_2 x^2}\Big)^{\frac{\gamma}{2}}.\end{aligned}$$

This completes the proof of the Theorem 4.3. ■

Now, we introduce local approximation in terms of r^{th} order Lipschitz-type maximal function given by Lenze(1988) as

$$\widetilde{\omega}_r(f;x) = \sup_{t\neq x, t\in(0,\infty)} \frac{|f(t)-f(x)|}{|t-x|^r},\ x\in[0,\infty) \text{ and } r\in(0,1]. \tag{14}$$

Then, we get the next result

Theorem 4.4.

Let $f \in C_B[0,\infty)$ and $r \in (0,1]$. Then, for all $x \in [0,\infty)$, we have

$$|K_n^*(f;x) - f(x)| \leq \widetilde{\omega}_r(f;x)\Big(\eta_n(x)\Big)^{\frac{r}{2}}.$$

Proof:

We know that

$$|K_n^*(f;x) - f(x)| \leq K_n^*(|f(t)-f(x)|;x).$$

From equation (14), we have

$$|K_n^*(f;x) - f(x)| \leq \widetilde{\omega}_r(f;x) K_n^*(|t-x|^r;x).$$

From Hölder's inequality with $p = \frac{2}{r}$ and $q = \frac{2}{2-r}$, we have

$$|K_n^*(f;x) - f(x)| \leq \widetilde{\omega}_r(f;x)\big(K_n^*(|t-x|^2;x)\big)^{\frac{r}{2}},$$

which proves the desired result. ■

5. Global approximation

Here, we recall some notation from Gadziev(1976) to prove next result. Let $B_{1+x^2}[0,\infty) = \{f(x) : |f(x)| \leq M_f(1+x^2), 1+x^2$ is weight function, M_f is a constant depending on f and $x \in [0,\infty)\}$, $C_{1+x^2}[0,\infty)$ is the space of continuous function in $B_{1+x^2}[0,\infty)$ with the norm $\|f(x)\|_{1+x^2} = \sup\limits_{x\in[0,\infty)} \frac{|f(x)|}{1+x^2}$ and $C^k_{1+x^2}[0,\infty) = \{f \in C_{1+x^2} : \lim\limits_{|x|\to\infty} \frac{f(x)}{1+x^2} = k$, where k is a constant depending on $f\}$.

Theorem 5.1.

If the operators K_n^* defined by (4) from $C^k_{1+x^2}[0,\infty)$ to $B_{1+x^2}[0,\infty)$ satisfying the conditions

$$\lim_{n\to\infty} \|K_n^*(e_i;x) - x^i\|_{1+x^2} = 0,\ i = 0,1,2.$$

Then, for each $C^k_{1+x^2}[0,\infty)$

$$\lim_{n\to\infty} \|K_n^*(f;x) - f\|_{1+x^2} = 0.$$

Proof:

To prove Theorem 5.1, it is enough to show that

$$\lim_{n\to\infty} \|K_n^*(e_i;x) - x^i\|_{1+x^2} = 0,\ i = 0, 1, 2.$$

From Lemma 2.1, we have

$$\|K_n^*(e_0;x) - x^0\|_{1+x^2} = \sup_{x\in[0,\infty)} \frac{|K_n^*(1;x) - 1|}{1+x^2} = 0 \text{ for } i = 0.$$

For $i = 1$,

$$\begin{aligned}\|K_n^*(e_1;x) - x^1\|_{1+x^2} &= \sup_{x\in[0,\infty)} \frac{|\frac{a_n}{b_n}x + \frac{1}{b_n} - x|}{1+x^2}\\ &= \left(\frac{a_n}{b_n} - 1\right) \sup_{x\in[0,\infty)} \frac{x}{1+x^2} + \frac{1}{b_n} \sup_{x\in[0,\infty)} \frac{1}{1+x^2}.\end{aligned}$$

This implies that $\|K_n^*(e_1;x) - x^1\|_{1+x^2} \to 0$ an $n \to \infty$. For $i = 2$,

$$\begin{aligned}\|K_n^*(e_2;x) - x^2\|_{1+x^2} &= \sup_{x\in[0,\infty)} \frac{\left|\left(\frac{a_n^2}{b_n^2} - 1\right)x^2 + \frac{a_n}{b_n^2}\left(2 + 2\mu\frac{e_\mu(-a_nx)}{e_\mu(a_nx)}\right)x + \frac{1}{3b_n^2}\right|}{1+x^2}\\ &= \left(\frac{a_n^2}{b_n^2} - 1\right) \sup_{x\in[0,\infty)} \frac{x^2}{1+x^2}\\ &+ \frac{a_n}{b_n^2}\left(2 + 2\mu\frac{e_\mu(-a_nx)}{e_\mu(a_nx)}\right) \sup_{x\in[0,\infty)} \frac{x}{1+x^2}\\ &+ \frac{1}{3b_n^2} \sup_{x\in[0,\infty)} \frac{1}{1+x^2}.\end{aligned}$$

Which shows that $\|K_n^*(e_2;x) - x^2\|_{1+x^2} \to 0$ an $n \to \infty$. ∎

Next, we discuss a result to approximate each function belongs to $C^k_{1+x^2}[0,\infty)$. Similar result is investigated by Gadziev(1976) for locally integrable functions.

Theorem 5.2.

Let $f \in C^k_{1+x^2}[0,\infty)$ and $\gamma > 0$. Then we have,

$$\lim_{n\to\infty} \sup_{x\in[0,\infty)} \frac{|K_n^*(f;x)| - f(x)}{(1+x^2)^{1+\gamma}} = 0.$$

Proof:

For any fixed real number $x_0 > 0$, one has

$$\begin{aligned}\sup_{x\in[0,\infty)} \frac{|K_n^*(f;x)| - f(x)}{(1+x^2)^{1+\gamma}} &\leq \sup_{x\leq x_0} \frac{|K_n^*(f;x)| - f(x)}{(1+x^2)^{1+\gamma}} + \sup_{x\geq x_0} \frac{|K_n^*(f;x)| - f(x)}{(1+x^2)^{1+\gamma}} \\ &\leq \||K_n^*(f;x)| - f(x)\|_{C[0,x_0]} \\ &+ \|f\|_{1+x^2} \sup_{x\geq x_0} \frac{|K_n^*(1+t^2;x)|}{(1+x^2)^{1+\gamma}} + \sup_{x\geq x_0} \frac{|f(x)|}{(1+x^2)^{1+\gamma}} \\ &= J_1 + J_2 + J_3, say. \qquad (15)\end{aligned}$$

Since $|f(x)| \leq \|f\|_{1+x^2}(1+x^2)$, we have

$$\begin{aligned}J_3 &= \sup_{x\geq x_0} \frac{|f(x)|}{(1+x^2)^{1+\gamma}} \\ &\leq \sup_{x\geq x_0} \frac{\|f\|_{1+x^2}(1+x^2)}{(1+x^2)^{1+\gamma}} \leq \frac{\|f\|_{1+x^2}}{(1+x^2)^{\gamma}}.\end{aligned}$$

Let $\epsilon > 0$ be arbitrary real number. Then, from Theorem 3.2 there exists $n_1 \in \mathbb{N}$ such that

$$\begin{aligned}J_2 &< \frac{1}{(1+x^2)^{\gamma}} \|f\|_{1+x^2} \Big(1 + x^2 + \frac{\epsilon}{3\|f\|_{1+x^2}}\Big) \text{ for all } n_1 \geq n, \\ &< \frac{\|f\|_{1+x^2}}{(1+x^2)^{\gamma}} + \frac{\epsilon}{3} \text{ for all } n_1 \geq n.\end{aligned}$$

This implies that

$$J_2 + J_3 < 2\frac{\|f\|_{1+x^2}}{(1+x^2)^{\gamma}} + \frac{\epsilon}{3}.$$

Next, let for a large value of x_0, we have $\frac{\|f\|_{1+x^2}}{(1+x^2)^{\gamma}} < \frac{\epsilon}{6}$.

$$J_2 + J_3 < \frac{2\epsilon}{3} \text{ for all } n_1 \geq n. \qquad (16)$$

From Theorem 5.1, there exists $n_2 > n$ in such a way

$$J_1 = \|K_n^*(f) - f\|_{C[0,x_0]} < \frac{\epsilon}{3} \text{ for all } n_2 \geq n. \qquad (17)$$

Let $n_3 = max(n_1, n_2)$. Then, combining (15), (16) and (17), we have

$$\sup_{x\in[0,\infty)} \frac{|K_n^*(f;x)| - f(x)}{(1+x^2)^{1+\gamma}} < \epsilon.$$

Hence, the proof of Theorem 5.2 is completed. ■

6. Conclusion

The goal of this article is to give a better error estimation of convergence by modification of Szász Kantorovich operators via Dunkl analogue. We have defined a Szász-Kantorovich-Chlodowsky based on Dunkl analogue with the aid of two unbounded and increasing real numbers sequences $\{a_n\}$ and $\{b_n\}$. This type of modification enables better error estimation for a certain function in comparison to the Szász-Kantorovich operators based on Dunkl analogue. We investigated some approximation results by means of the well-known Korovkin-type theorem. We have also calculated the rate of convergence of operators by means of Peetre's K-functional and second order modulus of continuity. Lastly, we studied the global approximation results.

REFERENCES

Altomare, F. and Campiti, M. (1994). Korovkin-type approximation theory and its applications, Appendix A by Michael Pannenberg and Appendix B by Ferdinand Beckhoff. de Gruyter Studies in Mathematics, Vol. 17. Walter de Gruyter and Co., Berlin, pp.xii+627.

DeVore, R. A. and Lorentz, G. G. (1993). Constructive Approximation, Grudlehren der Mathematischen Wissenschaften [Fundamental principales of Mathematical Sciences], (Springer-Verlag, Berlin.

Gadziev, A. D. (1976). Theorems of the type of P.P. Korovkin's theorems, Mat. Zametki, Vol. 20, No. 5, pp. 781-786 (in Russian), Math. Notes Vol. 20, No. 5-6, pp. 995-998 (Engl. Trans.).

Icoz, G. and Cekim, B. (2015). Dunkl generalization of SzÃ¡sz operators via q-calculus, J Inequal Appl, 2015:284

Karaisa, A. and Karakoc, F. (2016). Stancu type generalization of Dunkl analogue of SzÃ sz Operators, Adv Appl Cliff Algebras, 26:1235. https://doi.org/10.1007/s00006-016-0643-4

Lenze, B. (1988). On Lipschitz type maximal functions and their smoothness spaces, Nederl Akad Indag Math., Vol. 50, pp. 53-63.

Rao, N. and Wafi, A. (2018). A Modifed Szasz-Integral type operators, Italian journal of pure and applied maths, (In press).

Rosenblum, M. (1994) Generalized Hermite polynomials and the Bose-like oscillator calculus, Oper. Theory Adv. Appl., Vol. 73, pp. 369-396.

Sucu, S. (2014). Dunkl analogue of Szász operators, Appl. Math. Comput., Vol. 244, pp. 42–48.

Wafi, A. and Rao, N. (2018) Sz*á*sz-Durrmeyer operators based on Dunkl analogue, Complex Anal. Oper. Theory, Vol. 12, No. 7, pp 1519-1536.

Wafi, A. and Rao, N. (2019). Sz*á*sz-Gamma Operators Based on Dunkl Analogue, Iran J Sci Technol Trans Sc, Vol. 43, No. 1, pp. 213-223.